Lagrangian Floer theory for monotone Lagrangians

Recall that a symplectic manifold (M,ω) is called monotone if there is a constant $\tau > 0$ s.t. for all maps $v: S^2 \to M$ we have

$$\int_{S^2} v^* c_1(TM) = \tau \cdot \int_{S^2} v^* \omega .$$

For weakly monotone manifolds we also allow $\tau = 0$.

For a Lagrangian submanifold $L \subseteq (M,\omega)$, we have relative analogues of these notions, namely

$$\begin{aligned} \mu : \pi_2(M,L) &\to \mathbb{Z} \\ [u] &\mapsto \text{Maslov index of the loop } t \mapsto T_{u(e^{2\pi i t})} L \\ &\qquad \text{in a trivialization of } u^*TM \end{aligned}$$

and

$$\begin{aligned} A : \pi_2(M,L) &\to \mathbb{R} \\ [u] &\mapsto \int_{D^2} u^* \omega . \end{aligned}$$

Def: A Lagrangian submanifold $L \subseteq (M,\omega)$ is called monotone if there is a constant $\kappa > 0$ s.t.

$$A = \kappa \cdot \mu \quad \text{on } \pi_2(M,L)$$

Remark:
* Note that we have switched the roles of the two terms.
* A monotone Lagrangian submanifold can only exist inside a monotone symplectic manifold. This is because changing the homotopy class of a disk
$$u: (D^2, S^1) \to (M, L)$$
by taking the connected sum with a sphere
$$v: S^2 \to M$$
changes the Maslov index by $\int_{S^2} 2 c_1(M)$.

In particular, the monotonicity constants are related by

$$\kappa = \frac{1}{2}\tau^{-1}$$

Note: I realized that my definition of monotone symplectic manifolds (taken from Salamon's lecture notes) uses the opposite convention to more recent texts, e.g. the book "J-holomorphic curves in symplectic topology" by McDuff and Salamon, where it is written in the form

$$(*) \qquad \int_{S^2} v^*\omega = C \cdot \int_{S^2} v^* c_1(TM) \quad , \; C > 0$$

The difference becomes important when one allows the boundary case where the constant vanishes. Of course (*) fits better with our current discussion.

Examples:

(1) A product torus $(S^1)^n \subseteq (\mathbb{R}^{2n}, \omega_{st})$ is monotone if and only if each of the circles $S^1 \subseteq \mathbb{R}^2$ encloses the same area.

(2)* Suppose $I : (M,\omega) \to (M,\omega)$ is an antisymplectic involution, i.e. $I^*\omega = -\omega$, and $L = \mathrm{Fix}(I)$. Then L is a monotone Lagrangian submanifold.

*(M,ω) is assumed monotone.

To see this, start with a disk

$$u : (D^2, S^1) \to (M, L).$$

Then $\hat{u}(z) = I \circ u(\bar{z})$ is another disk whose boundary traverses the curve $u|_{S^1}$ in the opposite direction. It is obvious that

$$A(\hat{u}) = A(u),$$

and one checks similarly that $\mu(\hat{u}) = \mu(u)$. Gluing u and $\hat{u}$ along their common boundary gives rise to a map

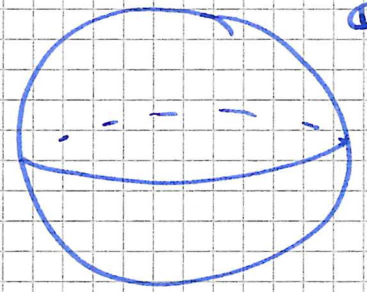
$$v : S^2 \to M$$

with $A(v) = 2A(u)$ and $\langle [v], c_1(TM) \rangle = \mu(u)$. So if $C > 0$ is the monotonicity constant

of (M,ω) in convention ⊛, then L is monotone with constant
$$\kappa = \frac{1}{2} c$$
as required.

As a specific example, consider $\mathbb{R}P^n \subseteq \mathbb{C}P^n$.

(i) For $n=1$, we get an equator in S^2



$\pi_2(S^2, S^1)$ has rank 2 and is generated by the two "visible" disks, which have equal area and Maslov index 2 (when parametrized as orientation preserving embeddings, so that they correspond to u and $\hat{u}$ as above).
Monotonicity follows.

(ii) For $n \geq 2$, it follows from the long exact sequence of homotopy groups

$$\ldots \to \pi_2(\mathbb{R}P^n) \xrightarrow{0} \underset{\underset{\mathbb{Z}}{\parallel}}{\pi_2(\mathbb{C}P^n)} \to \pi_2(\mathbb{C}P^n, \mathbb{R}P^n) \to \overset{=\mathbb{Z}_2}{\pi_1(\mathbb{R}P^n)} \to \underset{=0}{\pi_1(\mathbb{C}P^n)}$$

that $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$.

Exercise: Check that in fact $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n) \cong \mathbb{Z}$.

A generator is given by the embedding of one of the hemispheres from the above $n=1$ situation. This time we have
$$\mu(u) = n+1.$$
Monotonicity is obvious since any two homomorphisms from $\mathbb{Z}$ to $\mathbb{R}$ which are positive on the same generator are clearly positively proportional.

3 The Clifford torus in $\mathbb{CP}^n$.

One can obtain $\mathbb{CP}^n$ from a ball $B^{2n}(r)$ by collapsing the Hopf circles in the boundary sphere $S^{2n-1}(r)$ under the projection

$$pr: B^{2n}(r) \longrightarrow \mathbb{CP}^n$$

a coordinate disk $D^2(r) \subseteq B^{2n}(r)$ will project to a generator of $\pi_2(\mathbb{CP}^n)$. Choosing $r = \frac{1}{\sqrt{\pi}}$, this generator will have area 1.
It is known that $c_1(\mathbb{CP}^n)$ evaluates to $n+1$ on this generator, so $(\mathbb{CP}^n, \omega_{FS})$ with this normalization has monotonicity constant $\frac{1}{n+1}$.

Now consider the product tori

$T^n(\rho) := (S^1(\rho))^n \subseteq \mathbb{R}^{2n}$.
As each coordinate circle bounds an obvious disk of Maslov index 2 and area $\pi\rho^2$, these tori are monotone with constant $\frac{\pi\rho^2}{2}$ in $\mathbb{R}^{2n}$.

Now choose $\rho_0 = \frac{1}{\sqrt{\pi(n+1)}}$, so that $\pi\rho_0^2 = \frac{1}{n+1}$.

Since $T^n(\rho) \subseteq S^{2n-1}(\sqrt{n}\cdot r)$, we conclude that

$$T^n(\rho_0) \subseteq B^{2n}\left(\frac{1}{\sqrt{\pi}}\right)$$

so this torus gives rise to a Lagrangian torus in $\mathbb{CP}^n$, called the <u>Clifford torus</u>.

<u>Claim:</u> The Clifford torus in $\mathbb{CP}^n$ is monotone.

<u>Pf:</u> The long exact sequence in homotopy reads

$$\begin{array}{ccccccccc} \pi_2(T^n) & \to & \pi_2(\mathbb{CP}^n) & \to & \pi_2(\mathbb{CP}^n, T^n) & \xrightarrow{\partial} & \pi_1(T^n) & \to & \pi_1(\mathbb{CP}^n) \\ \| & & \cong & & & & \cong & & \| \\ 0 & & \mathbb{Z} & & & & \mathbb{Z}^n & & 0 \end{array}$$

The coordinate disks give rise to a splitting

$$s: \pi_1(T^n) \longrightarrow \pi_2(\mathbb{CP}^n, T^n),$$

so that
$$\pi_2(\mathbb{CP}^n, T^n) \cong \pi_2(\mathbb{CP}^n) \oplus \pi_1(T^n)$$

The computations above (together with $\mu = 2c_1$ on the first summand) show that the Clifford torus is monotone in $\mathbb{CP}^n$ with monotonicity constant $\frac{1}{2(n+1)}$.

We are interested in moduli spaces of holomorphic disks with boundary on a Lagrangian submanifold $L \subseteq (M, \omega)$. For $\alpha \in \pi_2(M, L)$ we set

$$\mathcal{M}(\alpha) := \{ u : (D^2, S^1) \to (M, L) \;:\; \bar\partial_J u = 0, [u] = \alpha \} \Big/ \mathrm{Aut}(D^2)$$

and

$$\mathcal{M}_1(\alpha) := \{ u : (D^2, S^1) \to (M, L) \;:\; \bar\partial_J u = 0, [u] = \alpha \} \Big/ \mathrm{Aut}(D^2, 1)$$

Fact: The expected dimension of these spaces is

$$\dim \mathcal{M}(\alpha) = n - 3 + \mu(\alpha)$$
$$\dim \mathcal{M}_1(\alpha) = n - 2 + \mu(\alpha).$$

Prop: Suppose $\alpha \in \pi_2(M, L)$ is a primitive class, i.e. not an integer multiple of another class. Then for a generic a.c. structure J compatible with ω the spaces $\mathcal{M}(\alpha)$ and $\mathcal{M}_1(\alpha)$ are manifolds of the expected dimension.

Remark: In contrast to the space $\mathcal{M}(\alpha)$, the space $\mathcal{M}_1(\alpha)$ comes with a well-defined evaluation map

$$ev_1 : \mathcal{M}_1(\alpha) \to L$$
$$[u] \mapsto u(1).$$

In general, these spaces are not compact, as there can be sphere or disk bubbling.
Recall that the minimal Maslov number of a Lagrangian $L \subseteq (M, \omega)$ is the positive generator of $\mu(\pi_2(M, L)) \subseteq \mathbb{Z}$.

A crucial consequence of monotonicity is the following:

Thm: Suppose $L \subseteq (M,\omega)$ is a monotone Lagrangian submanifold and $\alpha \in \pi_2(M,L)$ is such that

$\mu(\alpha) = N_L$ = minimal Maslov number of L

Then the spaces $\mathcal{M}(\alpha)$ and $\mathcal{M}_1(\alpha)$ are compact.

Pf: If we had disk bubbling, there would be classes α_1 and α_2 containing nonconstant J-holomorphic disk s.t. $\alpha_1 + \alpha_2 = \alpha$.
Nonconstant disks have positive symplectic area, and so by monotonicity positive Maslov number. This contradicts the minimality of $\mu(\alpha)$.
Similarly, if we had sphere bubbling there would be classes $\alpha_1 \in \pi_2(M,L)$, $\beta \in \pi_2(M)$ s.t. $\alpha = \alpha_1 + \beta$ and α_1 contains a nonconstant J-holomorphic disk, and β contains a nonconstant holomorphic sphere.
Again monotonicity implies $\mu(\alpha_1) > 0$ and $\langle c_1(M), \beta\rangle > 0$. But

$$\mu(\alpha) = \mu(\alpha_1) + 2\langle c_1(M), \beta\rangle$$

and so again we contradict minimality of $\mu(\alpha)$.

(Technically, we would have to argue not just for simple bubbling, but for potential bubble trees with several disk and/or sphere components. Clearly, the argument generalizes to cover this as well).

Since we excluded all bubbling, our spaces must be compact.

$\square$

<u>Corollary:</u> Suppose $L \subseteq (M,\omega)$ is a monotone Lagrangian submanifold which is orientable and $\alpha \in \pi_2(M,L)$ satisfies
$$\mu(\alpha) = N_L = 2.$$
Then for every point $x \in L$, the algebraic count of J-holomorphic disks passing through x is independent of $x \in L$ and of the choice of J with the property that $\mathcal{M}_1(\alpha)$ is regular and $ev_1 : \mathcal{M}_1(\alpha) \to L$ is transverse to x (this is generic).

Indeed, for a given J this number is the degree of the map
$$ev_1 : \mathcal{M}_1(\alpha) \to L$$
To compare different J, connect them by a family $\{J_t\}_{t\in[0,1]}$ and observe that the resulting space
$$\mathcal{M}_1(\alpha, \{J_t\}) = \bigcup_{t\in[0,1]} \mathcal{M}_1(\alpha, J_t)$$
is still compact by the same bubbling analysis as before. This implies that the evaluation maps for J_0 and J_1 have the same degree.

□

We can now state the main result of todays lecture, originally due to Y.-G. Oh.

<u>Thm:</u> Suppose $L_0, L_1 \subseteq (M,\omega)$ are monotone closed Lagrangian submanifolds intersecting transversely.

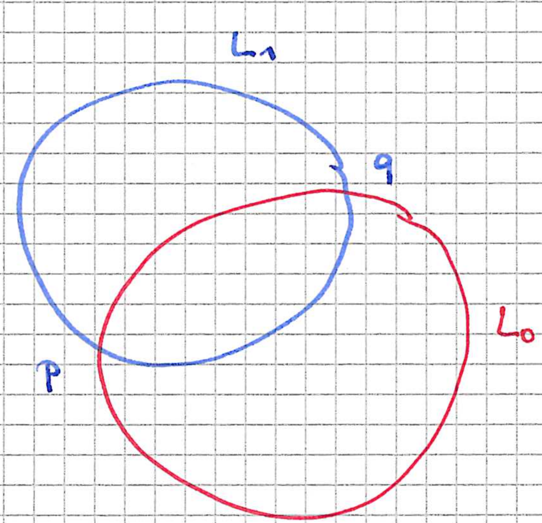
(a) The Floer homology $HF(L_0,L_1)$ is well-defined and invariant under moving the Lagrangian submanifolds by Hamiltonian isotopies provided that
$$\min(N_{L_0}, N_{L_1}) \geq 3.$$

(b) If $L_1 = \varphi(L_0)$ for a Hamiltonian diffeo morphism $\varphi \in \mathrm{Ham}(M,\omega)$, then one can weaken the assumption to

$$N_{L_0} = N_{L_1} \geq 2.$$

Rem: The improvement in (b) is due to Salamon. It is significant because it allows the definition of $HF(L,L)$ for any orientable closed Lagrangian.

Ex: We consider two circles $L_0, L_1 \in \mathbb{R}^2$ enclosing the same area (so they are Hamiltonianly isotopic). As a consequence of the invariance statement, we must have $HF(L_0,L_1) = 0$ because they can be made disjoint by moving one of them.
This is indeed the case:



$$\partial q = p$$
$$\partial p = 0$$

Sketch of proof for theorem:

It follows from monotonicity that all Floer strips with the same index and the same asymptotics have the same area.
To get a well-defined boundary operator, we need to prove compactness (up to $\mathbb{R}$-shift) of the components of index 1 in moduli spaces of holomorphic strips.
In principle, a limiting configuration of a sequence of index 1 strips could consist of a broken strip $(u^1, \dots, u^k)$ with J-holomorphic spheres $v_1, \dots, v_r$ and J-holomorphic disks $w_1, \dots, w_s$ attached.

As the Maslov index is additive under taking connected sum with a disk and adds $2\langle c_1(TM), [v_j]\rangle$ for each sphere, we would need to have

$$1 = \sum \mu(u^i) + 2\sum \langle c_1(TM), [v_j]\rangle + \sum \mu(w_j)$$

By regularity, the summands in the first sum are ≥ 0. By monotonicity, if there are summands in the other two sums they must be strictly positive, and $\geq N_{L_0}$ or N_{L_1} respectively.

So we conclude that in fact the second and third sums are empty, and the first sum has exactly one nontrivial summand, i.e. the limiting configuration is a single strip. This shows that components $\mathcal{M}^1(p,q)$ of index 1 are compact, i.e. finite.

The argument for

$$\partial^2 = 0$$

is quite similar, but looking at the compactification of moduli spaces of strips of index 2.

If the two asymptotic intersection points p and q are distinct, then there must be at least one nontrivial strip, and then the assumption $N_{L_i} \geq 2$ suffices to rule out bubbling off of disks in this case.

If $p=q$, then we could have a limiting configuration of a constant strip at p and either

(i) a J-holomorphic sphere of Chern class 1

or (ii) a J-holomorphic disk of Maslov index 2.

To exclude (i), we observe that the moduli space of J-holomorphic spheres in a homotopy class A has dimension

$$2(n-3) + 2\langle c_1(TM), A\rangle = 2n-4$$

in our case

So the union of their images forms a subset of codimension 2 in M, and so for generic J it will miss the finitely many intersection points of L_0 and L_1

In case (a) of the theorem, the case (ii) is excluded by the assumption that the minimal Maslov number of either Lagrangian is at least 3.
In case (b) of the theorem, this phenomenon can and will typically occur. Instead, one observes that the number of index 2 disks passing through a given intersection point $p \in L_0 \cap L_1$ with boundary on L_0 is the same as those with boundary on L_1. One now argues that indeed all these configurations really can be glued to nearby index 2 strips, so each one of them adds one boundary point. As the total number is even, we find that the total number of other boundary points, which correspond to once broken strips, is also even, and so in this case $\partial^2 = 0$ as well.

The moduli spaces one considers in the proof of invariance of Lagrangian Floer homology under Hamiltonian isotopy are of index 0 (for the definition of the chain map) and 1 (for proving the property of being a chain map), so the argument proceeds as in the definition of ∂.

$\square$