

Spectral disjointness of powers of diffeomorphisms with arbitrary Liouvillean rotation behavior

PHILIPP KUNDE*

Abstract

We show that on any smooth compact connected manifold of dimension $m \geq 2$ admitting a smooth non-trivial circle action preserving a smooth volume ν the set of $(\kappa_1, \dots, \kappa_k)$ -weakly mixing C^∞ -diffeomorphisms is generic in $\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} : h \in \text{Diff}^\infty(M, \nu)\}}^{C^\infty}$ for every Liouvillean number α , every $k \in \mathbb{N}$ and specific tuples $(\kappa_1, \dots, \kappa_k) \in [0, 1]^k$. In particular, these diffeomorphisms have spectrally disjoint powers. The proof is based on a quantitative version of the Approximation by Conjugation-method with explicitly defined conjugation maps and partitions.

1 Introduction

For a start, we recall that a dynamical system (X, T, μ) on a probability space (X, μ) is said to be weakly mixing if there is no nonconstant function $h \in L^2(X, \mu)$ such that $h(Tx) = \lambda \cdot h(x)$ for some $\lambda \in \mathbb{C}$. Equivalently there is an increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that $\lim_{n \rightarrow \infty} |\mu(B \cap T^{-m_n}(A)) - \mu(A) \cdot \mu(B)| = 0$ for every pair of measurable sets $A, B \subseteq X$ (see [Sk167] or [AK70, Theorem 5.1]). A. Katok and A. Stepin introduced the more general notion of κ -weak mixing ([Ka03], [St87]):

Definition 1.1. An automorphism T of a Lebesgue probability space (X, μ) is said to be κ -weakly mixing, $\kappa \in [0, 1]$, if there exists a strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that the weak convergence

$$U_T^{m_n} \longrightarrow_w (\kappa \cdot P_c + (1 - \kappa) \cdot Id)$$

holds, where $U_T : L^2(X, \mu) \rightarrow L^2(X, \mu)$, $f \mapsto f \circ T$, is the Koopman-operator induced by T and P_c is the projection on the subspace of constants.

By [St87, Proposition 3.1.] we can characterize this property in geometric language: A transformation T is κ -weakly mixing if and only if there is an increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that for all measurable sets A and B

$$\lim_{n \rightarrow \infty} \mu(A \cap T^{m_n} B) = \kappa \cdot \mu(A) \cdot \mu(B) + (1 - \kappa) \cdot \mu(A \cap B).$$

*Pennsylvania State University, Department of Mathematics, State College, PA 16802, USA. P.K. acknowledges financial support from a DFG Forschungsstipendium under Grant No. 405305501.

2010 Mathematics Subject classification: Primary: 37A30; Secondary: 37A05, 37C05, 37C40, 37E45, 47A35

Key words: Smooth ergodic theory, spectral disjointness, $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms, approximation-by-conjugation method

We recognize that 0-weakly mixing corresponds to rigidity and 1-weakly mixing to the usual notion of weak mixing. So, κ -weakly mixing expresses some recurrence and weak mixing properties. In fact for $\kappa > 0$ a κ -weakly mixing transformation has a continuous spectrum due to [St87, Proposition 3.4]. Thus it is weakly mixing.

The concept of κ -weakly mixing has implications on the spectral properties of the transformation (see [St87, Theorem 1]): If $\kappa \in (0, 1)$ and T is a κ -weakly mixing transformation, then a measure σ_0 of maximal spectral type of the operator U_T on the orthogonal complement $H_0 \subset L^2(X, \mu)$ to the subspace of constants satisfies that it and its convolutions σ_0^k are pairwise mutually singular. This property is called *disjointness of convolutions*. It is linked to a conjecture of Kolmogorov respectively Rokhlin and Fomin (after verifying that the property held for all dynamical systems known at that time, especially large classes of systems of probabilistic origin like Gaussian ones), namely that every ergodic transformation possesses the so-called group property, i.e. the maximal spectral type σ is symmetric and dominates its square $\sigma * \sigma$. This conjecture is an analogue of the well-known group property of the set of eigenvalues of an ergodic automorphism and was proven to be false. Indeed, in [St66] Stepin gave the first example of a dynamical system without the group property. Oseledets constructed an analogous example with continuous spectrum ([Os69]). Later Stepin showed that for a generic transformation all convolutions σ^k , $k \in \mathbb{N}$, of the maximal spectral type σ on $L_0^2(X, \mu)$ are mutually singular (see [St87]) exploiting the concept of κ -weak mixing.

In [JL92] del Junco and Lemanczyk introduced the following strengthening of the notion of κ -weak mixing:

Definition 1.2. Let T be an automorphism of a probability space (X, \mathcal{B}, μ) and $\kappa_1, \dots, \kappa_k \in [0, 1]$. Then T is called $(\kappa_1, \dots, \kappa_k)$ -weakly mixing, if there is an increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that for each $i = 1, \dots, k$ and for all $A, B \in \mathcal{B}$

$$\lim_{n \rightarrow \infty} \mu(A \cap T^{i \cdot m_n} B) = \kappa_i \cdot \mu(A) \cdot \mu(B) + (1 - \kappa_i) \cdot \mu(A \cap B).$$

In other words, each T^i is κ_i -weakly mixing along the common sequence $(m_n)_{n \in \mathbb{N}}$. With the aid of this concept del Junco and Lemanczyk were able to prove that for a generic automorphism for each $k(1), \dots, k(l) \in \mathbb{N}$, $k'(1), \dots, k'(l') \in \mathbb{N}$ the convolutions $\sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}}$ and $\sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l'')}}$ are mutually singular unless $(k(1), \dots, k(l))$ is a permutation of $(k'(1), \dots, k'(l'))$ ([JL92, Theorem 1]). Hereby, they were also able to get a description of the centralizer of $T^{k_1} \times T^{k_2} \times \dots$ as well as to reproduce several counterexamples (like non-disjoint transformations that have no common factor or automorphisms with no roots) of Rudolph ([Ru79, section 4]) in a broader context.

With regard to the *Smooth realization problem* one is interested in finding smooth diffeomorphisms with these properties. In general, it is an important question in Ergodic Theory (e.g. [OW91, p.89]) that dates back to the foundational paper [Ne32] of von Neumann.

The smooth realization problem. *Are there smooth versions to the objects and concepts of abstract ergodic theory?*

By a smooth version we mean a smooth diffeomorphism of a compact manifold preserving a measure equivalent to the volume element. The only known restriction is due to A. G. Kushnirenko who proved that such a diffeomorphism must have finite entropy. On the other hand, there is a lack of general results on the smooth realization problem.

One of the most powerful tools of constructing smooth diffeomorphisms with prescribed ergodic or topological properties is the so-called *approximation by conjugation*-method (also known as the *AbC*-method or *Anosov-Katok*-method) developed by D. Anosov and A. Katok in [AK70]. In fact,

on every smooth compact connected manifold M of dimension $m \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{S}^1}$ preserving a smooth volume ν this method enables the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weakly mixing ones in [AK70, section 5]) or non-standard smooth realizations of measure-preserving systems (e.g. [AK70, section 6] and [FSW07]). These diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2} \in \mathbb{Q}$, $H_n = H_{n-1} \circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $S_{\frac{1}{q_n}} \circ h_n = h_n \circ S_{\frac{1}{q_n}}$. In each step the conjugation map h_n and the parameter k_n are chosen such that the diffeomorphism f_n imitates the desired property with a certain precision. Then the parameter l_n is chosen large enough to guarantee closeness of f_n to f_{n-1} in the C^∞ -topology and so the convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ to a limit diffeomorphism is provided. It is even possible to keep this limit diffeomorphism within any given C^∞ -neighbourhood of the initial element S_{α_1} or, by applying a fixed diffeomorphism g first, of $g \circ S_{\alpha_1} \circ g^{-1}$. So the construction can be carried out in a neighbourhood of any diffeomorphism conjugate to an element of the action. Thus, $\mathcal{A}(M) = \overline{\{h \circ S_t \circ h^{-1} : t \in \mathbb{S}^1, h \in \text{Diff}^\infty(M, \nu)\}}^{C^\infty}$ is a natural space for the produced diffeomorphisms. Moreover, we will consider the restricted space $\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} : h \in \text{Diff}^\infty(M, \nu)\}}^{C^\infty}$ for $\alpha \in \mathbb{S}^1$. Another feature of the AbC-method is the possibility to deduce statements on the genericity of the constructed properties in $\mathcal{A}(M)$ or $\mathcal{A}_\alpha(M)$. As mentioned above Anosov and Katok proved that the set of weakly mixing diffeomorphisms is generic in $\mathcal{A}(M)$ in the $C^\infty(M)$ -topology. More specifically, for every Liouville number α the set of weakly mixing diffeomorphisms is generic in $\mathcal{A}_\alpha(M)$ ([FS05], [GK18]). See also [FK04] for more details and other results of this method.

Using a smooth variant of the method of approximation by periodic transformations Stepin constructed a smooth κ -weakly mixing diffeomorphism in [St87, section 4]. Another construction of a smooth diffeomorphisms without the group property even in the restricted space $\mathcal{A}_\alpha(M)$ for arbitrary Liouville number α was exhibited in [Ku16].

In this paper we construct the first smooth $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms under some additional technical assumption: For $k \in \mathbb{N}$ we define the set Π_k of tuples $(\kappa_1, \dots, \kappa_k) \in [0, 1]^k$ satisfying $\kappa_1 < \kappa_2 < \dots < \kappa_k$, $\kappa_k > k \cdot (\kappa_2 - \kappa_1)$ as well as $\kappa_k - \kappa_{k-1} < \kappa_{k-1} - \kappa_{k-2} < \dots < \kappa_2 - \kappa_1$.

Theorem. *Let M be a smooth compact connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume ν . Moreover, let $k \in \mathbb{N}$ and $(\kappa_1, \dots, \kappa_k) \in \Pi_k$. If $\alpha \in \mathbb{R}$ is Liouville, then the set of volume-preserving $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms contains a dense G_δ -set in the C^∞ -topology in $\mathcal{A}_\alpha(M)$.*

The additional condition $(\kappa_1, \dots, \kappa_k) \in \Pi_k$ is a shortcoming of our method of proof. We elaborate on this in Remark 3.3. It would be interesting to see if smooth $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms could be constructed without this condition.

However, our theorem implies the following Corollary.

Corollary 1. *If α is a Liouville number, then for a generic $T \in \mathcal{A}_\alpha(M)$ we have*

$$\sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l)'}}$$

for every $k(1), \dots, k(l), k'(1), \dots, k'(l)' \in \mathbb{N}$ unless $(k(1), \dots, k(l))$ is a permutation of $(k'(1), \dots, k'(l)')$. In particular, the powers of T are spectrally disjoint.

At this point, we recall that two automorphisms T and S are *spectrally disjoint* if the maximal spectral types σ_T and σ_S are mutually singular. In particular, spectral disjointness implies disjointness in the sense of Furstenberg ([Fu67]). Recently, spectral disjointness of rescalings of some surface flows is studied in [BK].

Another motivation to study spectral disjointness of different powers comes from Sarnak's conjecture stating that for every homeomorphism T of a compact metric space X with topological entropy zero, any $\varphi \in C(X)$ and any $x \in X$ the sequence $(\varphi(T^n x))_{n \in \mathbb{N}}$ is orthogonal to the Möbius function, i. e. we have

$$\frac{1}{N} \sum_{n=1}^N \varphi(T^n x) \boldsymbol{\mu}(n) \rightarrow 0 \text{ as } N \rightarrow \infty, \quad (1)$$

where the Möbius function $\boldsymbol{\mu} : \mathbb{N} \rightarrow \mathbb{Z}$ is defined by $\boldsymbol{\mu}(1) = 1$, $\boldsymbol{\mu}(n) = 0$ for non-square-free positive integers and $\boldsymbol{\mu}(n) = \pm 1$ depending on the parity of the number of prime factors for the remaining positive integers (see [Sa], [ELR14], and [FKL18] for a survey of recent results). By the very recent preprint [KLR] Sarnak's conjecture holds true for uniformly rigid systems like AbC diffeomorphisms. In fact, it is shown in [BSZ12] that the spectral disjointness of different prime powers implies the validity of a generalized version of equation (1), where $\boldsymbol{\mu}$ can be replaced by any bounded multiplicative function of the positive integers.

Remark 1.3. By some modifications using the conjugation maps as in [GK18] and [Ku20] we are even able to construct the $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms in such a way that they preserve a measurable Riemannian metric and that their projectivized derivative extension is ergodic with respect to a measure in the projectivization of the tangent bundle which is absolutely continuous in the fibers.

Remark 1.4. Recently, great progress has been made in extending the AbC-method to the real-analytic category in case of tori \mathbb{T}^m , $m \geq 2$, in a series of papers ([Ba17], [Ku17], [BK19]). All these constructions base on the concept of *block-slide type maps* on the torus and their sufficiently precise approximation by volume-preserving real-analytic diffeomorphisms. By this approach it is possible to adapt the constructions of this paper to show that for every $m \in \mathbb{N}$, $m \geq 2$, $k \in \mathbb{N}$, $\rho > 0$ and $(\kappa_1, \dots, \kappa_k) \in \Pi_k$ there are real-analytic $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms $T \in \text{Diff}_\rho^\omega(\mathbb{T}^m, \mu)$.

2 Preliminaries

We start by showing how the proof for the general case of a compact connected smooth manifold follows from constructions on $\mathbb{S}^1 \times [0, 1]^{m-1}$. Then we give an outline of this construction.

2.1 First steps of the proof

First of all we show how constructions on $\mathbb{S}^1 \times [0, 1]^{m-1}$ can be transferred to a general compact connected smooth manifold M with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$, $S_{t+1} = S_t$. By [AK70, Proposition 2.1.] we can assume that 1 is the smallest positive number satisfying $S_t = \text{id}$. Hence, we can assume \mathcal{S} to be effective. We denote the set of fixed points of \mathcal{S} by F and for $q \in \mathbb{N}$ F_q is the set of fixed points of the map $S_{\frac{1}{q}}$. On the other hand, we consider $\mathbb{S}^1 \times [0, 1]^{m-1}$ with Lebesgue measure μ . Furthermore let $\mathcal{R} = \{R_\alpha\}_{\alpha \in \mathbb{S}^1}$ be the standard action of \mathbb{S}^1 on $\mathbb{S}^1 \times [0, 1]^{m-1}$, where the map R_α is given by $R_\alpha(\theta, r_1, \dots, r_{m-1}) = (\theta + \alpha, r_1, \dots, r_{m-1})$. Hereby we can formulate the following result (see [FSW07, Proposition 1]):

Proposition 2.1. *Let M be a m -dimensional smooth, compact and connected manifold admitting an effective circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume ν . Let $B :=$*

$\partial M \cup F \cup \left(\bigcup_{q \geq 1} F_q \right)$. There exists a continuous surjective map $G : \mathbb{S}^1 \times [0, 1]^{m-1} \rightarrow M$ with the following properties:

1. The restriction of G to $\mathbb{S}^1 \times (0, 1)^{m-1}$ is a C^∞ -diffeomorphic embedding.
2. $\nu \left(G \left(\partial \left(\mathbb{S}^1 \times [0, 1]^{m-1} \right) \right) \right) = 0$
3. $G \left(\partial \left(\mathbb{S}^1 \times [0, 1]^{m-1} \right) \right) \supseteq B$
4. $G_*(\mu) = \nu$
5. $\mathcal{S} \circ G = G \circ \mathcal{R}$

By the same reasoning as in [FSW07, section 2.2] this proposition allows us to carry a construction from $\left(\mathbb{S}^1 \times [0, 1]^{m-1}, \mathcal{R}, \mu \right)$ to the general case (M, \mathcal{S}, ν) :

Suppose $f : \mathbb{S}^1 \times [0, 1]^{m-1} \rightarrow \mathbb{S}^1 \times [0, 1]^{m-1}$ is a $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphism sufficiently close to R_α in the C^∞ -topology obtained by $f = \lim_{n \rightarrow \infty} f_n$ with $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $f_n = R_{\alpha_{n+1}}$ in a neighbourhood of the boundary (in Proposition 2.2 we will see that these conditions can be satisfied in the constructions of this article). Then we define a sequence of diffeomorphisms:

$$\tilde{f}_n : M \rightarrow M \quad \tilde{f}_n(x) = \begin{cases} G \circ f_n \circ G^{-1}(x) & \text{if } x \in G \left(\mathbb{S}^1 \times (0, 1)^{m-1} \right) \\ S_{\alpha_{n+1}}(x) & \text{if } x \in G \left(\partial \left(\mathbb{S}^1 \times (0, 1)^{m-1} \right) \right) \end{cases}$$

Constituted in [FK04, section 5.1] (which bases upon [Ka79, Proposition 1.1]), this sequence is convergent in the C^∞ -topology to the diffeomorphism

$$\tilde{f} : M \rightarrow M \quad \tilde{f}(x) = \begin{cases} G \circ f \circ G^{-1}(x) & \text{if } x \in G \left(\mathbb{S}^1 \times (0, 1)^{m-1} \right) \\ S_\alpha(x) & \text{if } x \in G \left(\partial \left(\mathbb{S}^1 \times (0, 1)^{m-1} \right) \right) \end{cases}$$

provided the closeness from f to R_α in the C^∞ -topology.

We observe that f and \tilde{f} are metrically isomorphic. Then \tilde{f} is $(\kappa_1, \dots, \kappa_k)$ -weakly mixing because the $(\kappa_1, \dots, \kappa_k)$ -weak mixing-property is invariant under isomorphisms.

Altogether the construction done in the case of $\left(\mathbb{S}^1 \times [0, 1]^{m-1}, \mathcal{R}, \mu \right)$ is transferred to (M, \mathcal{S}, ν) . Hence it suffices to consider constructions on $M = \mathbb{S}^1 \times [0, 1]^{m-1}$ with circle action \mathcal{R} subsequently. In this case we will prove the following result:

Proposition 2.2. *For every $k \in \mathbb{N}$, $(\kappa_1, \dots, \kappa_k) \in \Pi_k$ and every Liouvillean number α there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of rational numbers $\alpha_n = \frac{p_n}{q_n}$ satisfying $\lim_{n \rightarrow \infty} |\alpha - \alpha_n| = 0$ monotonically and a sequence of measure-preserving diffeomorphisms $(h_n)_{n \in \mathbb{N}}$ satisfying $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$ such that the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $H_n = h_1 \circ h_2 \circ \dots \circ h_n$, coincide with $R_{\alpha_{n+1}}$ in a neighbourhood U_n of the boundary shrinking with n , converge in the $\text{Diff}^\infty(M)$ -topology and the diffeomorphism $f = \lim_{n \rightarrow \infty} f_n$ is $(\kappa_1, \dots, \kappa_k)$ -weakly mixing and satisfies $f \in \mathcal{A}_\alpha(M)$.*

Furthermore for every $\varepsilon > 0$ the parameters in the construction can be chosen in such a way that $d_\infty(f, R_\alpha) < \varepsilon$.

2.2 Outline of the proof

The constructions are based on the “approximation by conjugation”-method developed by D.V. Anosov and A. Katok in [AK70]. Here one constructs successively a sequence of volume-preserving diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where the conjugation maps $H_n = h_1 \circ \dots \circ h_n$ and the rational numbers $\alpha_n = \frac{p_n}{q_n}$ are chosen in such a way that the functions f_n converge to a diffeomorphism f with the aimed properties.

First of all, we will define the conjugation map h_n as a composition of two volume-preserving diffeomorphisms $h_n = g_n \circ \phi_n$. Here, g_n is the shear $g_n(\theta, r_1, \dots, r_{m-1}) = (\theta + nq_n \cdot r_1, r_1, \dots, r_{m-1})$ and ϕ_n is a step-by-step defined smooth volume-preserving diffeomorphism. Its construction in section 3.1 bases on maps of the form $\phi_\lambda^{(j)} = C_\lambda^{-1} \circ \varphi_{1,j} \circ C_\lambda$ with C_λ being a stretching by $\lambda \in \mathbb{N}$ in the first coordinate and $\varphi_{1,j}$ a “quasi-rotation”, i.e. a rotation by $\frac{\pi}{2}$ in the x_1 - x_j -coordinates on large part of the domain. In fact, ϕ_n will be of the form $\phi_{\lambda_m}^{(m)} \circ \dots \circ \phi_{\lambda_2}^{(2)}$ with explicitly chosen parameters $\lambda_j \in \mathbb{N}$, $\lambda_j < \lambda_{j+1}$, on the different sections. Moreover, we define a sequence of partial partitions η_n whose elements have such a small diameter that even the image under $H_{n-1} \circ g_n$ converges to the decomposition into points. We will prove the $(\kappa_1, \dots, \kappa_k)$ -weak mixing property on those partition elements.

Like in the criteria for weak mixing in [FS05], [GK18] and [Ku17] we use the concept of “almost uniform distribution”. Descriptively, a set of small diameter is “almost uniformly distributed” under a diffeomorphism Φ if it is mapped to a set of small width in the θ -coordinate and almost full length in the r_1, \dots, r_{m-1} -coordinates in a uniform way (see Definition 4.1 for the precise definition). In our case we use this concept for the maps $\Phi_n^i = \phi_n \circ R_{\alpha_{n+1}}^{im_n} \circ \phi_n^{-1}$ with $i = 1, \dots, k$ and a specific sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that $R_{\alpha_{n+1}}^{m_n}$ causes a translation by $\frac{1}{nq_n}$ to the adjacent domain of definition of the map ϕ_n (see section 4). In Lemma 4.3 we make the key observation that

$$\phi_{\mu_m}^{(m)} \circ \dots \circ \phi_{\mu_2}^{(2)} \circ R_{\alpha_{n+1}}^{i \cdot m_n} \circ \left(\phi_{\lambda_2}^{(2)} \right)^{-1} \circ \left(\phi_{\lambda_m}^{(m)} \right)^{-1}$$

almost uniformly distributes an element of the partition η_n if $\mu_j > \lambda_j$ for each $j = 2, \dots, m$. On the other hand, we observe that Φ_n^i acts approximately as the translation by $R_{\frac{i}{nq_n}}$ if $\mu_j = \lambda_j$. On this account, the choice of parameters λ_j in the definition of ϕ_n is exactly done in such a way that after a translation by $R_{\frac{i}{nq_n}}$ we have an increase of the λ -values on a portion of about κ_i of the partition element (see Lemma 4.4). Thus, we will have “almost uniform distribution” under Φ_n^i on a portion of about κ_i of the partition element. The subsequent application of the shear g_n will cause that this is almost uniformly distributed on the whole manifold $\mathbb{S}^1 \times [0, 1]^{m-1}$. In Section 5 we use this to establish the $(\kappa_1, \dots, \kappa_k)$ -weak mixing property in our construction.

In section 6 we will show convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ in \mathcal{A}_α for a given Liouville number α by the same approach as in [FS05] and [GK18]. For this purpose, we have to estimate the norms $\|H_n\|_k$ very carefully.

Finally, we prove the genericity statement of the Theorem in Section 7 and present some applications of the result in Section 8

3 Explicit constructions

We fix $(\kappa_1, \dots, \kappa_k) \in \Pi_k$ and an arbitrary Liouvillean number $\alpha \in \mathbb{S}^1$. We also introduce the notation $\tilde{\beta}_1 = \kappa_1$, $\tilde{\beta}_i = \kappa_i - \kappa_{i-1}$ for $i = 2, \dots, k$. Obviously, $\sum_{i=1}^d \tilde{\beta}_i = \kappa_d$. In particular, we have

$\sum_{i=1}^k \tilde{\beta}_i = \kappa_k \leq 1$. By the requirements on tuples in Π_k we also have

$$\sum_{i=1}^k \tilde{\beta}_i > k \cdot \tilde{\beta}_2, \quad \tilde{\beta}_i > \tilde{\beta}_{i+1} \text{ for every } i = 2, \dots, k-1.$$

Moreover, let $(\tilde{\beta}_{i,n})_{n \in \mathbb{N}}$ be a sequence of rational numbers $\tilde{\beta}_{i,n} = \frac{c_{i,n}}{d_n}$ with some common denominator d_n and satisfying $\tilde{\beta}_{i,n} \rightarrow \tilde{\beta}_i$ as $n \rightarrow \infty$ as well as the relations

$$\sum_{i=1}^k \tilde{\beta}_{i,n} > k \cdot \tilde{\beta}_{2,n}, \quad \tilde{\beta}_{i,n} > \tilde{\beta}_{i+1,n} \text{ for every } i = 2, \dots, k-1, \quad \sum_{i=1}^k \tilde{\beta}_{i,n} \leq 1.$$

We note $k \cdot c_{2,n} < d_n$. With the aid of these numbers $\tilde{\beta}_{i,n}$ we also introduce the numbers $u_{i,n} \in \mathbb{Z}$, $i = 0, \dots, k-1$ such that

$$u_{i,n} = d_n \cdot (\tilde{\beta}_{2,n} - \tilde{\beta}_{2+i,n}) \text{ for } i = 0, \dots, k-2, \quad u_{k-1,n} = d_n \cdot \tilde{\beta}_{2,n} = c_{2,n}. \quad (2)$$

In particular, we have $u_{0,n} = 0$ and $u_{i+1,n} > u_{i,n}$. These numbers will be used in the definition of the conjugation map ϕ_n to attribute particular mapping behavior on the right proportion of its domain (see Remark 3.3).

Finally, we also introduce the numbers

$$\kappa_{i,n} = \sum_{d=1}^i \tilde{\beta}_{d,n} \quad \text{for } i = 1, \dots, k. \quad (3)$$

Since $\tilde{\beta}_{d,n} \rightarrow \tilde{\beta}_d$ for $n \rightarrow \infty$, we also have $\kappa_{i,n} \rightarrow \kappa_i$, i. e. the numbers $\kappa_{i,n}$ are rational approximations of κ_i .

3.1 The conjugation map ϕ_n

The construction of the conjugation map ϕ_n bases on the following ‘‘pseudo-rotations’’ from [FS05].

Lemma 3.1. *For every $\varepsilon \in (0, \frac{1}{4})$ and every $i, j \in \{1, \dots, m\}$ there exists a smooth measure-preserving diffeomorphism $\varphi_{\varepsilon, i, j}$ on \mathbb{R}^m , which is the rotation in the $x_i - x_j$ -plane by $\pi/2$ about the point $(\frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^m$ on $[2\varepsilon, 1 - 2\varepsilon]^m$ and coincides with the identity outside of $[\varepsilon, 1 - \varepsilon]^m$.*

Proof. See [Ku16, Lemma 4.1]. □

Furthermore, for $\lambda \in \mathbb{N}$ we define the maps $C_\lambda(x_1, x_2, \dots, x_m) = (\lambda \cdot x_1, x_2, \dots, x_m)$. Using these maps we build the smooth measure-preserving diffeomorphism

$$\phi_{\lambda, \varepsilon, j} : \left[0, \frac{1}{\lambda}\right] \times [0, 1]^{m-1} \rightarrow \left[0, \frac{1}{\lambda}\right] \times [0, 1]^{m-1}, \quad \phi_{\lambda, \varepsilon, j} = C_\lambda^{-1} \circ \varphi_{\varepsilon, 1, j} \circ C_\lambda$$

Afterwards $\phi_{\lambda, \varepsilon, j}$ is extended to a diffeomorphism on $\mathbb{S}^1 \times [0, 1]^{m-1}$ by the description

$$\phi_{\lambda, \varepsilon, j} \left(x_1 + \frac{1}{\lambda}, x_2, \dots, x_m \right) = \left(\frac{1}{\lambda}, 0, \dots, 0 \right) + \phi_{\lambda, \varepsilon, j} (x_1, x_2, \dots, x_m).$$

By construction the map $\phi_{\lambda, \varepsilon, j}$ satisfies the following properties:

Proposition 3.2. *Let $j \in \{2, \dots, m\}$, $\varepsilon \in (0, \frac{1}{4})$ and $\lambda \in \mathbb{N}$. Moreover, let $t_0 \in \mathbb{Z}$, $\mu_s \in \mathbb{N}$, $t_s \in \mathbb{Z}$, $\lceil 2\varepsilon\mu_s \rceil \leq t_s < \mu_s - \lceil 2\varepsilon\mu_s \rceil$, for $s = 1, \dots, m$. Then we have*

1.

$$\begin{aligned} & \phi_{\lambda, \varepsilon, j}^{-1} \left(\left[\frac{t_0}{\lambda} + \frac{t_1}{\lambda \cdot \mu_1}, \frac{t_0}{\lambda} + \frac{t_1 + 1}{\lambda \cdot \mu_1} \right] \times \prod_{s=2}^m \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right] \right) \\ &= \left[\frac{t_0}{\lambda} + \frac{t_j}{\lambda \mu_j}, \frac{t_0}{\lambda} + \frac{t_j + 1}{\lambda \mu_j} \right] \times \prod_{s=2}^{j-1} \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right] \times \left[1 - \frac{t_1 + 1}{\mu_1}, 1 - \frac{t_1}{\mu_1} \right] \times \prod_{s=j+1}^m \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right]. \end{aligned}$$

2.

$$\begin{aligned} & \phi_{\lambda, \varepsilon, j} \left(\left[\frac{t_0}{\lambda} + \frac{t_1}{\lambda \cdot \mu_1}, \frac{t_0}{\lambda} + \frac{t_1 + 1}{\lambda \cdot \mu_1} \right] \times \prod_{s=2}^m \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right] \right) \\ &= \left[\frac{t_0 + 1}{\lambda} - \frac{t_j + 1}{\lambda \mu_j}, \frac{t_0 + 1}{\lambda} - \frac{t_j}{\lambda \mu_j} \right] \times \prod_{s=2}^{j-1} \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right] \times \left[\frac{t_1}{\mu_1}, \frac{t_1 + 1}{\mu_1} \right] \times \prod_{s=j+1}^m \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right]. \end{aligned}$$

In particular, we get

$$\begin{aligned} & \phi_{\lambda, \varepsilon, j} \left(\left[\frac{t_0 + 2\varepsilon}{\lambda}, \frac{t_0 + 1 - 2\varepsilon}{\lambda} \right] \times \prod_{s=2}^m \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right] \right) \\ &= \left[\frac{t_0 + 1}{\lambda} - \frac{t_j + 1}{\lambda \mu_j}, \frac{t_0 + 1}{\lambda} - \frac{t_j}{\lambda \mu_j} \right] \times \prod_{s=2}^{j-1} \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right] \times [2\varepsilon, 1 - 2\varepsilon] \times \prod_{s=j+1}^m \left[\frac{t_s}{\mu_s}, \frac{t_s + 1}{\mu_s} \right]. \end{aligned}$$

We start to define the diffeomorphism ϕ_n on the sector $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2} \right] \times [0, 1]^{m-1}$ for each $l, v \in \mathbb{Z}$, $0 \leq l < n$, $0 \leq v < nq_n$. For convenience we will use the notation $\phi_\lambda^{(j)} = \phi_{\lambda, \frac{1}{40n^4}, j}$. In a first step, we consider domains of the form

$$\Delta_{l, v, s, u} = \left[\frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{s \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2} + \frac{u}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{s \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2} + \frac{u+1}{d_n \cdot n^2 \cdot q_n^2} \right] \times [0, 1]^{m-1},$$

where $s, u \in \mathbb{Z}$, $0 \leq s < k$ and $0 \leq u < c_{2,n}$.

We define $\tilde{s} \equiv -l \pmod{k}$. Then for every $s \in \mathbb{Z}$, $0 \leq s < k$, there is a unique $t \in \{0, 1, \dots, k-1\}$ such that $s \equiv \tilde{s} + t \pmod{k}$. Recall the numbers $u_{i,n}$, $0 \leq i < k-1$, from equation (2).

- If the number $u \in \mathbb{Z}$, $0 \leq u < c_{2,n}$, satisfies $u_{i,n} \leq u < u_{i+1,n}$ with $i \geq t-1$, then we put

$$\phi_n = \phi_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot (t+k-t)}}^{(m)} \circ \dots \circ \phi_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot (t+k-t)}}^{(3)} \circ \phi_{d_n \cdot (nq_n)^{2 \cdot (t+k-t)}}^{(2)}$$

on the domain $\Delta_{l, v, s, u}$.

- If the number $u \in \mathbb{Z}$, $0 \leq u < c_{2,n}$, satisfies $u_{i,n} \leq u < u_{i+1,n}$ with $i < t - 1$, then we put

$$\phi_n = \phi_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot (l+k)}}^{(m)} \circ \cdots \circ \phi_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot (l+k)}}^{(3)} \circ \phi_{d_n \cdot (nq_n)^{2 \cdot (l+k)}}^{(2)}$$

on the domain $\Delta_{l,v,s,u}$.

In the next step we recall the requirements $\sum_{i=1}^k \tilde{\beta}_{i,n} > k \cdot \tilde{\beta}_{2,n}$ and $\sum_{i=1}^k \tilde{\beta}_{i,n} \leq 1$ which translate into $\sum_{i=1}^k c_{i,n} > k \cdot c_{2,n}$ and $\sum_{i=1}^k c_{i,n} \leq d_n$. Then we put

$$\phi_n = \phi_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot (l+k)}}^{(m)} \circ \cdots \circ \phi_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot (l+k)}}^{(3)} \circ \phi_{d_n \cdot (nq_n)^{2 \cdot (l+k)}}^{(2)}$$

on the domain

$$\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{k \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{\sum_{i=1}^k c_{i,n}}{d_n \cdot n^2 \cdot q_n^2} \right] \times [0, 1]^{m-1}.$$

Finally, we put

$$\phi_n = \text{id}$$

on the domain $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{\sum_{i=1}^k c_{i,n}}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2} \right] \times [0, 1]^{m-1}$.

This is a smooth map because ϕ_n coincides with the identity in a neighbourhood of the different sections. Hereby, we have defined the diffeomorphism ϕ_n on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0, 1]^{m-1}$. Now we extend ϕ_n to a diffeomorphism on $\mathbb{S}^1 \times [0, 1]^{m-1}$ using the description $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$.

Remark 3.3. In the definition of the map ϕ_n the partition of domains and the (nq_n) -powers are chosen in such a way that when passing from $\left[\frac{l}{nq_n}, \frac{l+1}{nq_n}\right] \times [0, 1]^{m-1}$ to $\left[\frac{l+i}{nq_n}, \frac{l+i+1}{nq_n}\right] \times [0, 1]^{m-1}$ the (nq_n) -powers are increasing on a proportion of $\kappa_{i,n}$ of the domain (see Lemma 4.4 for a proof). This will allow us to prove uniform distribution on this proportion with the aid of Lemma 4.3 which will approximate κ_i -weak mixing in Lemma 5.7. To have this flexibility to attribute the right length proportion to distinct domains of the conjugation map ϕ_n we used the conditions on the $\tilde{\beta}_{i,n}$'s and $\kappa_{i,n}$'s from the beginning of section 3. Those conditions were an immediate consequence of $(\kappa_1, \dots, \kappa_k) \in \Pi_k$ which motivates this additional technical assumption in our main theorem.

Example 3.4. Let $\kappa_1 = \frac{1}{2}$, $\kappa_2 = \frac{3}{4}$ and $\kappa_3 = \frac{7}{8}$. Since $\kappa_3 - \kappa_2 = \frac{1}{8} < \frac{1}{4} = \kappa_2 - \kappa_1$ and $3 \cdot (\kappa_2 - \kappa_1) = \frac{3}{4} < \frac{7}{8} = \kappa_3$ we have $(\kappa_1, \kappa_2, \kappa_3) \in \Pi_3$. According to this we have $\tilde{\beta}_1 = \frac{1}{2}$, $\tilde{\beta}_2 = \frac{1}{4}$, and $\tilde{\beta}_3 = \frac{1}{8}$. This yields $\tilde{\beta}_{i,n} = \tilde{\beta}_i$, i. e. $d_n = 8$, $u_{0,n} = 0$, $u_{1,n} = 1$, $u_{2,n} = c_{2,n} = 2$. In Figure 1 we list the powers γ of $\phi_{d_n \cdot (nq_n)^\gamma}^{(2)}$ on a domain $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2}\right] \times [0, 1]$ for different values of $l \in \mathbb{Z}$.

3.2 The conjugation map h_n

We define the conjugation map as the composition

$$h_n = g_n \circ \phi_n,$$

where

$$g_n(\theta, r_1, \dots, r_{m-1}) = (\theta + nq_n \cdot r_1, r_1, \dots, r_{m-1}).$$

	$\frac{1}{8}$	$\frac{1}{4}$		$\frac{1}{2}$		$\frac{3}{4}$		
$l = 0$	$2 \cdot 3$	$2 \cdot 3$	$2 \cdot 2$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 1$	$2 \cdot 3$	id
$l = 1$	$2 \cdot 3$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 2$	$2 \cdot 4$	$2 \cdot 4$	$2 \cdot 4$	id
$l = 2$	$2 \cdot 5$	$2 \cdot 3$	$2 \cdot 5$	$2 \cdot 5$	$2 \cdot 4$	$2 \cdot 4$	$2 \cdot 5$	id
$l = 3$	$2 \cdot 6$	$2 \cdot 6$	$2 \cdot 5$	$2 \cdot 5$	$2 \cdot 6$	$2 \cdot 4$	$2 \cdot 6$	id
$l = 4$	$2 \cdot 6$	$2 \cdot 6$	$2 \cdot 7$	$2 \cdot 5$	$2 \cdot 7$	$2 \cdot 7$	$2 \cdot 7$	id
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Figure 1: List of powers γ of $\phi_{d_n \cdot (nq_n)}^{(2)\gamma}$ on a domain $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2}\right] \times [0, 1]$ for an arbitrary value of v and different values of $l \in \mathbb{Z}$ in the setting of Example 3.4. In the horizontal direction we have the length $\frac{1}{n^2 \cdot q_n^2}$ of the domain. Note that the powers γ increase by κ_i when passing from l to $l + i$ for any $i = 1, 2, 3$.

3.3 Partial partition η_n

In this subsection we define the announced sequence of partial partitions $(\eta_n)_{n \in \mathbb{N}}$ of $M = \mathbb{S}^1 \times [0, 1]^{m-1}$. For convenience we will use the notation $\prod_{i=2}^m [a_i, b_i]$ for $[a_2, b_2] \times \dots \times [a_m, b_m]$

Initially η_n will be constructed on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0, 1]^{m-1}$. We start by considering the following sets: In the θ -coordinate we define

$$\tilde{I}_{l,v,u,t_1^{(2)}, \dots, t_1^{(m \cdot (n-1+k))}} := \left[\frac{l}{nq_n} + \frac{v}{(nq_n)^2} + \frac{u}{d_n \cdot (nq_n)^2} + \frac{t_1^{(2)}}{d_n \cdot (nq_n)^{2 \cdot 2}} + \dots + \frac{t_1^{(m \cdot (n-1+k))}}{d_n \cdot (nq_n)^{2m \cdot (n-1+k)}} + \frac{1}{10n^4 \cdot d_n \cdot (nq_n)^{2m \cdot (n-1+k)}}, \right. \\ \left. \frac{l}{nq_n} + \frac{v}{(nq_n)^2} + \frac{u}{d_n \cdot (nq_n)^2} + \frac{t_1^{(2)}}{d_n \cdot (nq_n)^{2 \cdot 2}} + \dots + \frac{t_1^{(m \cdot (n-1+k))} + 1}{d_n \cdot (nq_n)^{2m \cdot (n-1+k)}} - \frac{1}{10n^4 \cdot d_n \cdot (nq_n)^{2m \cdot (n-1+k)}} \right]$$

and

$$\tilde{I}_{l,v}^1 := \left[\frac{l}{n \cdot q_n} + \frac{v + \sum_{d=1}^k \tilde{\beta}_{d,n}}{n^2 \cdot q_n^2} + \frac{1 - \sum_{d=1}^k \tilde{\beta}_{d,n}}{2 \cdot n^2 \cdot q_n^2} \cdot \left(1 - \left(1 - \frac{1}{5n^4}\right)^{m \cdot (n-1+k) - 1}\right), \right. \\ \left. \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2} - \frac{1 - \sum_{d=1}^k \tilde{\beta}_{d,n}}{2 \cdot n^2 \cdot q_n^2} \cdot \left(1 - \left(1 - \frac{1}{5n^4}\right)^{m \cdot (n-1+k) - 1}\right) \right].$$

In the \vec{r} -coordinates we define

$$\begin{aligned}
& W_{j_2^{(1)}, j_2^{(2)}, t_2^{(2)}, \dots, t_2^{(n-1+k)}, j_3, t_3^{(2)}, \dots, t_3^{(n-1+k)}, \dots, j_m, t_m^{(2)}, \dots, t_m^{(n-1+k)}} := \\
& \left[\frac{j_2^{(1)}}{n \cdot q_n^2} + \frac{j_2^{(2)}}{n^2 \cdot q_n^2} + \frac{t_2^{(2)}}{(nq_n)^{2 \cdot 2}} + \dots + \frac{t_2^{(n-1+k)}}{(nq_n)^{2 \cdot (n-1+k)}} + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}}, \right. \\
& \left. \frac{j_2^{(1)}}{n \cdot q_n^2} + \frac{j_2^{(2)}}{n^2 \cdot q_n^2} + \frac{t_2^{(2)}}{(nq_n)^{2 \cdot 2}} + \dots + \frac{t_2^{(n-1+k)} + 1}{(nq_n)^{2 \cdot (n-1+k)}} - \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}} \right] \\
& \times \prod_{i=3}^m \left[\frac{j_i}{n^2 \cdot q_n^2} + \frac{t_i^{(2)}}{(nq_n)^{2 \cdot 2}} + \dots + \frac{t_i^{(n-1+k)}}{(nq_n)^{2 \cdot (n-1+k)}} + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}}, \right. \\
& \left. \frac{j_i + 1}{n^2 \cdot q_n^2} + \frac{t_i^{(2)}}{(nq_n)^{2 \cdot 2}} + \dots + \frac{t_i^{(n-1+k)} + 1}{(nq_n)^{2 \cdot (n-1+k)}} - \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}} \right].
\end{aligned}$$

In case of $l, v \in \mathbb{Z}$, $0 \leq l < n$ and $0 \leq v < nq_n$, we define

$$\begin{aligned}
& I_{l, v, j_2^{(1)}, j_2^{(2)}, j_3, \dots, j_m} := \\
& \bigcup \left(\tilde{I}_{l, v, t_1^{(2)}, \dots, t_1^{(m \cdot (n-1+k))}} \cup \tilde{I}_{l, v}^1 \right) \times W_{j_2^{(1)}, j_2^{(2)}, t_2^{(2)}, \dots, t_2^{(n-1+k)}, j_3, t_3^{(2)}, \dots, t_3^{(n-1+k)}, \dots, j_m, t_m^{(2)}, \dots, t_m^{(n-1+k)}},
\end{aligned}$$

where the union is taken over

- $u \in \mathbb{Z}$, $0 \leq u < \sum_{d=1}^k c_{d,n}$,
- $t_1^{(s)} \in \mathbb{Z}$, $\left\lceil \frac{q_n^2}{10n^2} \right\rceil \leq t_1^{(s)} \leq n^2 q_n^2 - \left\lceil \frac{q_n^2}{10n^2} \right\rceil - 1$, for $s = 2, \dots, m \cdot (n-1+k)$,
- $t_i^{(s)} \in \mathbb{Z}$, $\left\lceil \frac{q_n^2}{10n^2} \right\rceil \leq t_i^{(s)} \leq n^2 q_n^2 - \left\lceil \frac{q_n^2}{10n^2} \right\rceil - 1$, for $s = 2, \dots, n-1+k$ and $i = 2, \dots, m$.

In Lemma 6.6 we will choose q_n as a multiple of $10n^2$. In particular, $\left\lceil \frac{q_n^2}{10n^2} \right\rceil = \frac{q_n^2}{10n^2}$.

Remark 3.5. Descriptively $I_{l, v, j_2^{(1)}, j_2^{(2)}, j_3, \dots, j_m}$ is the cube

$$\left[\frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{nq_n} + \frac{v+1}{n^2 \cdot q_n^2} \right] \times \left[\frac{j_2^{(1)}}{n \cdot q_n^2} + \frac{j_2^{(2)}}{n^2 \cdot q_n^2}, \frac{j_2^{(1)}}{n \cdot q_n^2} + \frac{j_2^{(2)} + 1}{n^2 \cdot q_n^2} \right] \times \prod_{i=3}^m \left[\frac{j_i}{n^2 \cdot q_n^2}, \frac{j_i + 1}{n^2 \cdot q_n^2} \right]$$

with some holes. These holds are inserted in order to guarantee that $I_{l, v, j_2^{(1)}, j_2^{(2)}, j_3, \dots, j_m}$ belongs to the "good domain" of ϕ_n^{-1} and also of ϕ_n after a translation by $\frac{i}{nq_n}$ on the θ -axis. This property will be exploited in the proof of Lemma 4.3.

With the aid of these sets we define our partial partition η_n :

- On $\left[\frac{l}{n \cdot q_n}, \frac{l+1}{n \cdot q_n} \right] \times [0, 1]^{m-1}$, $0 \leq l < n - k$, the partial partition η_n consists of all such sets $I_{l, v, j_2^{(1)}, j_2^{(2)}, j_3, \dots, j_m}$, at which $j_i \in \mathbb{Z}$ and $\left\lceil \frac{q_n^2}{10n^2} \right\rceil \leq j_i \leq n^2 q_n^2 - \left\lceil \frac{q_n^2}{10n^2} \right\rceil - 1$ for $i = 3, \dots, m$ and $v \in \mathbb{Z}$, $0 \leq v \leq n \cdot q_n - 1$, as well as $j_2^{(1)} \in \mathbb{Z}$, $\left\lceil \frac{q_n^2}{10n^3} \right\rceil \leq j_2^{(1)} \leq nq_n^2 - \left\lceil \frac{q_n^2}{10n^3} \right\rceil - 1$ as well as $j_2^{(2)} \in \mathbb{Z}$, $0 \leq j_2^{(2)} \leq n - 1$ apart from those $j_2^{(2)}$ satisfying

$$l + j_2^{(2)} \equiv n - s \pmod{n} \quad \text{for } s = 1, 2, \dots, k + 1. \quad (4)$$

- On $\left[\frac{n-k}{n \cdot q_n}, \frac{1}{q_n}\right] \times [0, 1]^{m-1}$ there are no elements of the partial partition η_n .

As the image under R_{a/q_n} with $a \in \mathbb{Z}$ this partial partition of $\left[0, \frac{1}{q_n}\right] \times [0, 1]^{m-1}$ is extended to a partial partition of $\mathbb{S}^1 \times [0, 1]^{m-1}$.

By construction this sequence of partial partitions converges to the decomposition into points.

Remark 3.6. In the following we will often write a partition element in the comprehensive form $\hat{I}_n = \bigcup_{j=0}^N I_j \times W$, at which $W = \pi_{\bar{r}}(\hat{I}_n)$ is the $m - 1$ -dimensional projection of \hat{I}_n in the r_1, \dots, r_{m-1} -coordinates and we have the following sections I_j on the θ -axis: If $\phi_n = \phi_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot T} \circ \dots \circ \phi_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot T}} \circ \phi_{d_n \cdot (nq_n)^{2 \cdot T}}^{(2)}$ on $\left[\frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{u}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{u+1}{d_n \cdot n^2 \cdot q_n^2}\right] \times [0, 1]^{m-1}$, then sections on this domain are given by $I_j = \bigcup \tilde{I}_{l,v,u,t_1^{(2)}, \dots, t_1^{(m \cdot (n-1+k))}}$, where the union is taken over the allowed values of $t_1^{(T+1)}, \dots, t_1^{(m \cdot (n-1+k))}$. Finally, I_N corresponds to the section $\tilde{I}_{l,v}^1$, i. e.

$$\left[\frac{l}{n \cdot q_n} + \frac{v + \sum_{d=1}^k \tilde{\beta}_{d,n}}{n^2 \cdot q_n^2} + \frac{1 - \sum_{d=1}^k \tilde{\beta}_{d,n}}{2 \cdot n^2 \cdot q_n^2} \cdot \left(1 - \left(1 - \frac{1}{5n^4}\right)^{m \cdot (n-1+k) - 1}\right), \right. \\ \left. \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2} - \frac{1 - \sum_{d=1}^k \tilde{\beta}_{d,n}}{2 \cdot n^2 \cdot q_n^2} \cdot \left(1 - \left(1 - \frac{1}{5n^4}\right)^{m \cdot (n-1+k) - 1}\right) \right]$$

Note that the partition elements are constructed in such a way that

$$\mu(I_N \times W) = \left(1 - \sum_{d=1}^k \tilde{\beta}_{d,n}\right) \cdot \mu(\hat{I}_n).$$

Remark 3.7. For an element $\hat{I}_n = \bigcup_{j=0}^N I_j \times W$ of the partition η_n we observe:

$$\phi_n|_{I_N \times W} = \text{id}$$

Remark 3.8. The additional restrictions on $j_2^{(2)}$ in equation (4) will prove useful in Remark 5.5.

4 $(\gamma, \delta, s_1, s, \epsilon)$ -distribution

We introduce the central notion in the proof of the criterion for $(\kappa_1, \dots, \kappa_k)$ -weak mixing deduced in the next section:

Definition 4.1. Let $\Phi : M \rightarrow M$ be a diffeomorphism. We say Φ $(\gamma, \delta, s_1, s, \epsilon)$ -distributes a set \hat{I} , if the following properties are satisfied:

- $\Phi(\hat{I})$ is contained in a set of the form $[c, c + \gamma] \times [\delta, 1 - \delta]^{m-1}$ for some $c \in \mathbb{S}^1$.
- For every $(m - 1)$ -dimensional cuboid $\tilde{J} \subseteq [\delta, 1 - \delta]^{m-1}$ of r_1 -length at least s_1 and of side length s in the r_2, \dots, r_{m-1} -coordinates it holds:

$$\left| \mu(\hat{I} \cap \Phi^{-1}(\mathbb{S}^1 \times \tilde{J})) - \mu(\hat{I}) \cdot \mu^{(m-1)}(\tilde{J}) \right| \leq \epsilon \cdot \mu(\hat{I}) \cdot \mu^{(m-1)}(\tilde{J}),$$

where $\mu^{(m-1)}$ is the Lebesgue measure on $[0, 1]^{m-1}$.

Inspired by [FS05] we will call the second property “almost uniform distribution” of \hat{I} in the r_1, \dots, r_{m-1} -coordinates.

In the next step we define the sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$:

$$m_n = \min \left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \quad \inf_{k \in \mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} + k \right| \leq \frac{10n^2}{q_{n+1}} \right\}.$$

The existence of m_n follows by the same arguments as in [FS05, section 4.2.1] and [GK18, Lemma 4.2].

Remark 4.2. We define

$$a_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} \right) \bmod 1.$$

By the above construction of m_n it holds: $|a_n| \leq \frac{10n^2}{q_{n+1}}$. In Lemma 6.6 we will see that it is possible to choose $q_{n+1} \geq 200 \cdot n^2 \cdot k \cdot d_n^2 \cdot n^{4 \cdot (n+k)} \cdot q_n^{4 \cdot (n-1+k)}$. Thus we get:

$$|a_n| \leq \frac{1}{20n^4 \cdot k \cdot d_n^2 \cdot (n \cdot q_n)^{4 \cdot (n-1+k)}}.$$

By this choice of the number m_n , $R_{\alpha_{n+1}}^{m_n}$ causes a translation to the adjacent $\frac{1}{nq_n}$ -domain of definition of the map ϕ_n . On such a domain the elements \hat{I}_n of the partial partition η_n are positioned in such a way that all $\varphi_{\varepsilon,1,j}^{-1}$ in ϕ_n^{-1} act as the particular rotations. On the adjacent section, the stretching parameters λ_j in $\phi_{\lambda_j}^{(j)}$ are chosen so that either ϕ_n is of the same form as before or ϕ_n maps $R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1} \left(\hat{I}_n \right)$ to a set of almost full length in the r_1, \dots, r_{m-1} -coordinates. We make this precise in the subsequent lemma.

Lemma 4.3. *We consider a set I_n of the form*

$$\begin{aligned} & \bigcup \left[\frac{t_1^{(T+1)}}{d_n \cdot (n \cdot q_n)^{2 \cdot (T+1)}} + \dots + \frac{t_1^{(m \cdot (n-1+k))}}{d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}} + \frac{1}{10n^4 \cdot d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}}, \right. \\ & \left. \frac{t_1^{(T+1)}}{d_n \cdot (n \cdot q_n)^{2 \cdot (T+1)}} + \dots + \frac{t_1^{(m \cdot (n-1+k))} + 1}{d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}} - \frac{1}{10n^4 \cdot d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}} \right] \\ & \times \prod_{i=2}^m \left[\frac{j_i}{(nq_n)^2} + \frac{t_i^{(2)}}{(nq_n)^{2 \cdot 2}} + \dots + \frac{t_i^{(n-1+k)}}{(nq_n)^{2 \cdot (n-1+k)}} + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}}, \right. \\ & \left. \frac{j_i}{(nq_n)^2} + \frac{t_i^{(2)}}{(nq_n)^{2 \cdot 2}} + \dots + \frac{t_i^{(n-1+k)} + 1}{(nq_n)^{2 \cdot (n-1+k)}} - \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}} \right], \end{aligned}$$

which belongs to a partition element of η_n and where the union is taken over all occurring $t_i^{(j)}$. Moreover, we assume that

$$\phi_n = \phi_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot T}}^{(m)} \circ \dots \circ \phi_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot T}}^{(3)} \circ \phi_{d_n \cdot (nq_n)^{2 \cdot T}}^{(2)}$$

on it. After an application of $R_{\alpha_{n+1}}^{l \cdot m_n}$ with some $l \in \{1, \dots, k\}$ the image $R_{\alpha_{n+1}}^{l \cdot m_n} \circ \phi_n^{-1} (I_n)$ lies in a domain where

$$\phi_n = \phi_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot U}}^{(m)} \circ \dots \circ \phi_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot U}}^{(3)} \circ \phi_{d_n \cdot (nq_n)^{2 \cdot U}}^{(2)}.$$

1. If $U = T$, then $\phi_n \circ R_{\alpha_{n+1}}^{l \cdot m_n} \circ \phi_n^{-1}(I_n)$ is contained in the cube

$$\left[\frac{l}{nq_n}, \frac{l}{nq_n} + \frac{1}{(nq_n)^2} \right] \times \prod_{i=2}^m \left[\frac{j_i}{n^2 \cdot q_n^2}, \frac{j_i + 1}{n^2 \cdot q_n^2} \right].$$

In particular, $\pi_{\bar{r}}(\phi_n \circ R_{\alpha_{n+1}}^{l \cdot m_n} \circ \phi_n^{-1}(I_n))$ is contained in the same $\frac{1}{(nq_n)^2}$ -cube as $\pi_{\bar{r}}(I_n)$.

2. If $U > T$, then $\phi_n \circ R_{\alpha_{n+1}}^{l \cdot m_n} \circ \phi_n^{-1}$ is $\left(\frac{1}{d_n \cdot (nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, \frac{1}{q_n}, \frac{4}{n} \right)$ -distributing the set I_n for n sufficiently large.

Proof. When applying the map ϕ_n^{-1} we observe that the set is positioned in such a way that all the occurring maps $\varphi_{\varepsilon,1,j}^{-1}$ act as the respective rotations. Then we compute $\phi_n^{-1}(I_n)$ with the aid of Proposition 3.2:

$$\begin{aligned} & \bigcup \left[v_1 + \frac{1}{10n^4 \cdot d_n \cdot (nq_n)^{2(n-1+k+T)}}, v_1 + \frac{1 - \frac{1}{10n^4}}{d_n \cdot (nq_n)^{2(n-1+k+T)}} \right] \\ & \times \prod_{i=2}^{m-1} \left[v_i + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k+T)}}, v_i + \frac{1 - \frac{1}{10n^4}}{(nq_n)^{2 \cdot (n-1+k+T)}} \right] \\ & \times \left[v_m + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot m \cdot (n-1+k) - 2 \cdot (m-1) \cdot T}}, v_m + \frac{1 - \frac{1}{10n^4}}{(nq_n)^{2 \cdot m \cdot (n-1+k) - 2 \cdot (m-1) \cdot T}} \right], \end{aligned}$$

where

$$\begin{aligned} v_1 &= \frac{j_2}{d_n \cdot (nq_n)^{2(T+1)}} + \frac{t_2^{(2)}}{d_n \cdot (nq_n)^{2(T+2)}} + \dots + \frac{t_2^{(n-1+k)}}{d_n \cdot (nq_n)^{2(n-1+k+T)}}, \\ v_i &= 1 - \frac{t_1^{((i-1)T+1)}}{(nq_n)^2} - \dots - \frac{t_1^{(iT)}}{(nq_n)^{2T}} - \frac{j_{i+1}}{(nq_n)^{2 \cdot (T+1)}} - \frac{t_{i+1}^{(2)}}{(nq_n)^{2 \cdot (T+2)}} - \dots - \frac{t_{i+1}^{(n-1+k)} + 1}{(nq_n)^{2(n-1+k+T)}}, \\ v_m &= 1 - \frac{t_1^{((m-1) \cdot T+1)}}{(nq_n)^2} - \dots - \frac{t_1^{(m \cdot (n-1+k))} + 1}{(nq_n)^{2 \cdot m \cdot (n-1+k) - 2 \cdot (m-1) \cdot T}}. \end{aligned}$$

By our choice of the number m_n the subsequent application of $R_{\alpha_{n+1}}^{l \cdot m_n}$ yields a shift by $\frac{l}{nq_n} + la_n$ on the θ -axis, at which a_n is the ‘‘error term’’ introduced in Remark 4.2. With the aid of the bound on la_n from Remark 4.2 we can compute the image of I_n under $\Phi_n := \phi_n \circ R_{\alpha_{n+1}}^{l \cdot m_n} \circ \phi_n^{-1}$. In the first case (i. e. $U = T$) we get

$$\begin{aligned} & \bigcup \left[w_1 + \frac{1}{10n^4 \cdot d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}}, w_1 + \frac{1 - \frac{1}{10n^4}}{d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}} - \frac{1}{10n^4 \cdot d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}} \right] \\ & \times \left[w_2 + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}} + d_n \cdot (nq_n)^{2T} \cdot la_n, w_2 + \frac{1 - \frac{1}{10n^4}}{(nq_n)^{2 \cdot (n-1+k)}} + d_n \cdot (nq_n)^{2T} \cdot la_n \right] \\ & \times \prod_{i=3}^m \left[w_i + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k)}}, w_i + \frac{1 - \frac{1}{10n^4}}{(nq_n)^{2 \cdot (n-1+k)}} \right], \end{aligned}$$

where

$$w_1 = \frac{l}{nq_n} + \frac{t_1^{(T+1)}}{d_n \cdot (n \cdot q_n)^{2 \cdot (T+1)}} + \dots + \frac{t_1^{(m \cdot (n-1+k))}}{d_n \cdot (n \cdot q_n)^{2 \cdot m \cdot (n-1+k)}},$$

$$w_i = \frac{j_i}{(nq_n)^2} + \frac{t_i^{(2)}}{(nq_n)^{2 \cdot 2}} + \dots + \frac{t_i^{(n-1+k)}}{(nq_n)^{2 \cdot (n-1+k)}} \quad \text{for } i = 2, \dots, m.$$

In the second case $U = T + S > T$ we calculate $\phi_n \circ R_{\alpha_{n+1}}^{l \cdot m_n} \circ \phi_n^{-1}(I_n)$ to

$$\bigcup \left[w_1 + \frac{1}{10n^4 \cdot d_n \cdot (nq_n)^{2m \cdot (n-1+k) + 2 \cdot (m-1) \cdot (U-T)}}, w_1 + \frac{1 - \frac{1}{10n^4}}{d_n \cdot (nq_n)^{2 \cdot m \cdot (n-1+k) + 2 \cdot (m-1) \cdot (U-T)}} \right]$$

$$\times \left[w_2 + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k-S)}} + d_n \cdot (nq_n)^{2U} \cdot la_n, w_2 + \frac{1 - \frac{1}{10n^4}}{(nq_n)^{2 \cdot (n-1+k-S)}} + d_n \cdot (nq_n)^{2U} \cdot la_n \right]$$

$$\times \prod_{i=3}^m \left[w_i + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k-S)}}, w_i + \frac{1 - \frac{1}{10n^4}}{(nq_n)^{2 \cdot (n-1+k-S)}} \right],$$

where

$$w_1 = \frac{l}{nq_n} + \frac{j_2}{d_n (nq_n)^{2 \cdot (T+1)}} + \frac{t_2^{(2)}}{d_n (nq_n)^{2 \cdot (T+2)}} + \dots + \frac{t_2^{(S)}}{d_n (nq_n)^{2U}} + \frac{t_1^{(T+1)}}{d_n (nq_n)^{2 \cdot (U+1)}} + \dots$$

$$+ \frac{t_1^{(2T)}}{d_n (nq_n)^{2 \cdot (U+T)}} + \frac{j_3}{d_n (nq_n)^{2 \cdot (U+T+1)}} + \frac{t_3^{(2)}}{d_n (nq_n)^{2 \cdot (U+T+2)}} + \dots + \frac{t_3^{(S)}}{d_n (nq_n)^{2 \cdot 2U}}$$

$$+ \frac{t_1^{(2T+1)}}{d_n (nq_n)^{2 \cdot (2U+1)}} + \dots + \frac{t_1^{(3T)}}{d_n (nq_n)^{2 \cdot (2U+T)}} + \frac{j_4}{d_n (nq_n)^{2 \cdot (2U+T+1)}} + \dots + \frac{t_1^{((m-1) \cdot T+1)}}{d_n (nq_n)^{2 \cdot ((m-1) \cdot U+1)}}$$

$$+ \dots + \frac{t_1^{(m \cdot (n-1+k))}}{d_n (nq_n)^{2 \cdot m \cdot (n-1+k) + 2 \cdot (m-1) \cdot (U-T)}},$$

$$w_i = \frac{t_i^{(S+1)}}{(nq_n)^2} + \dots + \frac{t_i^{(n-1+k)}}{(nq_n)^{2 \cdot (n-1+k-S)}} \quad \text{for } i = 2, \dots, m.$$

Thus such a set $\Phi_n(I_n)$ has a θ -width of at most $\frac{1}{d_n \cdot (n \cdot q_n)^{2 \cdot (T+1)}}$.

Let $J = \prod_{i=2}^m \left[\frac{1}{10n^4}, 1 - \frac{1}{10n^4} \right]$ and $\tilde{J} \subset J$ be any $(m-1)$ -dimensional cuboid of r_1 -length $l_1 \geq \frac{1}{nq_n^2}$ and of side length q_n^{-1} in the r_2, \dots, r_{m-1} -coordinates. In each of the coordinates r_2, \dots, r_{m-1} , \tilde{J} contains at least $\frac{((nq_n)^2 \cdot (1 - \frac{1}{10n^4}))^{n-1+k-S}}{q_n}$ and at most $\frac{(nq_n)^{2 \cdot (n-1+k-S)}}{q_n}$ intervals of the form

$$\left[\frac{t_i^{(S+1)}}{(nq_n)^2} + \dots + \frac{t_i^{(n-1+k)}}{(nq_n)^{2 \cdot (n-1+k-S)}} + \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k-S)}}, \right.$$

$$\left. \frac{t_i^{(S+1)}}{(nq_n)^2} + \dots + \frac{t_i^{(n-1+k)} + 1}{(nq_n)^{2 \cdot (n-1+k-S)}} - \frac{1}{10n^4 \cdot (nq_n)^{2 \cdot (n-1+k-S)}} \right].$$

while in the r_1 -coordinate \tilde{J} contains at least $\left(\lfloor l_1 (nq_n)^2 \rfloor - 2\right) \cdot \left((nq_n)^2 \cdot \left(1 - \frac{1}{10n^4}\right)\right)^{n-2+k-S}$ and at most $l_1 \cdot (nq_n)^{2 \cdot (n-1+k-S)}$ such intervals. Hereby, we estimate

$$\begin{aligned} & \mu\left(\Phi_n(I_n) \cap \mathbb{S}^1 \times \tilde{J}\right) \\ & \geq \frac{l_1 \cdot \left(1 - \frac{3}{l_1(nq_n)^2}\right)}{d_n (nq_n)^{2(T+m-1)+m-2}} \cdot \left(1 - \frac{1}{10n^4}\right)^{(m-1)(n-1+k-S)-1} \cdot \left(1 - \frac{1}{5n^4}\right)^{(m-1)(S-1)+m(n+k)-T} \\ & \geq \left(1 - \frac{4}{n}\right) \cdot \mu(I_n) \cdot \mu^{(m-1)}(\tilde{J}) \end{aligned}$$

exploiting $l_1 \geq \frac{1}{nq_n^2}$ and

$$\mu(I_n) = \frac{1}{d_n \cdot (nq_n)^{2 \cdot (T+m-1)}} \cdot \left(1 - \frac{1}{5n^4}\right)^{m \cdot (n+k) - T + (m-1) \cdot (n+k-2)}.$$

On the other hand, we have

$$\begin{aligned} \mu\left(\Phi_n(I_n) \cap \mathbb{S}^1 \times \tilde{J}\right) & \leq \frac{1}{d_n \cdot (nq_n)^{2 \cdot (T+m-1)+m-2}} \cdot l_1 \cdot \left(1 - \frac{1}{5n^4}\right)^{m \cdot (n+k) - T + (m-1) \cdot (S-1)} \\ & \leq \left(1 + \frac{1}{n}\right) \cdot \mu(I_n) \cdot \mu^{(m-1)}(\tilde{J}) \end{aligned}$$

for n sufficiently large. Hence, the properties of a $\left(\frac{1}{d_n \cdot (nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, q_n^{-1}, \frac{4}{n}\right)$ -distribution are fulfilled. \square

The previous lemma shows the significance of an understanding of how the (nq_n) -powers in the definition of ϕ_n evolve while passing from $\left[\frac{l}{nq_n}, \frac{l+1}{nq_n}\right] \times [0, 1]^{m-1}$ to $\left[\frac{l+i}{nq_n}, \frac{l+i+1}{nq_n}\right] \times [0, 1]^{m-1}$:

Lemma 4.4. *Let $l, v, i \in \mathbb{Z}$, $0 \leq l < n - k$, $0 \leq v < nq_n$ and $0 < i \leq k$. On every section $\left[\frac{l+i}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l+i}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2}\right] \times [0, 1]^{m-1}$ the ratio of domains, where the (nq_n) -power in the definition of ϕ_n is greater than the (nq_n) -power of ϕ_n in the corresponding domain in $\left[\frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{nq_n} + \frac{v+1}{n^2 \cdot q_n^2}\right] \times [0, 1]^{m-1}$, is equal to $\sum_{d=1}^i \tilde{\beta}_{d,n} = \kappa_{i,n}$.*

Proof. First of all, we notice that there is no decline of (nq_n) -powers when passing from $\left[\frac{l}{nq_n}, \frac{l+1}{nq_n}\right] \times [0, 1]^{m-1}$ to $\left[\frac{l+i}{nq_n}, \frac{l+i+1}{nq_n}\right] \times [0, 1]^{m-1}$. Obviously, on the domains of the form

$$\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{k \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{\sum_{d=1}^k c_{d,n}}{d_n \cdot n^2 \cdot q_n^2}\right] \times [0, 1]^{m-1},$$

i. e. on a corresponding θ -length of $\frac{\sum_{d=1}^k \tilde{\beta}_{d,n} - k \cdot \tilde{\beta}_{2,n}}{n^2 q_n^2}$, we always have an increasing (nq_n) -power while passing from l to $l+i$ for any $i \geq 1$.

A more careful analysis has to be executed on the domains of the form

$$\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{k \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2} \right] \times [0, 1]^{m-1}$$

in the definition of ϕ_n : When passing from l to $l + i$ the number \tilde{s} changes to $\tilde{s} - i \pmod k$. Accordingly for each $s \in \{0, 1, \dots, k-1\}$ the corresponding number t_s is changed to $t_s + i \pmod k$. On the one hand, we observe that on the domains $\Delta_{l+i,v,s,u}$ with $u < u_{t_s+i-1,n}$ the (nq_n) -power is larger than on $\Delta_{l,v,s,u}$. On the other hand, the (nq_n) -power remains the same on domains of the form $\Delta_{l+i,v,s,u}$ for $u \geq u_{t_s+i-1,n}$ due to $l + i + k - (t_s + i) = l + k - t_s$. Hence, this yields an increased (nq_n) -power on a corresponding θ -length of $\sum_{t_s=0}^{k-1} \frac{u_{t_s+i-1,n}}{d_n \cdot n^2 \cdot q_n^2}$, where we use the convention $u_{j,n} = c_{2,n}$ for $j \geq k$. By definition of the numbers $u_{i,n}$ in equation (2) we get

$$\sum_{t_s=0}^{k-1} \frac{u_{t_s+i-1,n}}{d_n \cdot n^2 \cdot q_n^2} = \frac{k \cdot \tilde{\beta}_{2,n} - \sum_{d=i+1}^k \tilde{\beta}_{d,n}}{n^2 \cdot q_n^2}$$

in case of $i < k$. In case of $i = k$ the fraction is equal to $\frac{k \cdot \tilde{\beta}_{2,n}}{n^2 \cdot q_n^2}$.

Altogether the (nq_n) -power in the definition of ϕ_n has increased on a corresponding θ -length of $\frac{1}{n^2 q_n^2} \sum_{d=1}^i \tilde{\beta}_{d,n} = \frac{1}{n^2 q_n^2} \kappa_{i,n}$ using equation (3). \square

Remark 4.5. For a partition element $\hat{I}_n = \bigcup_{j=0}^N I_j \times W$ we introduce subsets N_i , $i = 1, \dots, k$, of the set of indices $\{0, 1, \dots, N-1\}$ in the following way: $j \in N_i$ if the (nq_n) -power of ϕ_n on $R_{\frac{i}{nq_n}}(I_j \times W)$ is larger than the (nq_n) -power of ϕ_n on $I_j \times W$. By the previous lemma and the shape of partition elements in η_n we have

$$\mu \left(\bigcup_{j \in N_i} I_j \times W \right) = \kappa_{i,n} \cdot \mu(\hat{I}_n).$$

5 Proof of $(\kappa_1, \dots, \kappa_k)$ -weak mixing

In this section we will prove the $(\kappa_1, \dots, \kappa_k)$ -weak mixing property for the constructed diffeomorphisms on $M = \mathbb{S}^1 \times [0, 1]^{m-1}$. For the derivation we need a couple of lemmas, where especially Lemmas 5.1-5.3 are very similar to corresponding steps in the criterion for weak mixing in [FS05, section 3]. At first we examine a sequence of partial partitions that will be used in our criterion:

Lemma 5.1. *Consider the sequence of partial partitions $(\eta_n)_{n \in \mathbb{N}}$ constructed in section 3.3 and the diffeomorphisms g_n from section 3.2. Furthermore let $(H_n)_{n \in \mathbb{N}}$ be a sequence of measure-preserving smooth diffeomorphisms satisfying $\|DH_{n-1}\| \leq \frac{\ln(q_n)}{n}$ for every $n \in \mathbb{N}$ and define the partial partitions $\nu_n = \left\{ \Gamma_n = H_{n-1} \circ g_n(\hat{I}_n) : \hat{I}_n \in \eta_n \right\}$. Then we get $\nu_n \rightarrow \varepsilon$.*

Proof. By construction $\eta_n = \left\{ \hat{I}_n^i : i \in \Lambda_n \right\}$, where Λ_n is a countable set of indices. Because of $\eta_n \rightarrow \varepsilon$ it holds $\lim_{n \rightarrow \infty} \mu \left(\bigcup_{i \in \Lambda_n} \hat{I}_n^i \right) = 1$. Since $H_{n-1} \circ g_n$ is measure-preserving, we conclude:

$$\lim_{n \rightarrow \infty} \mu \left(\bigcup_{i \in \Lambda_n} \Gamma_n^i \right) = \lim_{n \rightarrow \infty} \mu \left(\bigcup_{i \in \Lambda_n} H_{n-1} \circ g_n(\hat{I}_n^i) \right) = \lim_{n \rightarrow \infty} \mu \left(H_{n-1} \circ g_n \left(\bigcup_{i \in \Lambda_n} \hat{I}_n^i \right) \right) = 1.$$

For any m -dimensional cube with sidelength l_n it holds: $\text{diam}(W_n) = \sqrt{m} \cdot l_n$. Because every element of the partition η_n is contained in a cube of side length $\frac{1}{n^2 \cdot q_n^2}$ it follows for every $i \in \Lambda_n$: $\text{diam}(\hat{I}_n^i) \leq \sqrt{m} \cdot \frac{1}{n^2 \cdot q_n^2}$. Hence, we have for every $\Gamma_n^i = H_{n-1} \circ g_n(\hat{I}_n^i)$:

$$\text{diam}(\Gamma_n^i) \leq \|DH_{n-1}\|_0 \cdot \|Dg_n\|_0 \cdot \frac{\sqrt{m}}{n^2 \cdot q_n^2} \leq \frac{\ln(q_n)}{n} \cdot n \cdot q_n \cdot \frac{\sqrt{m}}{n^2 \cdot q_n^2} \leq \sqrt{m} \cdot \frac{\ln(q_n)}{n^2 \cdot q_n}.$$

We conclude $\lim_{n \rightarrow \infty} \text{diam}(\Gamma_n^i) = 0$ and consequently $\nu_n \rightarrow \varepsilon$. \square

In the following the Lebesgue measures on \mathbb{S}^1 , $[0, 1]^{m-2}$, $[0, 1]^{m-1}$ are denoted by $\tilde{\lambda}$, $\mu^{(m-2)}$ and $\mu^{(m-1)}$ respectively. The next technical result is needed in the proof of Lemma 5.3.

Lemma 5.2. *Given an interval K on the r_1 -axis and a $(m-2)$ -dimensional interval Z in (r_2, \dots, r_{m-1}) $K_{c,\gamma}$ denotes the cuboid $[c, c+\gamma] \times K \times Z$ for some $\gamma > 0$. We consider an interval $L = [l_1, l_2]$ of \mathbb{S}^1 and the map $\tilde{g}_b : M \rightarrow M$, $\tilde{g}_b(\theta, r_1, \dots, r_{m-1}) = (\theta + b \cdot r_1, r_1, \dots, r_{m-1})$ with some $b \in \mathbb{N}$. Let $b \cdot \lambda(K) > 2$. Then $U := \pi_{r_1}(K_{c,\gamma} \cap \tilde{g}_b^{-1}(L \times K \times Z))$ consists of at least $b\lambda(K) - 2$ and at most $b\lambda(K) + 2$ intervals where the two side intervals are possibly shorter than the other intervals of length $\frac{\tilde{\lambda}(L)+\gamma}{b}$. Let U_1 be the superset of U obtained by completing the two side intervals and let U_2 be the subset of U obtained by removing the two partial side intervals. For either $Q = U_1 \times Z$ or $Q = U_2 \times Z$ we have*

$$\left| \mu^{(m-1)}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) \right| \leq \left(\frac{2}{b} \cdot \tilde{\lambda}(L) + \frac{2 \cdot \gamma}{b} + \gamma \cdot \lambda(K) \right) \cdot \mu^{(m-2)}(Z).$$

Proof. To see this we consider the set:

$$\begin{aligned} Q_b &:= \pi_{\vec{r}}(K_{c,\gamma} \cap \tilde{g}_b^{-1}(L \times K \times Z)) \\ &= \{(r_1, r_2, \dots, r_{m-1}) \in K \times Z : (\theta + b \cdot r_1, \vec{r}) \in L \times K \times Z, \theta \in [c, c+\gamma]\} \\ &= \{(r_1, r_2, \dots, r_{m-1}) \in K \times Z : b \cdot r_1 \in [l_1 - c - \gamma, l_2 - c] \pmod{1}\} \end{aligned}$$

The interval $b \cdot K$ seen as an interval in \mathbb{R} does not intersect more than $b \cdot \lambda(K) + 2$ and not less than $b \cdot \lambda(K) - 2$ intervals of the form $[i, i+1]$ with $i \in \mathbb{Z}$. Therefore we compute on the one side:

$$\begin{aligned} \mu^{(m-1)}(Q) &\leq (b \cdot \lambda(K) + 2) \cdot \frac{l_2 - (l_1 - \gamma)}{b} \cdot \mu^{(m-2)}(Z) \\ &= \left(\lambda(K) \cdot \tilde{\lambda}(L) + 2 \cdot \frac{\tilde{\lambda}(L)}{b} + \lambda(K) \cdot \gamma + \frac{2 \cdot \gamma}{b} \right) \cdot \mu^{(m-2)}(Z) \end{aligned}$$

and on the other side

$$\begin{aligned} \mu^{(m-1)}(Q) &\geq (b \cdot \lambda(K) - 2) \cdot \frac{l_2 - (l_1 - \gamma)}{b} \cdot \mu^{(m-2)}(Z) \\ &= \left(\lambda(K) \cdot \tilde{\lambda}(L) - 2 \cdot \frac{\tilde{\lambda}(L)}{b} + \lambda(K) \cdot \gamma - \frac{2 \cdot \gamma}{b} \right) \cdot \mu^{(m-2)}(Z). \end{aligned}$$

Both equations together yield:

$$\left| \mu^{(m-1)}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) - \gamma \cdot \lambda(K) \cdot \mu^{(m-2)}(Z) \right| \leq \left(\frac{2}{b} \cdot \tilde{\lambda}(L) + \frac{2 \cdot \gamma}{b} \right) \cdot \mu^{(m-2)}(Z).$$

The claim follows because

$$\begin{aligned} & \left| \mu^{(m-1)}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) \right| - \gamma \cdot \lambda(K) \cdot \mu^{(m-2)}(Z) \\ & \leq \left| \mu^{(m-1)}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) - \gamma \cdot \lambda(K) \cdot \mu^{(m-2)}(Z) \right|. \end{aligned}$$

□

Lemma 5.3. *Let n be sufficiently large, g_n as in section 3.2 and $\hat{I}_n = \bigcup_{j=0}^N I_j \times W \in \eta_n$, where η_n is the partial partition constructed in section 3.3. For the diffeomorphism ϕ_n constructed in section 3.1 and m_n as in chapter 4 we consider $\Phi_n = \phi_n \circ R_{\alpha_{n+1}}^{i \cdot m_n} \circ \phi_n^{-1}$ for some $i \in \{1, \dots, k\}$. We assume that $\Phi_n \left(\frac{1}{d_n \cdot (nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, q_n^{-1}, \frac{4}{n} \right)$ -uniformly distributes $I_j \times W$ and denote $\left[\frac{1}{10n^4}, 1 - \frac{1}{10n^4} \right]^{m-1}$ by J_n . Then for every m -dimensional cube S of side length q_n^{-1} lying in $\mathbb{S}^1 \times J_n$ we get*

$$\left| \mu(I_j \times W \cap \Phi_n^{-1} \circ g_n^{-1}(S)) - \mu(I_j \times W) \cdot \mu(S) \right| \leq \frac{20}{n} \cdot \mu(I_j \times W) \cdot \mu(S). \quad (5)$$

In other words this Lemma tells us that parts of a partition element contained in the “distribution part” of Φ_n are “almost uniformly distributed” under $g_n \circ \Phi_n$ on the whole manifold $M = \mathbb{S}^1 \times [0, 1]^{m-1}$.

Proof. Let S be a m -dimensional cube with sidelength q_n^{-1} lying in $\mathbb{S}^1 \times J_n$. Furthermore we denote:

$$S_\theta = \pi_\theta(S) \quad S_{r_1} = \pi_{r_1}(S) \quad S_{\vec{r}} = \pi_{(r_2, \dots, r_{m-1})}(S) \quad S_r = S_{r_1} \times S_{\vec{r}} = \pi_r(S).$$

Obviously: $\tilde{\lambda}(S_\theta) = \lambda(S_{r_1}) = q_n^{-1}$ and $\tilde{\lambda}(S_\theta) \cdot \lambda(S_{r_1}) \cdot \mu^{(m-2)}(S_{\vec{r}}) = \mu(S) = q_n^{-m}$.

By assumption $\Phi_n \left(\frac{1}{d_n \cdot (nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, q_n^{-1}, \frac{4}{n} \right)$ -distributes $I_j \times W$, in particular $\Phi_n(I_j \times W) \subseteq [c, c + \gamma] \times J_n$ for some $c \in \mathbb{S}^1$ and some $\gamma \leq \frac{1}{d_n \cdot (n \cdot q_n)^2}$. By construction of the map g_n it holds: $\Phi_n(I_j \times W) \cap g_n^{-1}(S) \subseteq [c, c + \gamma] \times S_r =: K_{c, \gamma}$.

Since $2\gamma \leq \frac{2}{d_n \cdot (n \cdot q_n)^2} < q_n^{-1}$, we can define a cuboid $S_1 \subseteq S$, where $S_1 := [s_1 + \gamma, s_2 - \gamma] \times S_r$ using the notation $S_\theta = [s_1, s_2]$. Let Q be the superset of $\pi_{\vec{r}}(K_{c, \gamma} \cap g_n^{-1}(S_\theta \times S_r))$ with completed side r_1 -intervals and Q_1 be the subset of $\pi_{\vec{r}}(K_{c, \gamma} \cap g_n^{-1}([s_1 + \gamma, s_2 - \gamma] \times S_r))$ with removed side r_1 -intervals like in Lemma 5.2. As seen above $\Phi_n(I_j \times W) \cap g_n^{-1}(S) \subseteq \Phi_n(I_j \times W) \cap g_n^{-1}(S) \cap K_{c, \gamma}$, which implies the inclusion $\Phi_n(I_j \times W) \cap g_n^{-1}(S) \subseteq \Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q)$.

Claim: On the other hand: $\Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q_1) \subseteq \Phi_n(I_j \times W) \cap g_n^{-1}(S)$.

Proof of the claim: For $(\theta, \vec{r}) \in \Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q_1)$ arbitrary it holds $(\theta, \vec{r}) \in \Phi_n(I_j \times W)$, i.e. $\theta \in [c, c + \gamma]$, and $\vec{r} \in \pi_{\vec{r}}(K_{c, \gamma} \cap g_n^{-1}([s_1 + \gamma, s_2 - \gamma] \times S_r))$. This implies the existence of $\bar{\theta} \in [c, c + \gamma]$ satisfying $(\bar{\theta}, \vec{r}) \in K_{c, \gamma} \cap g_n^{-1}(S_1)$. Hence, there are $\beta \in [s_1 + \gamma, s_2 - \gamma]$ and $\vec{r}_1 \in S_r$, such that $g_n(\bar{\theta}, \vec{r}) = (\beta, \vec{r}_1)$. Since g_n maps sets of the form $I \times \vec{r}$ to a set $\tilde{I} \times \vec{r}$, where $I, \tilde{I} \subset \mathbb{S}^1$ are intervals of the same length, and $|\theta - \bar{\theta}| \leq \gamma$, we have $g_n(\theta, \vec{r}) = (\bar{\beta}, \vec{r})$ for some $\bar{\beta} \in [s_1, s_2]$. So $(\theta, \vec{r}) \in \Phi_n(I_j \times W) \cap g_n^{-1}(S)$. □

Altogether the following inclusions are true:

$$\Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q_1) \subseteq \Phi_n(I_j \times W) \cap g_n^{-1}(S) \subseteq \Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q).$$

Thus we obtain:

$$\begin{aligned}
& \left| \mu(I_j \times W \cap \Phi_n^{-1}(g_n^{-1}(S))) - \mu(I_j \times W) \cdot \mu(S) \right| \\
& \leq \max \left(\left| \mu(I_j \times W \cap \Phi_n^{-1}(\mathbb{S}^1 \times Q)) - \mu(I_j \times W) \cdot \mu(S) \right|, \right. \\
& \quad \left. \left| \mu(I_j \times W \cap \Phi_n^{-1}(\mathbb{S}^1 \times Q_1)) - \mu(I_j \times W) \cdot \mu(S) \right| \right). \tag{6}
\end{aligned}$$

We want to apply Lemma 5.2 for $K = S_{r_1}$, $L = S_\theta$, $Z = S_{\tilde{r}}$ and $b = n \cdot q_n$ (note that for $n > 2$: $b \cdot \lambda(K) = n \cdot q_n \cdot q_n^{-1} = n > 2$):

$$\begin{aligned}
\left| \mu^{(m-1)}(Q) - \mu(S) \right| & \leq \left(\frac{2}{n \cdot q_n} \cdot \tilde{\lambda}(S_\theta) + \frac{2\gamma}{n \cdot q_n} + \gamma \cdot \lambda(S_{r_1}) \right) \cdot \mu^{(m-2)}(S_{\tilde{r}}) \\
& \leq \left(\frac{2}{nq_n} \cdot \tilde{\lambda}(S_\theta) + \frac{4}{(nq_n)^3} + \frac{2 \cdot \lambda(S_{r_1})}{(nq_n)^2} \right) \cdot \mu^{(m-2)}(S_{\tilde{r}}) \\
& \leq \frac{8}{n} \cdot \mu(S).
\end{aligned}$$

In particular we receive from this estimate: $\frac{8}{n} \cdot \mu(S) \geq \mu^{(m-1)}(Q) - \mu(S)$, hence we have $\mu^{(m-1)}(Q) \leq (1 + \frac{8}{n}) \cdot \mu(S) \leq 2 \cdot \mu(S)$ for n sufficiently large.

Analogously we obtain: $|\mu^{(m-1)}(Q_1) - \mu(S_1)| \leq \frac{8}{n} \cdot \mu(S)$ as well as $\mu^{(m-1)}(Q_1) \leq 2 \cdot \mu(S)$.

Since Q as well as Q_1 are a finite union of disjoint $(m-1)$ -dimensional intervals contained in J_n of r_1 -length at least $\frac{1}{nq_n^2}$ as well as r_i -length $\frac{1}{q_n}$ for $i \geq 2$ and $\Phi_n \left(\frac{1}{d_n \cdot (nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, \frac{1}{q_n}, \frac{4}{n} \right)$ -distributes the interval $I_j \times W$, we get:

$$\left| \mu(I_j \times W \cap \Phi_n^{-1}(\mathbb{S}^1 \times Q)) - \mu(I_j \times W) \mu^{(m-1)}(Q) \right| \leq \frac{4}{n} \mu(I_j \times W) \mu^{(m-1)}(Q) \leq \frac{8}{n} \mu(I_j \times W) \mu(S)$$

as well as

$$\left| \mu(I_j \times W \cap \Phi_n^{-1}(\mathbb{S}^1 \times Q_1)) - \mu(I_j \times W) \mu^{(m-1)}(Q_1) \right| \leq \frac{8}{n} \mu(I_j \times W) \mu(S).$$

Now we can proceed

$$\begin{aligned}
& \left| \mu(I_j \times W \cap \Phi_n^{-1}(\mathbb{S}^1 \times Q)) - \mu(I_j \times W) \cdot \mu(S) \right| \\
& \leq \left| \mu(I_j \times W \cap \Phi_n^{-1}(\mathbb{S}^1 \times Q)) - \mu(I_j \times W) \cdot \mu^{(m-1)}(Q) \right| + \mu(I_j \times W) \cdot \left| \mu^{(m-1)}(Q) - \mu(S) \right| \\
& \leq \frac{8}{n} \cdot \mu(I_j \times W) \cdot \mu(S) + \mu(I_j \times W) \cdot \frac{8}{n} \cdot \mu(S) = \frac{16}{n} \cdot \mu(I_j \times W) \cdot \mu(S).
\end{aligned}$$

Noting that $\mu(S_1) = \mu(S) - 2\gamma \cdot \mu^{(m-1)}(S_r)$ and so $\mu(S) - \mu(S_1) \leq 2 \cdot \frac{1}{n \cdot q_n} \cdot \mu^{(m-1)}(S_r) \leq \frac{2}{n} \cdot \mu(S)$ we obtain in the same way as above:

$$\left| \mu(I_j \times W \cap \Phi_n^{-1}(\mathbb{S}^1 \times Q_1)) - \mu(I_j \times W) \cdot \mu(S) \right| \leq \frac{18}{n} \cdot \mu(I_j \times W) \cdot \mu(S).$$

Using equation (6) this yields:

$$|\mu(I_j \times W \cap \Phi_n^{-1}(g_n^{-1}(S))) - \mu(I_j \times W) \cdot \mu(S)| \leq \frac{20}{n} \cdot \mu(I_j \times W) \cdot \mu(S).$$

□

In the proof of the criterion we will consider specific cubes:

Definition 5.4. By S_{j_1, \dots, j_m} we denote the cube $\prod_{i=1}^m \left[\frac{j_i}{q_n}, \frac{j_i+1}{q_n} \right]$. Let $\tilde{\mathfrak{S}}_n$ be the family of cubes S_{j_1, \dots, j_m} satisfying $0 \leq j_1 \leq q_n - 1$ as well as $\lceil \frac{q_n}{10n^4} \rceil \leq j_i \leq q_n - \lceil \frac{q_n}{10n^4} \rceil - 1$ for $i = 2, \dots, m$. Then \mathfrak{S}_n is defined to be the partial partition $\mathfrak{S}_n := \left\{ H_{n-1}(S_n) : S_n \in \tilde{\mathfrak{S}}_n \right\}$.

Remark 5.5. A partition element $\hat{I}_n = I_{l, v, j_2^{(1)}, j_2^{(2)}, j_3, \dots, j_m}$ is contained in a cube

$$\left[\frac{s}{q_n} + \frac{l}{nq_n} + \frac{v}{(nq_n)^2}, \frac{s}{q_n} + \frac{l}{nq_n} + \frac{v+1}{(nq_n)^2} \right] \times \left[\frac{j_2^{(1)}}{nq_n^2} + \frac{j_2^{(2)}}{n^2 \cdot q_n^2}, \frac{j_2^{(1)}}{nq_n^2} + \frac{j_2^{(2)}+1}{n^2 \cdot q_n^2} \right] \times \prod_{i=3}^m \left[\frac{j_i}{(nq_n)^2}, \frac{j_i+1}{(nq_n)^2} \right].$$

Thus $g_n(\hat{I}_n)$ is contained in

$$\begin{aligned} & \left[\frac{s + j_2^{(1)}}{q_n} + \frac{l + j_2^{(2)}}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{s + j_2^{(1)}}{q_n} + \frac{l + j_2^{(2)} + 1}{n \cdot q_n} + \frac{v + 1}{n^2 \cdot q_n^2} \right] \\ & \times \left[\frac{j_2^{(1)}}{n \cdot q_n^2} + \frac{j_2^{(2)}}{n^2 \cdot q_n^2}, \frac{j_2^{(1)}}{n \cdot q_n^2} + \frac{j_2^{(2)} + 1}{n^2 \cdot q_n^2} \right] \times \prod_{i=3}^m \left[\frac{j_i}{n^2 \cdot q_n^2}, \frac{j_i + 1}{n^2 \cdot q_n^2} \right]. \end{aligned}$$

So we observe that $g_n(\hat{I}_n)$ is contained in a cube $S_n \in \tilde{\mathfrak{S}}_n$ completely or both have an empty intersection due to the restrictions on $j_2^{(2)}$ from equation (4). By these restrictions and the same reasoning we also have $\mu(R_{\alpha_{n+1}}^{i \cdot m_n}(\hat{I}_n) \cap g_n^{-1}(S_n)) = \mu(\hat{I}_n \cap g_n^{-1}(S_n))$ for every $\hat{I}_n \in \eta_n$, $i \in \{1, \dots, k\}$ and $S_n \in \tilde{\mathfrak{S}}_n$. With the aid of Lemma 4.3 and the bounds on a_n from Remark 4.2 this also yields

$$\mu(\phi_n \circ R_{\alpha_{n+1}}^{i \cdot m_n} \circ \phi_n^{-1}(I_j \times W) \cap g_n^{-1}(S_n)) = \mu(I_j \times W \cap g_n^{-1}(S_n))$$

for every $j \notin N_i$ (recall the definition of N_i in Remark 4.5).

Remark 5.6. Under the condition $\|DH_{n-1}\|_0 \leq \frac{\ln(q_n)}{n}$ we have

$$\text{diam}(H_{n-1}(S_n)) \leq \frac{\ln(q_n)}{n} \cdot \frac{\sqrt{m}}{q_n} \rightarrow 0$$

as $n \rightarrow \infty$. So we have $\mathfrak{S}_n \rightarrow \varepsilon$.

We investigate $(\kappa_{1,n}, \dots, \kappa_{k,n})$ -weak mixing of f_n on such sets $A_n \in \mathfrak{S}_n$ and partition elements $\Gamma_n \in \nu_n$:

Lemma 5.7. *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n \in \mathbb{N}}$ be constructed as in the previous sections. For $i = 1, \dots, k$, every $A_n \in \mathfrak{S}_n$ and $\Gamma_n \in \nu_n$ we have:*

$$\left| \mu(\Gamma_n \cap f_n^{-i \cdot m_n}(A_n)) - \kappa_{i,n} \cdot \mu(\Gamma_n) \cdot \mu(A_n) - (1 - \kappa_{i,n}) \cdot \mu(\Gamma_n \cap A_n) \right| < \frac{20}{n} \cdot \mu(\Gamma_n) \cdot \mu(A_n).$$

Proof. We write $A_n = H_{n-1}(S_n)$ and $\Gamma_n = H_{n-1} \circ g_n(\hat{I}_n)$, at which $\hat{I}_n = \bigcup_{j=0}^N I_j \times W$. Furthermore we note $f_n^{i \cdot m_n} = H_n \circ R_{\alpha_{n+1}}^{i \cdot m_n} \circ H_n^{-1} = H_{n-1} \circ g_n \circ \Phi_n \circ g_n^{-1} \circ H_{n-1}^{-1}$, where $\Phi_n = \phi_n \circ R_{\alpha_{n+1}}^{i \cdot m_n} \circ \phi_n^{-1}$. Then we calculate

$$\begin{aligned} & \left| \mu(\Gamma_n \cap f_n^{-i \cdot m_n}(A_n)) - \kappa_{i,n} \cdot \mu(\Gamma_n) \cdot \mu(A_n) - (1 - \kappa_{i,n}) \cdot \mu(\Gamma_n \cap A_n) \right| \\ &= \left| \mu(\hat{I}_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)) - \kappa_{i,n} \cdot \mu(\hat{I}_n) \cdot \mu(S_n) - (1 - \kappa_{i,n}) \cdot \mu(\hat{I}_n \cap g_n^{-1}(S_n)) \right|. \end{aligned}$$

We recall from Remark 4.5 that for every $i = 1, \dots, k$:

$$\mu\left(\bigcup_{j \in N_i} I_j \times W\right) = \kappa_{i,n} \cdot \mu(\hat{I}_n).$$

Hereby, we conclude

$$\begin{aligned} & \left| \mu(\Gamma_n \cap f_n^{-i \cdot m_n}(A_n)) - \kappa_{i,n} \cdot \mu(\Gamma_n) \cdot \mu(A_n) - (1 - \kappa_{i,n}) \cdot \mu(\Gamma_n \cap A_n) \right| \\ & \leq \left| \mu\left(\Phi_n\left(\bigcup_{j \notin N_i} I_j \times W\right) \cap g_n^{-1}(S_n)\right) - (1 - \kappa_{i,n}) \cdot \mu(\hat{I}_n \cap g_n^{-1}(S_n)) \right| \\ & \quad + \sum_{j \in N_i} \left| \mu(I_j \times W \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)) - \mu(I_j \times W) \cdot \mu(S_n) \right|. \end{aligned}$$

We start to examine the first term. By Remarks 4.5 and 5.5 we have

$$(1 - \kappa_{i,n}) \cdot \mu(\hat{I}_n \cap g_n^{-1}(S_n)) = \mu\left(\left(\bigcup_{j \notin N_i} I_j \times W\right) \cap g_n^{-1}(S_n)\right).$$

Additionally, Remark 5.5 tells us that $\mu(I_j \times W \cap g_n^{-1}(S_n)) = \mu(\Phi_n(I_j \times W) \cap g_n^{-1}(S_n))$ for each $j \notin N_i$. Hence, the first term is equal to 0.

In the next step, we examine the second term. We note that the cube S_n is contained in $\mathbb{S}^1 \times \left[\frac{1}{10n^4}, 1 - \frac{1}{10n^4}\right]^{m-1}$. So we can apply Lemma 5.3 for $j \in N_i$:

$$\begin{aligned} \sum_{j \in N_i} \left| \mu(I_j \times W \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)) - \mu(I_j \times W) \cdot \mu(S_n) \right| & \leq \sum_{j \in N_i} \frac{20}{n} \cdot \mu(I_j \times W) \cdot \mu(S_n) \\ & \leq \frac{20}{n} \cdot \mu(\hat{I}_n) \cdot \mu(S_n). \end{aligned}$$

□

Now we are able to prove that the constructed diffeomorphisms are $(\kappa_1, \dots, \kappa_k)$ -weakly mixing.

Proposition 5.8 (Proof of $(\kappa_1, \dots, \kappa_k)$ -weak mixing). *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n \in \mathbb{N}}$ be constructed as in the previous sections. Suppose additionally that for every $n \in \mathbb{N}$ $d_0(f^{i \cdot m_n}, f_n^{i \cdot m_n}) < \frac{1}{2^n}$ for $i = 1, \dots, k$, $\|DH_{n-1}\|_0 \leq \frac{\ln(q_n)}{n}$ and that the limit $f = \lim_{n \rightarrow \infty} f_n$ exists. Then f is $(\kappa_1, \dots, \kappa_k)$ -weakly mixing.*

Proof. Since every measurable set in $M = \mathbb{S}^1 \times [0, 1]^{m-1}$ can be approximated by a countable disjoint union of cubes in $\mathbb{S}^1 \times (0, 1)^{m-1}$ in arbitrary precision, we only have to prove the $(\kappa_1, \dots, \kappa_k)$ -weak mixing property in case that A and B are m -dimensional cubes in $\mathbb{S}^1 \times (0, 1)^{m-1}$. So let $A, B \subseteq \mathbb{S}^1 \times (0, 1)^{m-1}$ be m -dimensional cubes and $\varepsilon > 0$ be given.

According to Lemma 5.1 and Remark 5.6 respectively the partial partitions $\nu_n := H_{n-1} \circ g_n(\eta_n)$ and \mathfrak{S}_n converge to the decomposition into points. Thus we can approximate B by a countable disjoint union of sets $\Gamma_n \in \nu_n$ and A by a countable disjoint union of sets $C_n \in \mathfrak{S}_n$ in given precision, when n is chosen big enough. Consequently for n sufficiently large there are sets $B_1 = \bigcup_{i \in \Sigma_n^1} \Gamma_n^i$, $B_2 = \bigcup_{i \in \Sigma_n^2} \Gamma_n^i$ with countable sets Σ_n^1, Σ_n^2 of indices satisfying $B_1 \subseteq B \subseteq B_2$ and $\mu(B \Delta B_j) \leq \varepsilon \cdot \mu(A) \cdot \mu(B)$ for $j = 1, 2$. Furthermore, there are sets $A_1 = \bigcup_{i \in \Sigma_n^3} C_n^i$, $A_2 = \bigcup_{i \in \Sigma_n^4} C_n^i$ with countable sets Σ_n^3, Σ_n^4 of indices satisfying $A_1 \subseteq A \subseteq A_2$, $|\mu(A) - \mu(A_j)| \leq \varepsilon \cdot \mu(A) \cdot \mu(B)$ as well as $\text{dist}(\partial A, \partial A_j) > \frac{1}{2^n}$ for $j = 1, 2$, if n is chosen sufficiently large. Because of $d_0(f^{i \cdot m_n}, f_n^{i \cdot m_n}) < \frac{1}{2^n}$ for $i = 1, \dots, k$ the following relations are true:

$$\begin{aligned} f_n^{i \cdot m_n}(x) \in A_1 &\implies f^{i \cdot m_n}(x) \in A, \\ f_n^{i \cdot m_n}(x) \in A &\implies f_n^{i \cdot m_n}(x) \in A_2. \end{aligned}$$

Thus:

$$\begin{aligned} \mu(B_1 \cap f_n^{-i \cdot m_n}(A_1)) &\leq \mu(B \cap f_n^{-i \cdot m_n}(A_1)) \leq \mu(B \cap f^{-i \cdot m_n}(A)) \\ &\leq \mu(B \cap f_n^{-i \cdot m_n}(A_2)) \leq \mu(B_2 \cap f_n^{-i \cdot m_n}(A_2)). \end{aligned}$$

Additionally we choose n such that $\frac{20}{n} < \varepsilon$ as well as $|\kappa_i - \kappa_{i,n}| < \varepsilon \cdot \mu(A) \cdot \mu(B)$ hold. We can apply Lemma 5.7 on the sets A_2 and B_2 . Therewith we obtain the following estimate from above for every $i \in \{1, \dots, k\}$:

$$\begin{aligned} &\mu(B \cap f^{-i \cdot m_n}(A)) - \kappa_i \cdot \mu(A) \cdot \mu(B) - (1 - \kappa_i) \cdot \mu(A \cap B) \\ &\leq \mu(B_2 \cap f_n^{-i \cdot m_n}(A_2)) - \kappa_{i,n} \cdot \mu(A) \cdot \mu(B) - (1 - \kappa_{i,n}) \cdot \mu(A \cap B) + 2 \cdot |\kappa_i - \kappa_{i,n}| \\ &\leq \varepsilon \cdot \mu(A_2) \cdot \mu(B_2) + \kappa_{i,n} \cdot \mu(A_2) \cdot \mu(B_2) + (1 - \kappa_{i,n}) \cdot \mu(A_2 \cap B_2) - \kappa_{i,n} \cdot \mu(A) \cdot \mu(B) \\ &\quad - (1 - \kappa_{i,n}) \cdot \mu(A \cap B) + 2 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B) \\ &\leq \varepsilon \cdot (\mu(A) + \mu(A \Delta A_2)) \cdot (\mu(B) + \mu(B \Delta B_2)) + \kappa_{i,n} \cdot 3 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B) \\ &\quad + (1 - \kappa_{i,n}) \cdot (\mu(A \Delta A_2) + \mu(B \Delta B_2)) + 2 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B) \\ &\leq 8 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B). \end{aligned}$$

Analogously we estimate:

$$\mu(B \cap f^{-i \cdot m_n}(A)) - \kappa_i \cdot \mu(A) \cdot \mu(B) - (1 - \kappa_i) \cdot \mu(A \cap B) \geq -8 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B).$$

Both estimates combined enable us to conclude:

$$|\mu(B \cap f^{-i \cdot m_n}(A)) - \kappa_i \cdot \mu(A) \cdot \mu(B) - (1 - \kappa_i) \cdot \mu(A \cap B)| \leq 8 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B).$$

Since ε can be chosen arbitrarily small, the $(\kappa_1, \dots, \kappa_k)$ -weak mixing property is proven. \square

6 Convergence of $(f_n)_{n \in \mathbb{N}}$ in $\text{Diff}^\infty(M)$

In this section we show that the sequence of constructed measure-preserving smooth diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges. The proof follows along the lines of [FS05, section 5.3] and [GK18, section 5]. We also borrow the notations for the $\text{Diff}^k(M)$ norms $|||\cdot|||_k$ and metrics d_k from [GK18, subsection 2.1].

6.1 Properties of the conjugation maps ϕ_n and H_n

In this subsection we want to find estimates on the norms $|||H_n|||_l$ for any $l \in \mathbb{N}$. We start with an estimate of $|||\phi_n|||_l$.

Lemma 6.1. *For every $l \in \mathbb{N}$ it holds*

$$|||\phi_n|||_l \leq C \cdot q_n^{2 \cdot (m-1)^2 \cdot l \cdot (n-1+k)},$$

where C is a constant depending on m, l and n , but is independent of q_n .

Proof. We use [GK18, Lemma 5.3] in the setting of our explicit construction of the map ϕ_n in section 3.1 with $\varepsilon = \frac{1}{40 \cdot n^4}$ and $\lambda_{max} = d_n \cdot (n \cdot q_n)^{2 \cdot (m-1) \cdot (n-1+k)}$. Thus:

$$\begin{aligned} |||\phi_n|||_l &\leq \tilde{C}(m, l, n) \cdot \left(d_n \cdot (n \cdot q_n)^{2 \cdot (m-1) \cdot (n-1+k)} \right)^{(m-1) \cdot l} \\ &\leq C(m, l, n) \cdot q_n^{2 \cdot (m-1)^2 \cdot l \cdot (n-1+k)}, \end{aligned}$$

where $C(m, l, n)$ is a constant independent of q_n . \square

In the next step we consider the map $h_n = g_n \circ \phi_n$ defined in section 3.2:

Lemma 6.2. *For every $l \in \mathbb{N}$ it holds:*

$$|||h_n|||_l \leq \bar{C} \cdot q_n^{2 \cdot (m-1)^2 \cdot l \cdot (n+k)}$$

where \bar{C} is a constant depending on m, l and n , but is independent of q_n .

Proof. By definition of the map $h_n = g_n \circ \phi_n$ in section 3.2 we have:

$$\begin{aligned} h_n(x_1, \dots, x_m) &= g_n \circ \phi_n(x_1, \dots, x_m) \\ &= ([\phi_n(x_1, \dots, x_m)]_1 + n \cdot q_n \cdot [\phi_n(x_1, \dots, x_m)]_2, [\phi_n(x_1, \dots, x_m)]_2, \dots, [\phi_n(x_1, \dots, x_m)]_m) \end{aligned}$$

and

$$\begin{aligned} h_n^{-1}(x_1, \dots, x_m) &= \phi_n^{-1} \circ g_n^{-1}(x_1, \dots, x_m) \\ &= ([\phi_n^{-1}(x_1 - n \cdot q_n \cdot x_2, x_2, \dots, x_m)]_1, \dots, [\phi_n^{-1}(x_1 - n \cdot q_n \cdot x_2, x_2, \dots, x_m)]_m). \end{aligned}$$

We estimate:

$$\|h_n\|_l \leq 2 \cdot (n \cdot q_n)^l \cdot \|\phi_n\|_l \leq \bar{C}(m, l, n) \cdot q_n^l \cdot q_n^{2 \cdot (m-1)^2 \cdot l \cdot (n-1+k)} \leq \bar{C}(m, l, n) \cdot q_n^{2 \cdot (m-1)^2 \cdot l \cdot (n+k)}$$

with a constant $\bar{C}(m, l, n)$ independent of q_n . \square

Finally, we are able to prove an estimate on the norms of the map H_n :

Lemma 6.3. *For every $l \in \mathbb{N}$ we get:*

$$\|H_n\|_l \leq \check{C} \cdot q_n^{2 \cdot (m-1)^2 \cdot l \cdot (n+k)},$$

where \check{C} is a constant depending solely on m, l, n and H_{n-1} . Since H_{n-1} is independent of q_n in particular, the same is true for \check{C} .

Proof. The proof follows along the lines of [GK18, Lemma 5.5]. \square

In particular we see that this norm can be estimated by a power of q_n .

6.2 Proof of convergence

For the proof of the convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ in the $\text{Diff}^\infty(M)$ -topology the next result, that can be found in [FSW07, Lemma 4] is very useful.

Lemma 6.4. *Let $k \in \mathbb{N}_0$ and h be a C^∞ -diffeomorphism on M . Then we get for every $\alpha, \beta \in \mathbb{R}$:*

$$d_k(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}) \leq C_k \cdot \|h\|_{k+1}^{k+1} \cdot |\alpha - \beta|,$$

where the constant C_k depends solely on k and m . In particular $C_0 = 1$.

In the following Lemma we state that under some assumptions on the sequence $(\alpha_n)_{n \in \mathbb{N}}$ the sequence $(f_n)_{n \in \mathbb{N}}$ converges to $f \in \mathcal{A}_\alpha(M)$ in the $\text{Diff}^\infty(M)$ -topology. Afterwards we will show that we can fulfil these conditions (see Lemma 6.6).

Lemma 6.5 (Criterion for $\text{Diff}^\infty(M)$ convergence). *Let $\varepsilon > 0$ be arbitrary and $(k_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^\infty \frac{1}{k_n} < \varepsilon$. Furthermore we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha - \alpha_1| < \varepsilon \quad \text{and} \quad |\alpha - \alpha_n| \leq \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot \|H_n\|_{k_n+1}^{k_n+1}} \quad \text{for every } n \in \mathbb{N},$$

where C_{k_n} are the constants from Lemma 6.4.

1. *Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the $\text{Diff}^\infty(M)$ -topology to a measure-preserving smooth diffeomorphism f , for which $d_\infty(f, R_\alpha) < 3 \cdot \varepsilon$ holds.*
2. *Also the sequence of diffeomorphisms $\hat{f}_n = H_n \circ R_\alpha \circ H_n^{-1} \in \mathcal{A}_\alpha(M)$ converges to f in the $\text{Diff}^\infty(M)$ -topology. Hence $f \in \mathcal{A}_\alpha(M)$.*

Proof. See [Ku16, Lemma 5.8]. \square

As announced we show that we can satisfy the conditions from Lemma 6.5 in our constructions:

Lemma 6.6. *Let $(k_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty} \frac{1}{k_n} < \infty$ and C_{k_n} be the constants from Lemma 6.4. For any Liouvillean number α there exists a sequence $\alpha_n = \frac{p_n}{q_n}$ of rational numbers with $10n^2$ divides q_n , such that our conjugation maps H_n constructed in section 3.1 and 3.2 fulfil the following conditions:*

1. For every $n \in \mathbb{N}$:

$$|\alpha - \alpha_n| < \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}}.$$

2. For every $n \in \mathbb{N}$:

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot k \cdot q_n \cdot |||H_n|||_1}.$$

3. For every $n \in \mathbb{N}$

$$\|DH_{n-1}\|_0 < \frac{\ln(q_n)}{n}.$$

Proof. In Lemma 6.3 we saw $|||H_n|||_{k_n+1} \leq \check{C}_n \cdot q_n^{2 \cdot (m-1)^2 \cdot (n+k) \cdot (k_n+1)}$, where the constant \check{C}_n was independent of q_n . Thus we can choose $q_n \geq \check{C}_n$ for every $n \in \mathbb{N}$. Hence we obtain: $|||H_n|||_{k_n+1} \leq q_n^{2 \cdot m^2 \cdot (n+k) \cdot (k_n+1)}$.

Besides $q_n \geq \check{C}_n$ we keep the condition $q_n \geq 200 \cdot (n-1)^2 \cdot k \cdot d_{n-1}^2 \cdot (n-1)^{4 \cdot (n-1+k)} \cdot q_{n-1}^{4 \cdot (n-2+k)}$ from Remark 4.2 in mind. Furthermore we can demand $\|DH_{n-1}\|_0 < \frac{\ln(q_n)}{n}$ from q_n because H_{n-1} is independent of q_n . Since α is a Liouvillean number, we find a sequence of rational numbers $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$, \tilde{p}_n, \tilde{q}_n relatively prime, under the above restrictions (formulated for \tilde{q}_n) satisfying:

$$|\alpha - \tilde{\alpha}_n| = \left| \alpha - \frac{\tilde{p}_n}{\tilde{q}_n} \right| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1} \cdot k \cdot k_n \cdot C_{k_n} \cdot (10n^2)^{1+2 \cdot m^2 \cdot (n+k) \cdot (k_n+1)^2} \cdot \tilde{q}_n^{1+2 \cdot m^2 \cdot (n+k) \cdot (k_n+1)^2}}.$$

Put $q_n := 10n^2 \cdot \tilde{q}_n$ and $p_n := 10n^2 \cdot \tilde{p}_n$. Then we obtain:

$$|\alpha - \alpha_n| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1} \cdot k \cdot k_n \cdot C_{k_n} \cdot q_n^{1+2 \cdot m^2 \cdot (n+k) \cdot (k_n+1)^2}}.$$

So we have $|\alpha - \alpha_n| \xrightarrow{n \rightarrow \infty} 0$ monotonically. Because of $|||H_n|||_{k_n+1} \leq q_n^{2 \cdot m^2 \cdot (n+k) \cdot (k_n+1)^2}$ this yields: $|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot k \cdot q_n \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}}$. Thus the first property of this Lemma is fulfilled.

Furthermore, we note $k_n \geq 1$ and $C_{k_n} \geq 1$ by Lemma 6.4. Thus $q_n \cdot k_n \cdot C_{k_n} \geq q_n$. Moreover, $|||H_n|||_1 \geq |||H_n|||_0 = 1$, because $H_n : \mathbb{S}^1 \times [0, 1]^{m-1} \rightarrow \mathbb{S}^1 \times [0, 1]^{m-1}$ is a diffeomorphism. Hence $|||H_n|||_{k_n+1}^{k_n+1} \geq |||H_n|||_1$. Altogether we conclude $2^{n+1} \cdot q_n \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1} \geq 2^{n+1} \cdot q_n \cdot |||H_n|||_1$ and so:

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot k \cdot q_n \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}} \leq \frac{1}{2^{n+1} \cdot k \cdot q_n \cdot |||H_n|||_1}, \quad (7)$$

i.e. we verified the second property. \square

Lemma 6.6 shows that the conditions of Lemma 6.5 are satisfied. Therefore we conclude

Lemma 6.7. *The sequence of constructed diffeomorphisms f_n converges in the $\text{Diff}^\infty(M)$ -topology to a diffeomorphism $f \in \mathcal{A}_\alpha(M)$.*

To satisfy the assumptions of Proposition 5.8 we need a result akin to [GK18, Lemma 5.9]:

Lemma 6.8. *Let $(\alpha_n)_{n \in \mathbb{N}}$ be constructed as in Lemma 6.6. Then it holds for every $n \in \mathbb{N}$, for every $i \in \mathbb{N}$, $i \leq k$, and for every $\tilde{m} \leq q_{n+1}$:*

$$d_0(f^{i \cdot \tilde{m}}, f_n^{i \cdot \tilde{m}}) \leq \frac{1}{2^n}$$

Proof. According to our construction it holds $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$ and hence

$$\begin{aligned} f_{n-1} &= H_{n-1} \circ R_{\alpha_n} \circ H_{n-1}^{-1} = H_{n-1} \circ R_{\alpha_n} \circ h_n \circ h_n^{-1} \circ H_{n-1}^{-1} \\ &= H_{n-1} \circ h_n \circ R_{\alpha_n} \circ h_n^{-1} \circ H_{n-1}^{-1} = H_n \circ R_{\alpha_n} \circ H_n^{-1}. \end{aligned}$$

Hereby and with the aid of Lemma 6.4 we compute:

$$d_0(f_j^{i \cdot \tilde{m}}, f_{j-1}^{i \cdot \tilde{m}}) = d_0(H_j \circ R_{i \cdot \tilde{m} \cdot \alpha_{j+1}} \circ H_j^{-1}, H_j \circ R_{i \cdot \tilde{m} \cdot \alpha_j} \circ H_j^{-1}) \leq \|H_j\|_1 \cdot i \cdot \tilde{m} \cdot 2 \cdot |\alpha - \alpha_j|.$$

Since $\tilde{m} \leq q_{n+1} \leq q_j$ we conclude for every $j > n$ using equation (7):

$$d_0(f_j^{i \cdot \tilde{m}}, f_{j-1}^{i \cdot \tilde{m}}) \leq \|H_j\|_1 \cdot i \cdot \tilde{m} \cdot 2 \cdot |\alpha - \alpha_j| \leq \|H_j\|_1 \cdot i \cdot \tilde{m} \cdot 2 \cdot \frac{1}{2^{j+1} \cdot k \cdot q_j \cdot \|H_j\|_1} \leq \frac{i \cdot \tilde{m}}{k \cdot q_j} \cdot \frac{1}{2^j} \leq \frac{1}{2^j}.$$

Thus for every $\tilde{m} \leq q_{n+1}$ we get the claimed result:

$$d_0(f^{i \cdot \tilde{m}}, f_n^{i \cdot \tilde{m}}) = \lim_{k \rightarrow \infty} d_0(f_k^{i \cdot \tilde{m}}, f_n^{i \cdot \tilde{m}}) \leq \lim_{k \rightarrow \infty} \sum_{j=n+1}^k d_0(f_j^{i \cdot \tilde{m}}, f_{j-1}^{i \cdot \tilde{m}}) \leq \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \left(\frac{1}{2}\right)^n.$$

□

Proof of Proposition 2.2. Note that the sequence $(m_n)_{n \in \mathbb{N}}$ defined in section 4 meets the mentioned condition $m_n \leq q_{n+1}$ and hence Lemma 6.8 can be applied on it. Concluding we have checked that all the assumptions of Proposition 5.8 are satisfied. Thus this criterion guarantees that the constructed diffeomorphism $f \in \mathcal{A}_\alpha(M)$ is $(\kappa_1, \dots, \kappa_k)$ -weakly mixing. In addition, for every $\varepsilon > 0$ we can choose the parameters by Lemma 6.5 in such a way, that $d_\infty(f, R_\alpha) < \varepsilon$ holds. This completes the proof of Proposition 2.2. □

In order to prove the genericity results in section 7 we have to compute the weak distance between f and f_n .

Lemma 6.9. *Let $\nu_n := H_{n-1} \circ g_n(\eta_n)$ and $(m_n)_{n \in \mathbb{N}}$ be the sequence of natural numbers defined in section 4. Then we have for every $i \in \{1, \dots, k\}$*

$$\sum_{c \in \nu_n} \mu(f_n^{i \cdot m_n}(c) \Delta f^{i \cdot m_n}(c)) < \frac{1}{n} \cdot \frac{1}{n^{2m} \cdot q_n^{3m}}$$

Proof. At first we observe for $\Gamma_n = H_{n-1} \circ g_n(\hat{I}_n) \in \nu_n$

$$\mu(f_n^{i \cdot m_n}(\Gamma_n) \Delta f_{n+1}^{i \cdot m_n}(\Gamma_n)) = \mu\left(R_{\alpha_{n+1}}^{i \cdot m_n}\left(\phi_n^{-1}\left(\hat{I}_n\right)\right) \Delta h_{n+1} \circ R_{\alpha_{n+2}}^{i \cdot m_n} \circ h_{n+1}^{-1}\left(\phi_n^{-1}\left(\hat{I}_n\right)\right)\right)$$

and introduce the notation $b_{n,i} := d_0 \left(R_{\alpha_{n+1}}^{i \cdot m_n}, h_{n+1} \circ R_{\alpha_{n+2}}^{i \cdot m_n} \circ h_{n+1}^{-1} \right)$.

In case of $0 \leq l < n - k$ there are

$$n \cdot q_n \cdot \left(nq_n^2 - 2 \cdot \left[\frac{q_n^2}{10n^3} \right] \right) \cdot (n - k - 1) \cdot \left(n^2q_n^2 - 2 \cdot \left[\frac{q_n^2}{10n^2} \right] \right)^{m-2} \leq (n \cdot q_n)^{2 \cdot (m-1)+1}$$

elements of the partition η_n on $\left[\frac{l}{n \cdot q_n}, \frac{l+1}{n \cdot q_n} \right] \times [0, 1]^{m-1}$. By the shape of elements in η_n and the calculations in the proof of Lemma 4.3 the image of such an element under $R_{\alpha_{n+1}}^{i \cdot m_n} \circ \phi_n^{-1}$ consists of

$$1 + \sum_{d=1}^k c_{d,n} \left(n^2q_n^2 - 2 \left[\frac{q_n^2}{10n^2} \right] \right)^{m(n-1+k)-1} \left(n^2q_n^2 - 2 \left[\frac{q_n^2}{10n^2} \right] \right)^{(m-1)(n-2+k)} \leq \sum_{d=1}^k c_{d,n} (nq_n)^{4m(n-1+k)}$$

blocks. For such a block we denote its sidelengths by w_θ in the θ -coordinate, w in the coordinates r_1, \dots, r_{m-2} and w_{m-1} in the r_{m-1} -coordinate. The part of $h_{n+1} \circ R_{\alpha_{n+2}}^{i \cdot m_n} \circ h_{n+1}^{-1} \left(\phi_n^{-1} \left(\hat{I}_n \right) \right)$ corresponding to any block of $R_{\alpha_{n+1}}^{i \cdot m_n} \left(\phi_n^{-1} \left(\hat{I}_n \right) \right)$ surrounds a block of sidelengths $w_\theta - 2b_{n,i}$ in the θ -coordinate, $w - 2b_{n,i}$ in the r_1, \dots, r_{m-2} -coordinates and $w_{m-1} - 2b_{n,i}$ in the r_{m-1} -coordinate as well as it is contained in a block of sidelengths $w_\theta + 2b_{n,i}$ in the θ -coordinate, $w + 2b_{n,i}$ in the r_1, \dots, r_{m-2} -coordinates and $w_{m-1} + 2b_{n,i}$ in the r_{m-1} -coordinate. Since

$$\begin{aligned} b_{n,i} = d_0 \left(R_{\alpha_{n+1}}^{i \cdot m_n}, h_{n+1} \circ R_{\alpha_{n+2}}^{i \cdot m_n} \circ h_{n+1}^{-1} \right) &\leq \| |h_{n+1}| \|_1 \cdot k \cdot m_n \cdot |\alpha_{n+1} - \alpha_{n+2}| \\ &< \| |H_{n+1}| \|_1 \cdot q_{n+1} \cdot 2 \cdot |\alpha_{n+1} - \alpha| < \frac{1}{\frac{2 \cdot m^2 \cdot (n+k+1) \cdot k_{n+1}^2}{q_{n+1}}} \end{aligned}$$

by Lemma 6.6, we conclude

$$\begin{aligned} \sum_{c \in \nu_n} \mu \left(f_n^{i \cdot m_n}(c) \triangle f_{n+1}^{i \cdot m_n}(c) \right) &\leq (n - k) \cdot q_n \cdot (n \cdot q_n)^{2 \cdot (m-1)+1} \cdot \sum_{d=1}^k c_{d,n} \cdot (nq_n)^{4 \cdot m \cdot (n-1+k)} \cdot 4 \cdot m \cdot b_{n,i} \\ &< \frac{1}{2n} \cdot \frac{1}{n^{2m} \cdot q_n^{3m}}, \end{aligned}$$

which implies the claim. \square

7 Proof of genericity

First of all, Proposition 2.2 yields the denseness of $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms in $\mathcal{A}_\alpha(M)$:

Because of $\mathcal{A}_\alpha(M) = \overline{\{h \circ R_\alpha \circ h^{-1} : h \in \text{Diff}^\infty(M, \mu)\}}^{C^\infty}$ it is enough to show that for every diffeomorphism $h \in \text{Diff}^\infty(M, \mu)$ and every $\epsilon > 0$ there is a $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphism \tilde{f} such that $d_\infty(\tilde{f}, h \circ R_\alpha \circ h^{-1}) < \epsilon$. For this purpose, let $h \in \text{Diff}^\infty(M, \mu)$ and $\epsilon > 0$ be arbitrary. Since $\text{Diff}^\infty(M)$ is a Lie group, the conjugating map $g \mapsto h \circ g \circ h^{-1}$ is continuous with respect to the metric d_∞ . Continuity in the point R_α yields the existence of $\delta > 0$, such that $d_\infty(g, R_\alpha) < \delta$ implies $d_\infty(h \circ g \circ h^{-1}, h \circ R_\alpha \circ h^{-1}) < \epsilon$. By Proposition 2.2 we can find a

$(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphism f with $d_\infty(f, R_\alpha) < \delta$. Hence $\tilde{f} := h \circ f \circ h^{-1}$ satisfies $d_\infty(\tilde{f}, h \circ R_\alpha \circ h^{-1}) < \epsilon$. Note that \tilde{f} is $(\kappa_1, \dots, \kappa_k)$ -weakly mixing.

In order to prove the genericity statement in the Theorem we consider all sequences of constructed diffeomorphisms $(f_n)_{n \in \mathbb{N}}$ satisfying the requirements from the previous sections. Let $U_n(f_n)$ be the subsequent neighbourhood of the diffeomorphism f_n :

$$U_n(f_n) := \left\{ g \in \text{Diff}^\infty(M, \mu) : \right. \\ \left. d_{k_{n+1}}(f_n, g) < \frac{2}{k_{n+1}}, \sum_{\Gamma_n \in \nu_n} \mu(g^{im_n}(\Gamma_n) \Delta f_n^{im_n}(\Gamma_n)) < \frac{1}{n} \cdot \frac{1}{n^{2m} q^{3m}} \text{ for } i = 1, \dots, k \right\}$$

By Θ_n we denote the union of all neighbourhoods $U_n(f_n)$ over all the n -th diffeomorphisms in the above mentioned sequences. Since the neighbourhoods $U_n(f_n)$ are open, the sets Θ_n are open as well. Then

$$\Theta := \bigcap_{n \in \mathbb{N}} \bigcup_{s \geq n} \Theta_s$$

is a G_δ -set as the countable intersection of open sets.

- For all the sequences $(f_n)_{n \in \mathbb{N}}$ the respective limit diffeomorphism $f \in \mathcal{A}_\alpha(M)$ belongs to Θ , because it belongs to $U_n(f_n)$ for every $n \in \mathbb{N}$ by construction (cf. Lemma 6.9). So Θ contains all the constructed diffeomorphisms with the aimed properties. Hence, it is dense in $\mathcal{A}_\alpha(M)$ due to the above considerations.
- In the next step we want to show that $f \in \Theta$ is a $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphism: For any $f \in \bigcap_{n \in \mathbb{N}} \bigcup_{s \geq n} \Theta_s$ there is a sequence $(n_l)_{l \in \mathbb{N}}$ with $n_l \rightarrow \infty$ as $l \rightarrow \infty$, such that $f \in \Theta_{n_l}$. So there is a sequence $(f_{n_l})_{l \in \mathbb{N}}$ of diffeomorphisms, at which f_{n_l} is the n_l -th element of one of the above mentioned sequences of constructed diffeomorphisms, such that $f \in U_{n_l}(f_{n_l})$. We observe that $\nu_{n_l} \rightarrow \varepsilon$ as $l \rightarrow \infty$, where ν_{n_l} is the partition belonging to the diffeomorphism f_{n_l} . Moreover, we have

$$\left| \mu(\Gamma_{n_l} \cap f^{-i \cdot m_{n_l}}(A_{n_l})) - \kappa_{i, n_l} \cdot \mu(\Gamma_{n_l}) \cdot \mu(A_{n_l}) - (1 - \kappa_{i, n_l}) \cdot \mu(\Gamma_{n_l} \cap A_{n_l}) \right| \\ < \frac{23}{n_l} \cdot \mu(\Gamma_{n_l}) \cdot \mu(A_{n_l})$$

for every $\Gamma_{n_l} \in \nu_{n_l}$ and $A_{n_l} \in \mathfrak{S}_{n_l}$ by the definition of the neighbourhoods $U_{n_l}(f_{n_l})$. Then we can conclude that f is $(\kappa_1, \dots, \kappa_k)$ -weakly mixing.

Thus the set of $(\kappa_1, \dots, \kappa_k)$ -weakly mixing diffeomorphisms in $\mathcal{A}_\alpha(M)$ contains a dense G_δ -set. Hence, the Theorem is deduced from Proposition 2.2.

8 Applications

As a first application of the $(\kappa_1, \dots, \kappa_k)$ -weak mixing property we state [JL92, Lemma 1.3]:

Lemma 8.1. *Suppose that T is $(1 - \kappa_1, \dots, 1 - \kappa_K)$ -weakly mixing with $\kappa_1, \dots, \kappa_K \in (0, 1)$ and $\{\log(\kappa_1), \dots, \log(\kappa_K)\}$ linearly independent over \mathbb{Q} . Then*

$$\sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l)'}}$$

for all numbers $0 < k(1), \dots, k(l), k'(1), \dots, k'(l)' \leq K$ unless $(k(1), \dots, k(l))$ is a permutation of $(k'(1), \dots, k'(l)')$.

Hereby, Corollary 1 follows from the Theorem.

Moreover, we can follow the lines of [JL92] and [Ru79, Example 2] to get a smooth weakly mixing diffeomorphism with no measurable square root: Let $T \in \mathcal{A}_\alpha(M)$ be a diffeomorphism as constructed in Corollary 1 and $S = T \times T \times T$. By [JL92, Corollary 3] any measurable square root must have the form $(T^{l(1)} \times T^{l(2)} \times T^{l(3)}) U_\pi$ with $l(i) \in \mathbb{Z}$ and $\pi^2 = \text{id}$, where $U_\pi : M^3 \rightarrow M^3$ is defined by $U_\pi(x)(j) = x(\pi(j))$. Thus, for some i we would have $\pi(i) = i$ and then $2l(i) = 1$ which is impossible for $l(i) \in \mathbb{Z}$.

Acknowledgement: The author would like to thank the referee for helpful comments.

9 References

- [AK70] D. V. Anosov and A. Katok: *New examples in smooth ergodic theory. Ergodic diffeomorphisms.* Trudy Moskov. Mat. Obsc., 23: 3–36, 1970.
- [Ba17] S. Banerjee: *Non-standard real-analytic realization of some rotations of the circle.* Ergodic Theory & Dynamical Systems, 37 (5): 1369–1386, 2017.
- [BK19] S. Banerjee and P. Kunde: *Real-analytic AbC constructions on the torus.* Ergodic Theory & Dynamical Systems, 39 (10): 2643–2688, 2019.
- [BK] P. Berk and A. Kanigowski: *Spectral disjointness of rescalings of some surface flows.* Preprint, arXiv:1901.04724.
- [BSZ12] J. Bourgain, P. Sarnak, T. Ziegler: *Disjointness of Möbius from horocycle flows.* In: *From Fourier and Number theory to Radon Transforms and Geometry*, Developments in Mathematics, 28: 67–83, 2012.
- [CS96] G. M. Constantine and T. H. Savits: *A multivariate Faà di Bruno formula with applications.* Trans. Amer. Math. Soc., 348 (2): 503–520, 1996.
- [ELR14] H. El Abdalaoui, M. Lemanczyk and T. de la Rue: *On spectral disjointness of powers for rank-one transformations and Möbius orthogonality.* J. Funct. Anal., 266 (1): 284–317, 2014.
- [FK04] B. Fayad and A. Katok: *Constructions in elliptic dynamics.* Ergodic Theory Dynam. Systems, 24 (5): 1477–1520, 2004.
- [FKL18] S. Ferenczi, J. Kulaga-Przymus, M. Lemanczyk: *Sarnak’s Conjecture: What’s New.* In: *Ergodic Theory and Dynamical Systems in their Interactions with Arithmetics and Combinatorics*, Lecture Notes in Mathematics (2213), Springer, 2018.

- [FS05] B. Fayad and M. Saprykina: *Weak mixing disc and annulus diffeomorphisms with arbitrary Liouville rotation number on the boundary*. Ann. Scient. École. Norm. Sup.(4), 38(3): 339–364, 2005.
- [FSW07] B. Fayad, M. Saprykina and A. Windsor: *Nonstandard smooth realizations of Liouville rotations*. Ergodic Theory Dynam. Systems, 27: 1803–1818, 2007.
- [Fu67] H. Furstenberg: *Disjointness in ergodic theory, minimal sets and diophantine approximation*. Math. Syst. Th., 1: 1–49, 1967.
- [GK18] R. Gunesch and P. Kunde: *Weakly mixing diffeomorphisms preserving a measurable Riemannian metric with prescribed Liouvillean rotation behavior*. Journal of Discrete and Continuous Dynamical Systems A, 38 (4): 1615–1655, 2018.
- [JL92] A. del Junco and M. Lemanczyk: *Generic spectral properties of measure-preserving maps and applications*. Proc. Amer. Math. Soc., 115 (3): 725–736, 1992.
- [Ka79] A. Katok: *Bernoulli diffeomorphisms on surfaces*. Ann. of Math., 110: 529–547, 1979.
- [Ka03] A. Katok: *Combinatorial Constructions in Ergodic Theory and Dynamics*. American Mathematical Society, Providence, 2003.
- [KLR] A. Kanigowski, M. Lemanczyk, M. Radziwiłł: *Rigidity in dynamics and Möbius disjointness*. Preprint, arXiv:1905.13256.
- [Ku16] P. Kunde: *Smooth diffeomorphisms with homogeneous spectrum and disjointness of convolutions*. Journal of Modern Dynamics, 10: 439–481, 2016.
- [Ku17] P. Kunde: *Real-analytic weak mixing diffeomorphism preserving a measurable Riemannian metric*. Ergodic Theory & Dynamical Systems, 37 (5): 1547–1569, 2017.
- [Ku20] P. Kunde: *Smooth zero-entropy diffeomorphisms with ergodic derivative extension*. Commentarii Mathematici Helvetici, 95: 1–25, 2020.
- [Ne32] J. von Neumann: *Zur Operatorenmethode in der klassischen Mechanik*. Ann. of Math., 33(3): 587–642, 1932.
- [Os69] V. I. Oseledets: *An automorphism with simple continuous spectrum not having the group property*. Mat. Zametki 5: 323–326, 1969.
- [OW91] D. S. Ornstein and B. Weiss: *Statistical properties of chaotic systems*. Bull. Amer. Math. Soc., 24: 11–116, 1991.
- [Ru79] D. J. Rudolph: *An example of a measure-preserving map with minimal self-joinings and applications*. J. Anal. Math. 35: 97–122, 1979.
- [Sa] P. Sarnak: *Three lectures on the Möbius function - randomness and dynamics*. publications.ias.edu/sarnak

- [Sk167] M. D. Sklover: *Classical dynamical systems on the torus with continuous spectrum*. Izv. Vys. Uceb. Zaved. Mat., 10: 113–124, 1967.
- [St66] A. M. Stepin: *Spectral properties of ergodic dynamical systems with locally compact time*. Dokl. Akad. Nauk SSSR, 169(4): 773–776, 1966.
- [St87] A. M. Stepin: *Spectral properties of generic dynamical systems*. Math. USSR-Izv. 29: 159–192, 1987.