Today we want to analyze the (failure of) compactness of our spaces

$$\mathcal{F}(p,q) = \widetilde{\mathcal{F}}(p,q)/_{\mathbb{R}}, \quad \mathcal{K}(p^-,p^+) \text{ and } \mathcal{H}(p^-,p^+).$$

Again we first focus on $\mathcal{F}(p,q)$.

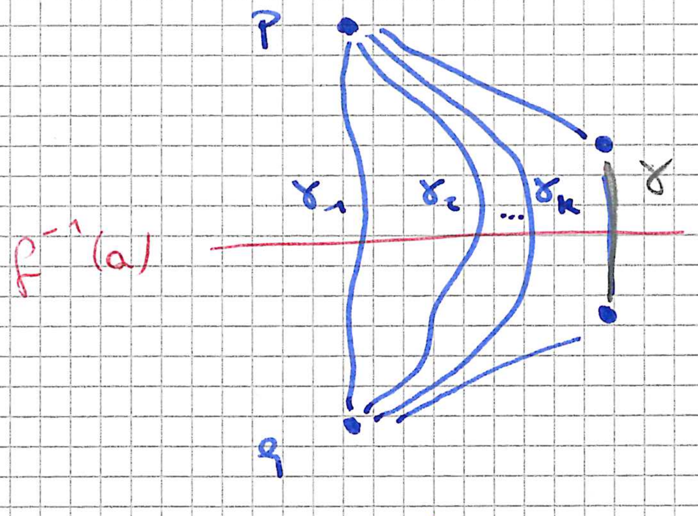
<u>Lemma:</u> $f: M \to \mathbb{R}$ Morse, M closed, g any metric, $p,q \in \mathrm{Crit}(f)$.
Let $\{\gamma_k\}_{k\geq 1}$ be any sequence of elements in $\widetilde{\mathcal{F}}(p,q)$, and let $f(p) > a > f(q)$ be a regular value.
Then, after passing to a subsequence, we can find real numbers $t_k \in \mathbb{R}$ s.t. the shifted trajectories

$$t_k \cdot \gamma_k = \gamma_k(\cdot + t_k)$$

converge in $C^\infty_{loc}(\mathbb{R}, M)$ to a flow line

$$\gamma: \mathbb{R} \to M$$

passing through $f^{-1}(a)$.



<u>Pf:</u> Since each γ_k runs from p to q, we find $t_k \in \mathbb{R}$ s.t. $\quad \gamma_k(t_k) \in f^{-1}(a)$.

By compactness of M, we find a subsequence s.t.

$$\gamma_k(t_k) \longrightarrow x_0 \in f^{-1}(a)$$

Set $\tilde{\gamma}_k(t) := \gamma_k(t+t_k)$, so that $\tilde{\gamma}_k(0) \in f^{-1}(a)$

and let $\gamma : \mathbb{R} \to M$ be the flow line with $\gamma(0) = x_0$.

Since all the curves we consider are solutions of the equation

$$\dot{\alpha}(t) = X(\alpha(t))$$

and the vector field X is uniformly bounded (since M is compact), we can use the Arzela-Ascoli theorem to get, for each $R > 0$, a further subsequence converging uniformly on $[-R, R]$.

Taking a final diagonal subsequence completes the proof.

$\square$

Proposition: M closed, (f,g) Morse-Smale pair
$p, q \in \mathrm{Crit}(f)$ $\quad r := \mathrm{ind}(p) - \mathrm{ind}(q)$

The space $\mathcal{F}(p,q)$ is compact up to broken trajectories with at most $r-1$ breaks, in the following sense:

Suppose $\{[\gamma_k]\}_{k \geq 1} \subseteq \mathcal{F}(p,q)$ is a sequence of trajectories. Then either the sequence converges in $\mathcal{F}(p,q)$, or there is a subsequence $\{\gamma_{k_n}\}_{n \geq 1}$,

a sequence

$$p = x_0, x_1, \ldots, x_s = q \qquad \text{with } 2 \leq s \leq r$$

of critical points,

trajectories $\gamma^j \in \tilde{\mathcal{F}}(x_{j-1}, x_j)$ for $1 \leq j \leq r$

and sequences of real numbers $\{t^j_{k_n}\}$ for $1 \leq j \leq r$

such that

$$\gamma_{k_n}(\cdot + t^j_{k_n}) \longrightarrow \gamma^j \quad \text{in } C^\infty_{loc}.$$

Moreover, we must have

$$f(p) > f(x_1) > \ldots > f(x_{s-1}) > f(q) \text{ and}$$

$$\mathrm{ind}(p) > \mathrm{ind}(x_1) > \ldots > \mathrm{ind}(x_{s-1}) > \mathrm{ind}(q).$$

Remark: It will be clear from the proof (as ~~should~~ should already be apparent from the statement) that

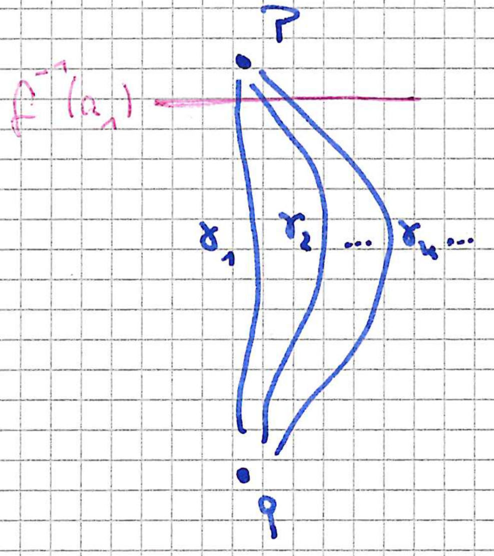
$$t^1_{k_n} < t^2_{k_n} < \dots < t^r_{k_n}$$

and $$t^{i+1}_{k_n} - t^i_{k_n} \longrightarrow \infty \quad \text{as } n \to \infty$$

Pf: (Sketch)

Using the exponential convergence at $\pm\infty$, it is easy to see that if $\gamma_k \to \gamma$ in C^∞_{loc} and $\gamma \in \mathcal{F}(p,q)$, i.e. the limit has the same asymptotics, then $\gamma_k \longrightarrow \gamma$ in $\mathcal{B}^{1,2}_{p,q}$ (i.e. we have convergence in the $W^{1,2}$-topology).

So suppose the sequence does not converge in $\mathcal{B}^{1,2}_{p,q}$.



Choose a regular value $a_1 < f(p)$ such that $(a_1, f(p))$ contains no critical value.

Applying the previous lemma, we get t^1_k, $k \geq 1$ and a flow line

$$\gamma^1 : \mathbb{R} \longrightarrow M$$

such that for a subsequence we have

$$\gamma_{k_n}(\cdot + t^1_{k_n}) \longrightarrow \gamma^1 \text{ in } C^\infty_{loc}.$$

Now γ^1 must connect two critical points p' and q' with

$$f(p) \geq f(p') > a_1 > f(q') \geq f(q) \qquad \text{(why?)}$$

It now follows from our choice of a_1 that we must have $f(p') = f(p)$ and therefore $p' = p$.

If $q' = q$, then we are done. Otherwise, set $x_1 := q'$, and note that $\operatorname{ind}(x_1) < \operatorname{ind}(p)$ by our Morse-Smale assumption.

Now we repeat the process, picking $a_2 < f(x_1)$ such that $(a_2, f(x_1))$ contains no critical ~~pa~~ values.

With the same argument as before, we find $t^2_{k_n} > t^1_{k_n}$ and $\gamma^2 : \mathbb{R} \to M$ such that

$$\gamma_{k_{n_m}}(\cdot + t^2_{k_{n_m}}) \longrightarrow \gamma^2 \quad \text{in } C^\infty_{loc}$$

and

$$\lim_{t \to -\infty} \gamma^2(t) = x_1 .$$

This procedure will stop after at most $r = \operatorname{ind}(p) - \operatorname{ind}(q)$ steps, because

$$\operatorname{ind}(x_{i+1}) < \operatorname{ind}(x_i)$$

$$\text{and} \quad \operatorname{ind}(x_i) \geq \operatorname{ind}(q) \quad \text{for all } i. \quad \square$$

The above proposition describes the possible limit points of $\mathcal{F}(p,q)$. ~~To prove that indeed any, for~~

$\cancel{\operatorname{ind}(p) - \operatorname{ind}(p)}$

In particular, it proves that for $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$, our spaces $\mathcal{F}(p,q)$ must be compact, since there are no critical points with index lying strictly between $\operatorname{ind}(p)$ and $\operatorname{ind}(q)$.

It remains to treat the case of index difference 2. Our goal is to show that for every critical point $r \in \operatorname{Crit}(f)$ ~~we~~ with

$$\operatorname{ind}(p) > \operatorname{ind}(r) > \operatorname{ind}(q)$$

every configuration in ~~$\mathcal{F}(p,r)$~~ $\mathcal{F}(p,q) \times \mathcal{F}(r,q)$ indeed appears as a boundary point in an essentially unique way.

Gluing Theorem (Thm 3 in sec. 2.5. in Schwarz)

Let M be closed, (f,g) a Morse-Smale pair, $p,r,q \in \mathrm{Crit}(f)$

Thm:

(a) If $\tilde{K} \subseteq \hat{\tilde{\mathcal{F}}}(p,r) \times \tilde{\mathcal{F}}(r,q)$ is compact, there exists $\tilde{\rho}_K > 0$ and a smooth map

$$\# : \tilde{K} \times [\tilde{\rho}_K, \infty) \to \hat{\tilde{\mathcal{F}}}(p,q)$$

$$(\gamma_1, \gamma_2, \rho) \longmapsto \gamma_1 \#_\rho \gamma_2$$

such that for any fixed $\rho \geq \rho_K$ the map

$$\#_\rho : \tilde{K} \to \hat{\tilde{\mathcal{F}}}(p,q)$$

is an embedding.

(b) Given a compact subset $K \subseteq \mathcal{F}(p,r) \times \mathcal{F}(r,q)$, the map $\#$ above induces a smooth embedding

$$\# : K \times [\rho_K, \infty) \to \mathcal{F}(p,q)$$

such that

$$[\gamma_1] \#_\rho [\gamma_2] \xrightarrow[\rho \to \infty]{} ([\gamma_1], [\gamma_2])$$

in C^∞_{loc} in the sense of the previous proposition, i.e. with appropriate time shifts t^1_ρ and t^2_ρ.

(c) Every sequence in $\mathcal{F}(p,q)$ converging to a broken trajectory $([\gamma_1],[\gamma_2]) \in K$ eventually lies in the image of $\#$.

Remarks:

(1) Many of the complications in Schwarz' proof arise from the fact that he considers gluing in a family K, so he has to worry about smooth dependence of the construction on the parameters.

The case we are interested in is $\mathrm{ind}(p) - \mathrm{ind}(q) = 2$, and so

$$K = \mathcal{F}(p,r) \times \mathcal{F}(r,q),$$

which is a compact 0-manifold, i.e. a finite set.

We will describe the steps of the proof now, assuming for simplicity $K = \{([\gamma_1], [\gamma_2])\}$ is a single point.

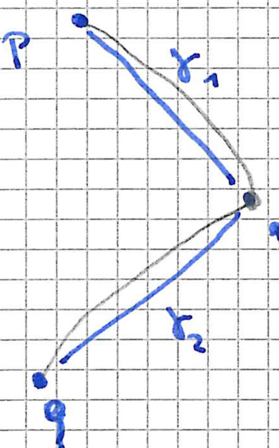
Step 1: (pre-gluing)

Let $(\gamma_1, \gamma_2) \in \tilde{\mathcal{F}}(p,r) \times \tilde{\mathcal{F}}(r,q)$ be given.

We choose curves $\alpha \in C^\infty_{p,r}$ and $\beta \in C^\infty_{r,q}$ such that $\gamma_1 \in \mathcal{U}_\alpha$ and $\gamma_2 \in \mathcal{U}_\beta$, the coordinate charts defined for α and β.

Since we are free to choose α and β, we may arrange that

$$\alpha(t) = r \quad \text{for } t \geq T \quad \text{and}$$
$$\beta(t) = r \quad \text{for } t \leq -T$$



$\gamma_1 \in \mathcal{U}_\alpha$ means $\gamma_1(t) = \exp_{\alpha(t)} s_1(t)$

$\gamma_2 \in \mathcal{U}_\beta$ means $\gamma_2(t) = \exp_{\beta(t)} s_2(t)$

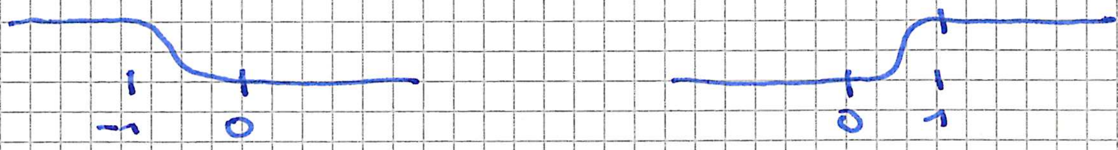
By our choice of α and β, for $\rho > T$ the curve

$$(\alpha \#_\rho \beta)(t) := \begin{cases} \alpha(t+\rho) & t \leq 0 \\ \beta(t-\rho) & t \geq 0 \end{cases}$$

is in $C^\infty_{p,q}$.

Fix cutoff functions $\lambda^\pm : \mathbb{R} \to [0,1]$ such that

$$\lambda^-(t) = \begin{cases} 1 & \text{if } t \leq -1 \\ 0 & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad \lambda^+(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq 1 \end{cases}$$



Now we define the pregluing map $(\rho_0 = T+1)$

$$\#^0 : [\rho_0, \infty) \longrightarrow \mathcal{P}^{1,2}_{p,q}$$
$$\rho \longmapsto \gamma_1 \#^0_\rho \gamma_2$$

as

$$\gamma_1 \#_\rho^0 \gamma_2(t) = \exp_{\alpha \#_\rho \beta(t)}\left(\lambda^-(t) s_1(t+\rho) + \lambda^+(t) s_2(t-\rho)\right)$$

$$= \begin{cases} \gamma_1(t+\rho) & \text{if } t \leq -1 \\ \exp_r\left(\lambda^-(t) s_1(t+\rho) + \lambda^+(t) s_2(t-\rho)\right) & \text{if } |t| \leq 1 \\ \gamma_2(t-\rho) & \text{if } t \geq 1 \end{cases}$$

<u>Step 2:</u> (Linearization)

Recall that $\widetilde{F}(p,q)$ is the zero set of the section

$$F : \mathcal{P}^{1,2}_{p,q} \longrightarrow L^2(\mathcal{P}^*_{p,q} TM)$$
$$\gamma \longmapsto \dot{\gamma} + \nabla f \circ \gamma$$

We will denote by $D_\rho := D_{\gamma_\rho} F : W^{1,2}_{\gamma_\rho} \to L^2_{\gamma_\rho}$

its linearization at $\gamma_\rho = \gamma_1 \#_\rho^0 \gamma_2$.

We also have the linearizations $D_{\gamma_1} = D_{\gamma_1} F$ and $D_{\gamma_2} = D_{\gamma_2} F$

Using the same cutoff functions as above, we get a linear version of pregluing

$$\#_\rho : W^{1,2}_{\gamma_1} \times W^{1,2}_{\gamma_2} \longrightarrow W^{1,2}_{\gamma_\rho}$$

which glues shifted tangent vectors (i.e. vector fields along γ_1 and γ_2).

~~We define~~

<u>Proposition:</u> (Prop. 2.50 in Schwarz)

$\exists \rho_1 \geq \rho_0$ such that for all $\rho \geq \rho_1$ the linearization

$$D_{\gamma_\rho} : W^{1,2}_{\gamma_\rho} \to L^2_{\gamma_\rho} \quad \text{is onto.}$$

Moreover, the composition of linear pregluing with the L^2-orthogonal projection $W^{1,2}_{\gamma_\rho} \to \ker D_{\gamma_\rho}$

induces an isomorphism

$$\phi_\rho : \ker D_{\gamma_1} \times \ker D_{\gamma_2} \to \ker D_{\gamma_\rho} .$$

Set
$$L^1_\rho := \{ \vartheta \in W^{1,2}_{\gamma_\rho} : \langle \vartheta, \xi \#_\rho \eta \rangle = 0 \text{ for all } (\xi, \eta) \in \ker D_{\gamma_1} \times \ker D_{\gamma_2} \}$$

It follows from the proposition that L^1_ρ maps isomorphically onto $L^2_{\gamma_\rho}$ under $D_{\gamma_\rho}|_{L^1_\rho} : L^1_\rho \to L^2_{\gamma_\rho}$.

We denote by $G_\rho : L^2_\rho \to L^1_\rho$ the inverse of this isomorphism.

<u>Lemma</u>: $\exists\, \rho_2 \geq \rho_1$ such that there is a constant $C_1 > 0$ satisfying
$$\| G_\rho \xi \|_{1,2} \leq C_1 \| \xi \|_{L^2}$$
for all $\xi \in L^2_\rho$.

To summarize:

The linearizations of F along the preglued curves $\gamma_\rho = \gamma_1 \#^0_\rho \gamma_2$ have uniformly bounded right inverses for ρ sufficiently large.