Further remarks and examples

Last time we saw that Floer homology of a pair of closed monotone Lagrangian submanifolds $L_0, L_1 \subseteq (M,\omega)$ can be defined whenever

(a) both Lagrangians have minimal Maslov number at least 3,

or (b) L_0 and L_1 are Hamiltonianly isotopic and have minimal Maslov number ≥ 2.

Moreover, in both cases the resulting Floer homology is invariant under moving one of the Lagrangians by a Hamiltonian isotopy.
It immediately follows that if $HF(L_0, L_1) \neq 0$, then

$$L_0 \cap \varphi(L_1) \neq \emptyset$$

for every $\varphi \in \mathrm{Ham}(M,\omega)$.

Ex: We saw last time that $\mathbb{RP}^n \subseteq \mathbb{CP}^n$ is a monotone Lagrangian submanifold (with minimal Maslov number $n+1$) and computed its Floer homology as

$$HF(\mathbb{RP}^n, \mathbb{RP}^n; \mathbb{Z}_2) \cong H(\mathbb{RP}^n; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{n+1}$$

Ex: Reviewing the argument in case (b) above, one finds that in fact $HF(L_0, L_1)$ can be defined whenever both minimal Maslov numbers are at least 2 and the total number of Maslov index 2 disks through a generic point with boundary on either L_0 or L_1 is even.
Now consider the pair $(\mathbb{RP}^k, T^k)$ in $\mathbb{CP}^k$, where $T^k \subseteq \mathbb{CP}^k$ is the Clifford torus. Then $N_{\mathbb{RP}^k} = k+1$ and $N_{T^k} = 2$. One can check that the number of Maslov index 2 disks through a point on the Clifford torus is $k+1$. So if k is odd, this number is even and $HF(\mathbb{RP}^k, T^k)$ can be defined.

G. Alston proved that in this situation
$$HF(\mathbb{RP}^k, T^k) \cong (\mathbb{Z}_2)^{2^r} \text{ where } k = 2r-1.$$

As we have seen in examples, it is not true in general that for a monotone Lagrangian submanifold L we have an isomorphism between $HF_*(L,L)$ and $H_*(L,\mathbb{Z}_2)$.
However, Oh observed that the Floer differential can be split as
$$\partial = \partial_0 + \partial'$$
where ∂_0 counts "local" strips that stay close to L and its perturbed copy $\varphi(L)$, and the contributions to ∂' increase the grading of intersection points by Morse index. In fact, Floer's local computation in T^*L can be adapted to prove that
$$H_*(CF(L,\varphi(L)), \partial_0) \cong H_*(L;\mathbb{Z}_2).$$
Then ∂' defines a boundary operator there which preserves the filtration
$$F_p H_*(L;\mathbb{Z}_2) = \bigoplus_{k\geq p} H_k(L;\mathbb{Z}_2)$$
One gets a spectral sequence with initial page equal to $H_*(L;\mathbb{Z}_2)$ which converges to $HF_*(L,L)$.
He derives several consequences of this result.

Thm: Suppose L is a monotone Lagrangian submanifold in some symplectic manifold (M,ω). Then

(a) If the minimal Maslov number $N_L \geq n+2$, then $HF_*(L,L) \cong H_*(L;\mathbb{Z}_2)$.

(b) If $N_L = n+1$, the same is true for $* \neq 0, n \mod n+1$

Cor: For a closed monotone Lagrangian submanifold $L \subseteq \mathbb{C}^n$ we have
$$1 \leq N_L \leq n.$$

~~G. Alston proved that in this situation~~

$$\cancel{HF(\mathbb{RP}^k, T^k) \cong (\mathbb{Z}_2)^{2^r} \quad \text{where } k = 2r-1.}$$

The assumption of transverse intersection between L_0 and L_1 in the definition of the Floer complex means that even in the case of Floer homology of one monotone Lagrangian with itself we have to perturb one of the copies of L to get started with the theory as discussed so far.

There is an alternative approach, developed by Biran and Cornea, which we discuss next.
For a survey, see:

P. Biran, O. Cornea : A Lagrangian quantum homology
in: New perspectives and challenges in symplectic field theory
CRM Proceedings and Lecture Notes vol. 49
AMS, 2009, 1–44

Assumptions: (M,ω) closed or convex at infinity
$L \subseteq M$ closed monotone Lagrangian submanifold with $N_L \geq 2$.

Set $\Lambda := \mathbb{Z}_2[t, t^{-1}]$, the ring of Laurent polynomials, with grading $|t| = -N_L$.

Pick a Morse-Smale pair of a function $f: L \to \mathbb{R}$ and a metric g on L, and consider the negative gradient flow φ_t.
Also choose a generic ω-compatible a.c. structure J on M.

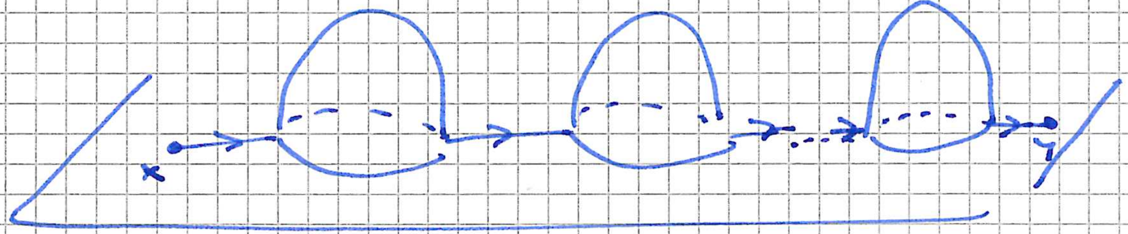
In this situation, Biran and Cornea define the pearl complex as follows:

$$\mathcal{C} := \underbrace{\mathbb{Z}_2\langle \mathrm{Crit}\, f\rangle}_{\substack{\text{vector space generated}\\ \text{by } \mathrm{Crit}(f)}} \otimes \Lambda$$

$\mathcal{C}$ is graded, where $|x| = $ Morse index for $x \in \mathrm{Crit}(f)$.

Given $x, y \in L$ and a class $0 \neq A \in H_2^D(M,L)$ = image of $\pi_2(M,L)$ in $H_2(M,L)$, consider the space of sequences $(u_1, \dots, u_\ell)$ of finite length $\ell \geq 1$ s.t.

* each $u_j : (D, S^1) \to (M, L)$ is a <u>nonconstant</u> J-holomorphic disk
* There is some $-\infty \leq t' < 0$ s.t. $\varphi_{t'}(u_1(-1)) = x$
* For every $1 \leq i \leq \ell - 1$ there is some $t_i^\bullet > 0$ s.t.
$$\varphi_{t_i^\bullet}(u_i(1)) = u_{i+1}(-1)$$
* There is some $0 < t'' \leq \infty$ s.t. $\varphi_{t''}(u_\ell(1)) = y$
* $[u_1] + \dots + [u_\ell] = A$ in $H^2(M,L)$.



For $A = 0$ one just considers ordinary Morse trajectories from x to y.

When $x, y \in \mathrm{Crit}(f)$, one checks that the expected dimension of the space of such "pearly trajectories" is
$$|x| - |y| + \mu(A) - 1 =: \delta(x, y; A)$$

<u>Prop 1</u>: If J is generic and $(x, y; A)$ are such that $\delta(x, y; A) = 0$, then the space of pearly trajectories from x to y is finite.

Denote by $\rho(x, y; A; f, g, J)$ the count of these pearly trajectories in this case, and define

$$d : \mathcal{C} \to \mathcal{C} \quad \text{on generators as}$$
$$d(x) = \sum_{y, A} \rho(x, y; A; f, g, J) \cdot y \cdot t^{\frac{\mu(A)}{N_L}}$$

Note that in our grading convention, this map has degree -1.

<u>Thm:</u> Under the above assumptions, the map d just defined is a boundary operator, i.e. $d \circ d = 0$.
The homology
$$QH(L) := H_*(\mathcal{C}, d)$$
is independent of the choice of the generic triple (f, g, J) up to canonical isomorphism.
Moreover, there is an isomorphism
$$\Theta : HF(L, L; \Lambda) \to QH(L)$$
which is canonical up to a shift in grading.

<u>Rem:</u> So far, we have viewed $HF(L,L)$ as a $\mathbb{Z}_2$-module with a grading modulo N_L (which depends on some choices). By working over $\Lambda = \mathbb{Z}_2[t, t^{-1}]$ instead of $\mathbb{Z}_2$ one can instead get a periodic theory with an absolute grading well-defined up to a global shift.

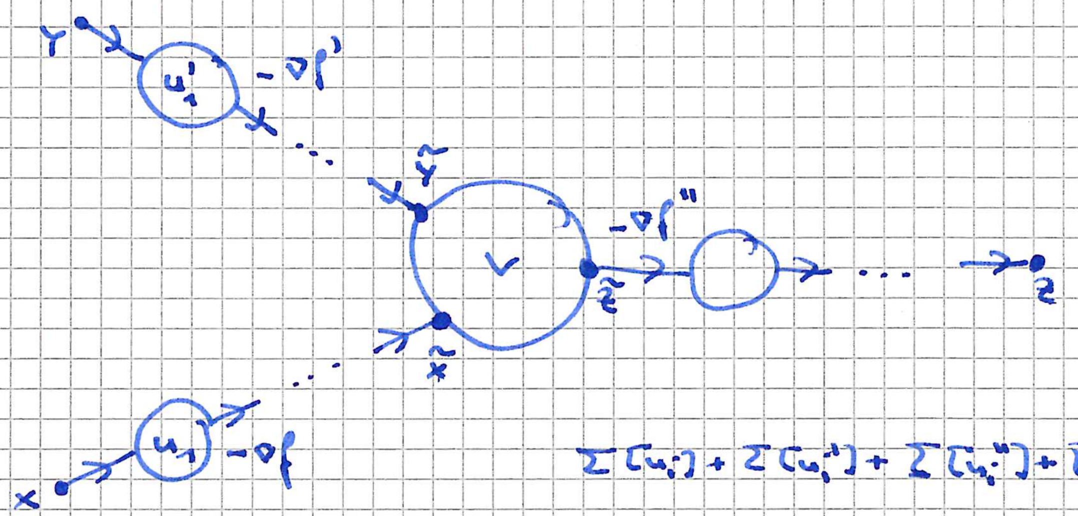
<u>Idea of proof:</u> To understand why an equation like $d \circ d = 0$ should hold, let us try to understand possible degenerations in a 1-parameter family of pearly trajectories:

(i) one of the finite length Morse trajectories could become longer and in the limit break at a critical point
(ii) one of the finite length Morse trajectories could shrink to zero length
(iii) one of the holomorphic disks could break into 2 disks (by bubbling)

One now argues that boundary points of type (ii) in a pearly trajectory of length l and of type (iii) in corresponding pearly trajectory of length $l-1$ come in pairs and so can be ignored, leaving us with boundary points of type (i) which we interpret as counting pairs of pearly trajectories appearing in $d \circ d$. ☐

There is a product structure on the module $QH(L)$ which is rather simple to describe:
Fix 3 Morse-Smale pairs (f,g), (f',g) and (f'',g) and a generic a.c. structure J.
Given $x \in Crit(f)$, $y \in Crit(f')$ and $z \in Crit(f'')$ and a class $A \in H_2^D(M,L)$ consider the following data:

* a J-holomorphic disk $v: (D, S^1) \to (M,L)$ (which may be constant)
* Set $\tilde{x} := v(e^{-\frac{2\pi i}{3}})$, $\tilde{y} := v(e^{\frac{2\pi i}{3}})$, $\tilde{z} := v(1)$
  Then we assume given pearly trajectories from x to $\tilde{x}$, from y to $\tilde{y}$ and from $\tilde{z}$ to z for the respective choice of Morse-Smale data:



$$\sum [u_i] + \sum [u_i'] + \sum [u_i''] + [v] = A$$

It turns out that the expected dimension of the space of such configurations is

$$\delta_{prod}(x,y,z;A) := |x| + |y| - |z| - n + \mu(A)$$

<u>Prop2</u>: For generic data and $\delta_{prod}(x,y,z;A) = 0$ there is a finite number $p(x,y,z;\dots)$ of configurations as above.

So we get a map

$$\circ : \mathcal{C}(f,g;J) \otimes \mathcal{C}(f',g;J) \to \mathcal{C}(f'',g;J)$$

defined on generators by

$$x \circ y = \sum_{z,A} p(x,y,z;A,\dots) \cdot z \cdot t^{\frac{\mu(A)}{N_L}}.$$

<u>Thm 2:</u> The map $\circ$ is a chain map, so it descends to homology to an operation compatible with the canonical isomorphisms, so one obtains a canonical operation

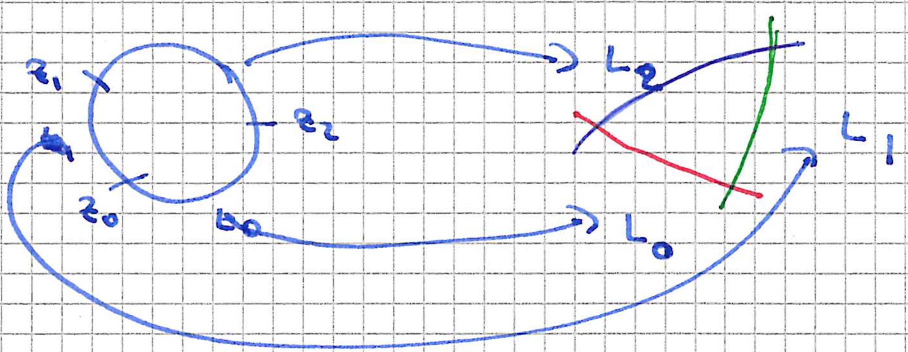
$$\circ : QH_i(L) \otimes QH_j(L) \to QH_{i+j-n}(L)$$

which turns $QH(L)$ into an associative ring with unit.

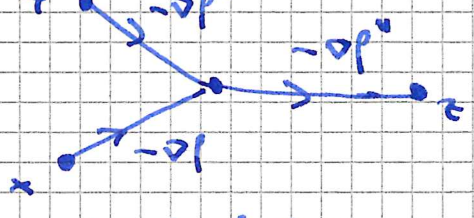
<u>Rem:</u>
- The unit of $\circ$ has degree n. If the Morse function used in the definition has a unique maximum, the this maximum defines a cycle which represents the unit.
- The isomorphism $HF(L,L) \cong QH(L)$ is an isomorphism of rings if on $HF(L,L)$ one considers the "triangle product",

$$\Delta : CF(L_0,L_1) \otimes CF(L_1,L_2) \to CF(L_0,L_2)$$

defined in terms of counting J-holomorphic maps $u : (D^2 \setminus \{z_0,z_1,z_2\}, S^1 \setminus \{z_0,z_1,z_2\}) \to M$



- The contribution of the class $A = 0 \in H_2^D(M,L)$ counts Morse configurations

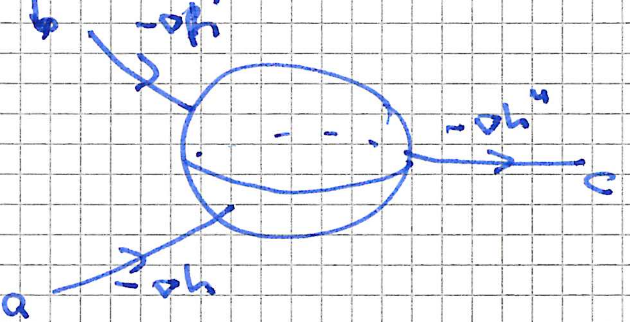


So one can think of this multiplication as a deformation of the usual ~~cap~~ intersection product (at least in cases when $QH(L) \cong H(L) \otimes \Lambda$).

The Lagrangian quantum homology $QH(L)$ of a monotone Lagrangian submanifold $L \subseteq (M,\omega)$ is a module over the quantum homology of the ambient symplectic manifold.

Set
$$\Gamma := \mathbb{Z}_2[s, s^{-1}] \quad \text{where } |s| = -2\min\{\langle c_1(TM), A\rangle > 0,\ A \in \pi_2(M)\}$$

Then
$$QH_*(M) = H_*(M;\mathbb{Z}_2) \otimes \Gamma$$
as a module. It carries a multiplication which deforms the usual intersection product by adding counts of configurations



For even more structure, see the survey by Biran and Cornea.

I want to end by mentioning some applications of the theory to symplectic topology.

Thm A: Suppose L_0 and L_1 are two monotone Lagrangians in $\mathbb{CP}^n$. If $QH(L_0) \neq 0$ and $QH(L_1) \neq 0$, then
$$L_0 \cap L_1 \neq \emptyset.$$

Thm B: Let $Q \subseteq \mathbb{CP}^n$ be the smooth complex quadric. Then any two Lagrangian spheres $L_0, L_1 \subset Q$ have nonempty intersection.

Thm C: Suppose $L \subseteq \mathbb{CP}^n$ is a closed Lagrangian submanifold with $2H_1(L;\mathbb{Z}) = 0$. Then

(i) There is a map $\varphi: L \to \mathbb{RP}^n$ inducing a ring isomorphism
$$\varphi_*: H_*(L;\mathbb{Z}_2) \to H_*(\mathbb{RP}^n;\mathbb{Z}_2)$$

(ii) Let $h = [\mathbb{CP}^{n-1}] \in H_{2n-2}(\mathbb{CP}^n;\mathbb{Z}_2)$. Then $h \,\tilde{\cap}\, [L]$ is the generator of $H_{n-2}(L;\mathbb{Z}_2)$

(iii) The inclusion $\iota : L \hookrightarrow \mathbb{CP}^n$ induces isomorphisms

$$\iota_* : H_j(L;\mathbb{Z}_2) \to H_j(\mathbb{CP}^n;\mathbb{Z}_2)$$

for all even $0 \leq j \leq n$.

Rem: From the point of view of all the structures here, L is indistinguishable from $\mathbb{RP}^n$. In fact, it is conceivable that any such L could be Hamiltonianly isotopic to $\mathbb{RP}^n \subseteq \mathbb{CP}^n$, but such a result is beyond these methods.

One can also prove results which can be formulated independently of Floer homology. Here is one example:

Thm D: (a) A symplectic ball $B \subseteq (\mathbb{CP}^n, \omega_{FS})$ disjoint from the Clifford torus fills at most $\frac{n}{n+1}$ of the total volume.

(b) A symplectic ball $B \subseteq (\mathbb{CP}^n, \omega_{FS})$ disjoint from $\mathbb{RP}^n$ fills at most half the volume.