Smooth diffeomorphisms with homogeneous spectrum and disjointness of convolutions

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Abstract

On any smooth compact connected manifold M of dimension $m \geq 2$ admitting a smooth non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{S}^1}$ and for every Liouville number $\alpha \in \mathbb{S}^1$ we prove the existence of a C^∞-diffeomorphism $f \in \mathcal{A}_\alpha = \overline{\{h \circ S_\alpha \circ h^{-1} \,:\, h \in \text{Diff}^\infty(M, \nu)\}}^{C^\infty}$ with a good approximation of type $(h, h+1)$, a maximal spectral type disjoint with its convolutions and a homogeneous spectrum of multiplicity two for the Cartesian square $f \times f$. This answers a question of Fayad and Katok ([FK04], Problem 7.11.). The proof is based on a quantitative version of the approximation by conjugation-method with explicitly defined conjugation maps and tower elements.

Introduction

Let M be a smooth compact connected manifold of dimension $m \geq 2$ admitting a smooth non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{S}^1}$ preserving a smooth volume ν. In case of a manifold with boundary by a smooth diffeomorphism we mean infinitely differentiable in the interior and such that all the derivatives can be extended to the boundary continuously. In this setting we consider the closure of conjugates $\mathcal{A} = \overline{\{h \circ S_t \circ h^{-1} \,:\, h \in \text{Diff}^\infty(M, \nu), t \in \mathbb{S}^1\}}^{C^\infty}$ and more precisely for $\alpha \in \mathbb{S}^1$ the restricted spaces $\mathcal{A}_\alpha = \overline{\{h \circ S_\alpha \circ h^{-1} \,:\, h \in \text{Diff}^\infty(M, \nu)\}}^{C^\infty}$. In [FK04], Problem 7.11., the following question is posed:

Question 1. *Given a circle action $\mathcal{S}$ and the corresponding space $\mathcal{A}$, is there a diffeomorphism $f \in \mathcal{A}$ with any of the following properties:*

1. *a good approximation of type $(h, h+1)$;*
2. *a maximal spectral type disjoint with its convolutions;*
3. *a homogeneous spectrum of multiplicity two for the Cartesian square $f \times f$?*

This question takes up problems in the category of measure-preserving transformations on a Lebesgue space (X, μ). For instance, there is extensive research on the *spectral multiplicity problem* about the construction of transformations possessing specific essential values $\mathcal{M}_{U_T}$ of the spectral multiplicities:

Question 2. *Given a subset $E \subset \mathbb{N} \cup \{\infty\}$, is there an ergodic transformation T such that $\mathcal{M}_{U_T} = E$?*

This question is a weak version of one of the main problems in the spectral theory of dynamical systems at the interface of unitary operator theory and ergodic theory:

Question 3. *What are possible spectral properties for a Koopman operator associated with a measure-preserving transformation?*

These two problems are open and no restrictions (except for the obvious ones) are known. However, there is an impressive progress concerning Question 2 (see [Da13] for a survey on spectral multiplicities of ergodic actions) and there exist two standard points of view: to consider the spectrum of T (and in particular $\mathcal{M}_{U_T}$) either on $L^2(X, \mu)$ or on the orthogonal complement $L^2_0(X, \mu)$ of the constant functions. In [KL95] it was proved that all possible subsets of $\mathbb{N} \cup \{\infty\}$ can be realized as $\mathcal{M}_{U_T}$ for some ergodic transformation T in the first case (since 1 is always an eigenvalue because of the constant functions, "possible" means any subset of $\mathbb{N} \cup \{\infty\}$ with 1 as an element). In the second case the Cartesian powers of a generic transformation provide a good opportunity for the construction of examples with the infimum of essential spectral multiplicities larger than 1. Although, it seems very unlikely that these Cartesian powers have finite maximal spectral multiplicity, this is the generic case: Independently Ageev and Ryzhikov proved the celebrated result, that for a generic automorphism T the Cartesian square $T \times T$ has homogeneous spectrum of multiplicity 2 (see [Ag99] resp. [Ry99a]). Ageev was even able to show that for the n-th power $T^n = T \times ... \times T$ of a generic transformation T it holds $M(T^n) = \{n, n \cdot (n-1), ..., n!\}$ (cf. [Ag99], Theorem 2). He also proved for every $n \in \mathbb{N}$ the existence of an ergodic transformation with homogeneous spectrum of multiplicity n in the orthogonal complement of the constant functions ([Ag05], Theorem 1) solving Rokhlin's problem on homogeneous spectrum in ergodic theory.
Here the property of admitting an approximation of type $(h, h+1)$ is often used to find an upper bound for $\mathcal{M}_{U_{T \times T}}$ like in [Ka03], Proposition 3.6. Moreover, it was used in [KS70] to construct homeomorphisms with continuous spectrum. In particular, this will enable us to conclude the ergodicity of $f \times f$ in the case of our constructions.
The second part of Question 1 is linked to a conjecture of Kolmogorov respectively Rokhlin and Fomin (after verifying that the property held for all dynamical systems known at that time, especially large classes of systems of probabilistic origin like Gaussian ones), namely that every ergodic transformation possesses the so-called group property, i.e. the maximal spectral type σ is symmetric and dominates its square $\sigma * \sigma$. This conjecture is an analogue of the well-known group property of the set of eigenvalues of an ergodic automorphism and was proven to be false. Indeed, in [St66] A.M. Stepin gave the first example of a dynamical system without the group property. V.I. Oseledets constructed an analogous example with continuous spectrum ([Os69]). Later Stepin showed that for a generic transformation all convolutions σ_0^k, $k \in \mathbb{N}$, of the maximal spectral type σ_0 on $L^2_0(X, \mu)$ are mutually singular (see [St87]).
In the smooth category there are only few results in this direction. In general, it is one of the most important problems in ergodic theory to find smooth models of ergodic transformations. Explicitly, Danilenko asks which subsets $E \neq \{1\}$ admit a smooth ergodic transformation T with $\mathcal{M}_{U_T} = E$ ([Da13], section 10). Blanchard and Lemanczyk showed that every set E containing 1 as well as $\mathrm{lcm}(e_1, e_2)$ for $e_1, e_2 \in E$ is realizable as the set of essential spectral multiplicities for a Lebesgue measure-preserving analytic diffeomorphism of a finite dimensional torus ([BL93]). In [St87], §4, Stepin constructed diffeomorphisms, for which the convolutions of the maximal spectral type on $L^2_0(M, \mu)$ are mutually singular, on manifolds M as above using a smooth variant of the method of approximation by periodic transformations.

In order to extend this result of Stepin and answering the beforehand cited question 1 affirmatively we prove the following Theorem:

Theorem 1. *Let M be a smooth compact connected manifold of dimension $m \geq 2$ admitting a smooth non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{S}^1}$ preserving a smooth volume ν and α a Liouvillean number. Then the set of smooth diffeomorphisms, that have a maximal spectral type disjoint with its convolutions, a homogeneous spectrum of multiplicity 2 for $f \times f$ and admit a good approximation of type $(h, h+1)$, is residual (i.e. it contains a dense G_δ-set) in $\mathcal{A}_\alpha$ in the Diff$^\infty(M)$-topology.*

The proof is based on the so-called "approximation by conjugation-method" introduced in [AK70]: The diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \in \mathbb{Q}$, $H_n = H_{n-1} \circ \phi_n$ and ϕ_n is a measure-preserving diffeomorphism satisfying $S_{\frac{1}{q_n}} \circ \phi_n = \phi_n \circ S_{\frac{1}{q_n}}$. In each step of the construction, the conjugation map ϕ_n as well as specific partial partitions of the manifold have to be chosen in such a way that f_n imitates the aimed properties with a certain precision. Moreover, the conjugation maps must allow explicit norm estimates. Then we will exploit the fact that α is a Liouville number in order to prove convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_\alpha$.
Further applications of this method are the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weak mixing ones in [AK70], section 5, or [GK00]) or non-standard smooth realizations of measure preserving systems (e.g. [AK70], section 6, [Be13] and [FSW07]). See [FK04] for more details and other results of this method.

1 Preliminaries

1.1 Definitions and notations

1.1.1 C^∞-Topology

In this chapter we want to introduce advantageous definitions and notations. Initially we discuss topologies on the space of smooth diffeomorphisms on the manifold $M = \mathbb{S}^1 \times [0,1]^{m-1}$. Note that for diffeomorphisms $f = (f_1, ..., f_m) : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ the coordinate function f_1 understood as a map $\mathbb{R} \times [0,1]^{m-1} \to \mathbb{R}$ has to satisfy the condition $f_1(\theta + n, r_1, ..., r_{m-1}) = f_1(\theta, r_1, ..., r_{m-1}) + l$ for $n \in \mathbb{Z}$, where either $l = n$ or $l = -n$. Moreover, for $i \in \{2, ..., m\}$ the coordinate function f_i has to be $\mathbb{Z}$-periodic in the first component, i.e. $f_i(\theta + n, r_1, ..., r_{m-1}) = f_i(\theta, r_1, ..., r_{m-1})$ for every $n \in \mathbb{Z}$.
For defining explicit metrics on Diff$^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ and in the following the subsequent notations will be useful:

Definition 1.1.
1. For a sufficiently differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ and a multiindex $\vec{a} = (a_1, ..., a_m) \in \mathbb{N}_0^m$

$$D_{\vec{a}} f := \frac{\partial^{|\vec{a}|}}{\partial x_1^{a_1} ... \partial x_m^{a_m}} f,$$

where $|\vec{a}| = \sum_{i=1}^m a_i$ is the order of $\vec{a}$.

2. For a continuous function $F : (0,1)^m \to \mathbb{R}$

$$\|F\|_0 := \sup_{z\in(0,1)^m} |F(z)|.$$

Diffeomorphisms on $\mathbb{S}^1 \times [0,1]^{m-1}$ can be regarded as maps from $[0,1]^m$ to $\mathbb{R}^m$. In this spirit the expressions $\|f_i\|_0$ as well as $\|D_{\vec{a}} f_i\|_0$ for any multiindex $\vec{a}$ with $|\vec{a}| \leq k$ have to be understood for $f = (f_1, ..., f_m) \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$. Since such a diffeomorphism is a continuous map on the compact manifold and every partial derivative can be extended continuously to the boundary, all these expressions are finite. Thus, the subsequent definition makes sense:

Definition 1.2. 1. For $f, g \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ with coordinate functions f_i resp. g_i we define

$$\tilde{d}_0(f,g) = \max_{i=1,..,m} \left\{ \inf_{p \in \mathbb{Z}} \|(f-g)_i + p\|_0 \right\}$$

as well as

$$\tilde{d}_k(f,g) = \max\left\{ \tilde{d}_0(f,g), \|D_{\vec{a}}(f-g)_i\|_0 \ : \ i = 1,...,m \ , \ 1 \leq |\vec{a}| \leq k \right\}.$$

2. Using the definitions from 1. we define for $f, g \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$:

$$d_k(f,g) = \max\left\{ \tilde{d}_k(f,g) \ , \ \tilde{d}_k\left(f^{-1}, g^{-1}\right) \right\}.$$

Obviously d_k describes a metric on $\text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ measuring the distance between the diffeomorphisms as well as their inverses. As in the case of a general compact manifold the following definition connects to it:

Definition 1.3. 1. A sequence of $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$-diffeomorphisms is called convergent in $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ if it converges in $\text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ for every $k \in \mathbb{N}$.

2. On $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ we declare the following metric

$$d_\infty(f,g) = \sum_{k=1}^{\infty} \frac{d_k(f,g)}{2^k \cdot (1 + d_k(f,g))}.$$

It is a general fact that $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ is a complete metric space with respect to this metric d_∞.
Again considering diffeomorphisms on $\mathbb{S}^1 \times [0,1]^{m-1}$ as maps from $[0,1]^m$ to $\mathbb{R}^m$ we add the adjacent notation:

Definition 1.4. Let $f \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ with coordinate functions f_i be given. Then

$$\|Df\|_0 := \max_{i,j \in \{1,...,m\}} \|D_j f_i\|_0$$

and

$$|||f|||_k := \max\left\{ \|D_{\vec{a}} f_i\|_0, \left\|D_{\vec{a}}\left(f_i^{-1}\right)\right\|_0 \ : \ i = 1,...,m, \ \vec{a} \text{ multiindex with } 0 \leq |\vec{a}| \leq k \right\}.$$

Remark 1.5. By the above-mentioned observations for every multiindex $\vec{a}$ with $|\vec{a}| \geq 1$ and every $i \in \{1,...,m\}$ the derivative $D_{\vec{a}} h_i$ is $\mathbb{Z}$-periodic in the first variable. Since in case of a diffeomorphism $g = (g_1, ..., g_m)$ on $\mathbb{S}^1 \times [0,1]^{m-1}$ regarded as a map $[0,1]^m \to \mathbb{R}^m$ the coordinate functions g_j for $j \in \{2,...,m\}$ satisfy $g_j([0,1]^m) \subseteq [0,1]$, it holds:

$$\sup_{z \in (0,1)^m} |(D_{\vec{a}} h_i)(g(z))| \leq |||h|||_{|\vec{a}|}.$$

1.1.2 Partial partitions

Furthermore, we introduce the notion of a partial partition of a measure space (X, μ), which is a pairwise disjoint countable collection of measurable subsets of X.

Definition 1.6.
- A sequence of partial partitions ν_n converges to the decomposition into points if and only if for a given measurable set A and for every $n \in \mathbb{N}$ there exists a measurable set A_n, which is a union of elements of ν_n, such that $\lim_{n\to\infty} \mu(A \triangle A_n) = 0$. We often denote this by $\nu_n \to \varepsilon$.
- A partial partition ν is a refinement of a partial partition η if and only if for every $C \in \nu$ there exists a set $D \in \eta$ such that $C \subseteq D$. We write this as $\eta \leq \nu$.

Using the notion of a partition we can introduce the weak topology in the space of measure-preserving transformations on a Lebesgue space:

Definition 1.7.
1. For two measure-preserving transformations T, S and for a finite partition ξ the weak distance with respect to ξ is defined by $d(\xi, T, S) := \sum_{c \in \xi} \mu(T(c) \triangle \sigma(c))$.
2. The base of neighbourhoods of T in the weak topology consists of the sets

$$W(T, \xi, \varepsilon) = \{S \ : \ d(\xi, T, S) < \varepsilon\},$$

where ξ is a finite partition and ε is a positive number.

1.2 First steps of the proof

First of all, we show how constructions on $\mathbb{S}^1 \times [0,1]^{m-1}$ can be transferred to a general compact connected smooth manifold M with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$. By [AK70], Proposition 2.1., we can assume that 1 is the smallest positive number t satisfying $S_t = Id$. Hence, we can assume $\mathcal{S}$ to be effective. We denote the set of fixed points of $\mathcal{S}$ by F and for $q \in \mathbb{N}$ F_q is the set of fixed points of the map $S_{\frac{1}{q}}$.
On the other hand, we consider $\mathbb{S}^1 \times [0,1]^{m-1}$ with Lebesgue measure μ. Furthermore, let $\mathcal{R} = \{R_t\}_{t\in\mathbb{S}^1}$ be the standard action of $\mathbb{S}^1$ on $\mathbb{S}^1 \times [0,1]^{m-1}$, where the map R_t is given by $R_t(\theta, r_1, ..., r_{m-1}) = (\theta + t, r_1, ..., r_{m-1})$. Hereby, we can formulate the following result (see [FSW07], Proposition 1):

Proposition 1.8. *Let M be a m-dimensional smooth, compact and connected manifold admitting an effective circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume ν. We denote $B := \partial M \cup F \cup \left(\bigcup_{q\geq 1} F_q\right)$. There exists a continuous surjective map $G : \mathbb{S}^1 \times [0,1]^{m-1} \to M$ with the following properties:*

1. *The restriction of G to $\mathbb{S}^1 \times (0,1)^{m-1}$ is a C^∞-diffeomorphic embedding.*
2. $\nu\left(G\left(\partial\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)\right)\right) = 0.$
3. $G\left(\partial\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)\right) \supseteq B.$
4. $G_*(\mu) = \nu.$
5. $\mathcal{S} \circ G = G \circ \mathcal{R}.$

By the same reasoning as in [FSW07], section 2.2., this proposition allows us to carry a construction from $\left(\mathbb{S}^1 \times [0,1]^{m-1}, \mathcal{R}, \mu\right)$ to the general case $(M, \mathcal{S}, \nu)$:

Suppose $f : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ is a diffeomorphism sufficiently close to R_α in the C^∞-topology with the aimed properties obtained by $f = \lim_{n\to\infty} f_n$ with $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $f_n = R_{\alpha_{n+1}}$ in a neighbourhood of the boundary (in Proposition 1.9 we will see that these conditions can be satisfied in the constructions of this article). Then we define a sequence of diffeomorphisms:

$$\tilde{f}_n : M \to M \qquad \tilde{f}_n(x) = \begin{cases} G \circ f_n \circ G^{-1}(x) & \text{if } x \in G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right) \\ S_{\alpha_{n+1}}(x) & \text{if } x \in G\left(\partial\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)\right) \end{cases}$$

Constituted in [FK04], section 5.1. (which bases upon [Ka79], Proposition 1.1.), this sequence is convergent in the C^∞-topology to the diffeomorphism

$$\tilde{f} : M \to M \qquad \tilde{f}(x) = \begin{cases} G \circ f \circ G^{-1}(x) & \text{if } x \in G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right) \\ S_\alpha(x) & \text{if } x \in G\left(\partial\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)\right) \end{cases}$$

provided the closeness from f to R_α in the C^∞-topology.
We observe that f and $\tilde{f}$ are metrically isomorphic (Recall that two measure preserving dynamical systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ are metrically isomorphic if there exist $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that $T_1 B_1 \subseteq B_1$, $T_2 B_2 \subseteq B_2$, $\mu_1(B_1) = 1$, $\mu_2(B_2) = 1$ and there exists an automorphism $\phi : B_1 \to B_2$ satisfying $\phi \circ T = S \circ \phi$). Thus, $\tilde{f}$ admits a good approximation of type $(h, h+1)$ because the speed and the type of a periodic approximation are invariant under isomorphisms. Moreover, f and $\tilde{f}$ are unitarily equivalent (see Remark 3.1). Since the spectral types and multiplicities are spectral invariants, we conclude that $\tilde{f}$ has the aimed properties.
Hence, it is sufficient to prove the Theorem in case of $\left(\mathbb{S}^1 \times [0,1]^{m-1}, \mathcal{R}, \mu\right)$. In this setting we will show the subsequent statement:

Proposition 1.9. *For every Liouvillean number α there are a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of rational numbers $\alpha_n = \frac{p_n}{q_n}$ converging monotonically to α and a sequence of measure-preserving smooth diffeomorphisms ϕ_n, that coincide with the identity in a neighbourhood of the boundary and satisfy $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$, such that the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $H_n = H_{n-1} \circ \phi_n$, converge in the Diff$^\infty$-topology to a limit $f = \lim_{n\to\infty} f_n$, which satisfies $f \in \mathcal{A}_\alpha$ and admits a good linked approximation of type $(h, h+1)$ as well as a good cyclic approximation.*
Furthermore, for every $\varepsilon > 0$ the parameters in the construction can be chosen in such a way that $d_\infty(f, R_\alpha) < \varepsilon$.

In section 9 we will deduce the Theorem from this Proposition which is proven in section 8.

1.3 Outline of the proof

By the previous section it is enough to construct a diffeomorphism f with the aimed properties in the case of $M = \mathbb{S}^1 \times [0,1]^{m-1}$. Using results on periodic approximation as well as spectral theory of dynamical systems stated in the successive two sections we will be able to reduce this task to the construction of a diffeomorphism admitting a good linked approximation of type $(h, h+1)$ and a good cyclic approximation (see section 9). For this purpose, we will use the "approximation by conjugation"-method and obtain the aimed diffeomorphism as the limit of conjugates $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \in \mathbb{Q}$, $H_n = H_{n-1} \circ \phi_n$ and ϕ_n is a

measure-preserving diffeomorphism satisfying $R_{\frac{1}{q_n}} \circ \phi_n = \phi_n \circ R_{\frac{1}{q_n}}$. These conjugation maps ϕ_n are constructed very explicitly in section 4. We will sketch the construction of ϕ_n as well as its action on the explicitly defined tower elements in subsection 4.4.
Subsequently, we prove that the sequence $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the Diff$^\infty$-topology to a measure-preserving smooth diffeomorphism $f \in \mathcal{A}_\alpha$ under some conditions on the sequence $(\alpha_n)_{n\in\mathbb{N}}$ of rational numbers (cf. Lemma 5.9). Here, we require precise norm estimates on the conjugation maps. In the adjacent two sections 6 and 7 we show that this constructed limit f admits the required types of approximation with the respective speeds of approximation. Hereby, we will proof Proposition 1.9 in section 8.

2 Periodic approximation in Ergodic Theory

This section provides a short introduction to the concept of periodic approximation in Ergodic Theory. A more comprehensive presentation can be found in [Ka03].
Let (X, μ) be a Lebesgue space. A tower t of height $h(t) = h$ is an ordered sequence of disjoint measurable sets $t = \{c_1, ..., c_h\}$ of X having equal measure, which is denoted by $m\,(t)$. The sets c_i are called the levels of the tower, especially c_1 is the base. Associated with a tower there is a cyclic permutation σ sending c_1 to c_2, c_2 to c_3,... and c_h to c_1. Using the notion of a tower we can give the next definition:

Definition 2.1. A periodic process is a collection of disjoint towers covering the space X together with an equivalence relation among these towers identifying their bases.

There are two partial partitions associated with a periodic process: The partition ξ into all sets of all towers and the partition η consisting of the union of bases of towers in each equivalence class and their images under the iterates of σ, where when we go beyond the height of a certain tower in the class we drop this tower and continue until the highest tower in the equivalence class has been exhausted. Obviously, we have $\eta \leq \xi$.
A sequence $(\xi_n, \eta_n, \sigma_n)$ of periodic processes is called exhaustive if $\eta_n \to \varepsilon$. Such an exhaustive sequence of periodic processes is consistent if for every measurable subset $A \subseteq X$ the sequence $\sigma_n\,(A)$ converges to a set B, i.e. $\mu\,(\sigma_n\,(A) \triangle B) \to 0$ as $n \to \infty$. Moreover, we will call a sequence of towers $t^{(n)}$ from the periodic process $(\xi_n, \eta_n, \sigma_n)$ substantial if there exists $r > 0$ such that $h\left(t^{(n)}\right) \cdot m\left(t^{(n)}\right) > r$ for every $n \in \mathbb{N}$.

Definition 2.2. Let $T : (X, \mu) \to (X, \mu)$ be a measure-preserving transformation. An exhaustive sequence of periodic processes $(\xi_n, \eta_n, \sigma_n)$ forms a periodic approximation of T if

$$d\,(\xi_n, T, \sigma_n) = \sum_{c\in\xi_n} \mu\,(T\,(c) \triangle \sigma_n\,(c)) \to 0 \quad \text{ as } n \to \infty.$$

Given a sequence $g\,(n)$ of positive numbers we will say that the transformation T admits a periodic approximation with speed $g\,(n)$ if for a certain subsequence $(n_k)_{k\in\mathbb{N}}$ there exists an exhaustive sequence of periodic processes $(\xi_k, \eta_k, \sigma_k)$ such that $d\,(\xi_k, T, \sigma_k) < g\,(n_k)$.

In order to define the type of the periodic approximation we need the notion of equivalence for sequences of periodic processes:

Definition 2.3. Two sequences of periodic processes $P_n = (\xi_n, \eta_n, \sigma_n)$ and $P'_n = (\xi'_n, \eta'_n, \sigma'_n)$ are called equivalent if for every $n \in \mathbb{N}$ there is a bijective correspondence θ_n between subsets S_n and S'_n of the sets of towers of P_n respectively P'_n such that

- For $t \in S_n$: $h(\theta_n(t)) = h(t)$.
- $\sum_{t \in S_n} h(t) m(t) \to 1$ as $n \to \infty$.
- $\sum_{t \in S_n} h(t) \cdot |m(t) - m(\theta_n(t))| \to 0$ as $n \to \infty$.
- If two towers from S_n are equivalent in P_n, then their images under θ_n are equivalent in P'_n.

There are various types of approximation. We introduce the most important ones:

Definition 2.4. 1. A cyclic process is a periodic process which consists of a single tower of height h. An approximation by an exhaustive sequence of cyclic processes is called a cyclic approximation. More specifically we will refer to a cyclic approximation with speed $o\left(\frac{1}{h}\right)$ as a good cyclic approximation.

2. An approximation generated by periodic processes equivalent to periodic processes consisting of two substantial towers whose heights differ by one is said to be of type $(h, h+1)$. Equivalently the heights of the two towers t_1 and t_2 with base B_1 resp. B_2 are equal to h and $h+1$ and for some $r > 0$ we have $\mu(B_1) > \frac{r}{h}$ as well as $\mu(B_2) > \frac{r}{h+1}$. We will call the approximation of type $(h, h+1)$ with speed $o\left(\frac{1}{h}\right)$ good and with speed $o\left(\frac{1}{h \cdot (h+1)}\right)$ excellent.

3. An approximation of type $(h, h+1)$ will be called a linked approximation of type $(h, h+1)$ if the two towers involved in the approximation are equivalent. This insures that the sequence of partitions η_n generated by the union of the bases of the two towers and the iterates of this set converges to the decomposition into points.

Remark 2.5. As noted in [Ry06] a good linked approximation of type $(h, h+1)$ implies the convergence

$$U_T^{k \cdot (h+1)} \longrightarrow_w r \cdot U_T^k + (1-r) \cdot Id$$

in the weak operator topology for every $k \in \mathbb{N}$ and some $r \in (0,1)$, where U_T is the Koopman-operator of T (see section 3.1).

From the different types of approximations various ergodic properties can be derived. For example in [KS67], Corollary 2.1., the subsequent Lemma is proven.

Lemma 2.6. *Let $T : (X, \mu) \to (X, \mu)$ be a measure-preserving transformation. If T admits a good cyclic approximation, then T is ergodic.*

In [KS70] Katok and Stepin proved the genericity of automorphisms having a continuous spectrum in the set of measure-preserving homeomorphisms (recall that a transformation has a continuous spectrum, i.e. the corresponding operator U_T in the space $L^2(M, \mu)$ has no eigenfunctions other than constants, if and only if it is weak mixing). For this purpose, they deduced the following result ([KS70], Theorem 5.1.):

Lemma 2.7. *Let $T : (X, \mu) \to (X, \mu)$ be a measure preserving transformation. If T is ergodic and admits a good approximation of type $(h, h+1)$, then T has continuous spectrum.*

Moreover, the theory of periodic approximation can be used to prove genericity of constructed properties. The applied statement can be summarized as follows (cf. [Ka03], Theorem 2.1.):

Lemma 2.8. *Given a type τ and a speed $g(n)$, the set of all measure-preserving transformations of a Lebesgue space which admit a periodic approximation of type τ with speed $g(n)$ is a residual set (i.e. it contains a dense G_δ-set) in the weak topology.*

3 Spectral theory of dynamical systems

Besides the concept of periodic approximation we will need further mathematical tools. We refer to [Na98] and [Go99] for more details.

3.1 Spectral types

Let (X, μ) be a Lebesgue space and $T : (X, \mu) \to (X, \mu)$ be an automorphism. Then we define the induced Koopman-operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$ by $U_T f = f \circ T$. Since

$$\langle U_T f, U_T g\rangle = \int_X f \circ T \cdot \overline{g \circ T}\, d\mu = \int_X f \cdot \overline{g}\, d\mu = \langle f, g\rangle \qquad \text{for every } f, g \in L^2(X; \mu)$$

and $U_T^{-1} = U_{T^{-1}}$ this is an unitary operator on the Hilbert space $L^2(X, \mu)$.

Remark 3.1. If two measure-preserving dynamical systems (X_1, μ_1, T_1) and (X_2, μ_2, T_2) are metrically isomorphic, their isomorphism $h : X_1 \to X_2$ induces an isomorphism of Hilbert spaces $V_h : L^2(X_2, \mu_2) \to L^2(X_1, \mu_1)$ by $(V_h f) = f \circ h$. Then we have $U_{T_1} = V_h \circ U_{T_2} \circ V_h^{-1}$ and this relation is called unitary equivalence of operators. Hence, any invariant of unitary equivalence defines an invariant of isomorphisms. Such invariants are said to be spectral invariants or spectral properties.
Moreover, we note that 1 is always an eigenvalue of U_T because of the constant functions. So when we discuss the spectral properties of U_T we refer to its spectral properties that are restricted to the orthogonal complement of the constants. Hence, we consider the properties of U_T in the space $L_0^2(X, \mu)$ of all L^2-functions with zero integral.

One of the important spectral invariants are the so-called spectral measures: Let $f \in L_0^2(X, \mu)$ and $Z(f) := \overline{\operatorname{span}\{U_T^n f \,:\, n \in \mathbb{Z}\}}^{L_0^2(X,\mu)}$. Using Bochner's theorem one can prove the existence of a finite Borel measure σ_f defined on the unit circle $\mathbb{S}^1$ in the complex plane satisfying

$$\langle U_T^n f, f\rangle = \int_{\mathbb{S}^1} z^n\, d\sigma_f(z) \qquad \text{for every } n \in \mathbb{Z}.$$

Then σ_f is called the spectral measure of f with respect to U_T.
Moreover, by the Hahn-Hellinger Theorem, there is a sequence of functions $f_n \in L_0^2(X, \mu)$, $n \in \mathbb{N}$, for which

$$L_0^2(X, \mu) = \oplus_{n\in\mathbb{N}} Z(f_n) \quad \text{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \ldots$$

These measures are unique in the sense that for any other family of functions $g_n \in L_0^2(X, \mu)$, $n \in \mathbb{N}$, for which $L_0^2(X, \mu) = \oplus_{n\in\mathbb{N}} Z(g_n)$ and $\sigma_{g_1} \gg \sigma_{g_2} \gg \ldots$ we have $\sigma_{f_n} \sim \sigma_{g_n}$ for every $n \in \mathbb{N}$.

Definition 3.2. The spectral type of σ_{f_1} is called the maximal spectral type σ of U_T.

According to this we say that U_T has a continuous spectrum if σ_{f_1} is a continuous measure and U_T has a discrete spectrum if σ_{f_1} is a discrete measure.

3.2 Spectral multiplicities

Besides the maximal spectral type an important characterization of U_T is the **multiplicity function** $M_{U_T} : \mathbb{S}^1 \to \mathbb{N} \cup \{\infty\}$, which is σ_{f_1}- almost everywhere defined by

$$M_{U_T}(z) = \sum_{i=1}^{\infty} \chi_{A_i}(z), \text{ where } A_i = \left\{ z \in \mathbb{S}^1 \,:\, \frac{d\sigma_{f_i}}{d\sigma_{f_1}}(z) > 0 \right\}.$$

Here $\frac{d\sigma_{f_i}}{d\sigma_{f_1}}$ is the Radon-Nikodym derivative of σ_{f_i} with respect to σ_{f_1}.
Using this multiplicity function we establish the set $\mathcal{M}_{U_T}$ of **essential spectral multiplicities**, which is the essential range of M_{U_T} with respect to σ_{f_1}. Then we define the **maximal spectral multiplicity** m_{U_T} as the essential supremum (with respect to σ_{f_1}) of $\mathcal{M}_{U_T}$.

Definition 3.3. U_T is said to have homogeneous spectrum of multiplicity m if $\mathcal{M}_{U_T} = \{m\}$. In particular, U_T has a simple spectrum if $\mathcal{M}_{U_T} = \{1\}$. In all other cases U_T has a non-simple spectrum.

Another interpretation of the multiplicity function can be given by the the subsequent formulation on the canonical form of an unitary operator (see [CFS82], Appendix 2):
For any unitary operator U on a separable complex Hilbert space H and any $m \in \mathbb{N} \cup \{\infty\}$ we can find a Borel set A_m of the circle $\mathbb{S}^1$ and a sequence of vectors $h_{m,k} \in H$, $k = 1,..,m$ (respectively $k = 1, 2, ...$ in case of $m = \infty$), such that

- $\bigcup_{m\in\mathbb{N}} A_m = \mathbb{S}^1$ *and* $A_{m_1} \cap A_{m_2} = \emptyset$ *for* $m_1 \neq m_2$.
- $\oplus_{m\in\mathbb{N}} \oplus_{k=1}^{m} Z\left(h_{m,k}\right) = H$, $Z\left(h_{m,k}\right) \perp Z\left(h_{m_1,k_1}\right)$ *for* $(m,k) \neq (m_1,k_1)$.
- $\sigma^{(m)} := \sigma_{h_{m,k}} = \sigma_{h_{m,l}}$ *for* $1 \leq k,l \leq m$ *and* $\sigma^{(m)}\left(\mathbb{S}^1 \setminus A_m\right) = 0$.

The multiplicity function is defined on $\mathbb{S}^1$ by the relation $m(\lambda) = m$ for $\lambda \in A_m$. Note that the measures $\sigma^{(m)}$ are not the spectral measures, but the spectral type of the measure $\sigma^{(m)}$ is called a spectral type of multiplicity m.

In connection with the previous chapter 2 we state the following result ([KS67], Theorem 3.1.):

Lemma 3.4. *Let T be an automorphism of a Lebesgue space. If T admits a cyclic approximation of speed $\frac{\theta}{h}$, where $\theta < \frac{1}{2}$, then the spectrum of U_T is simple.*

For automorphisms with simple spectrum we have the subsequent theorem of Ryzhikov ([Ry99b], Theorem 2.1.):

Lemma 3.5. *Let (X,μ) be a Lebesgue space with $\mu(X) = 1$ and $T : (X,\mu) \to (X,\mu)$ be an automorphism with simple spectrum. Suppose that the weak convergence*

$$U_T^{k_n} \longrightarrow_w (a \cdot U_T + (1-a) \cdot Id)$$

holds for some $a \in (0,1)$ and some strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers. Then the Cartesian square $T \times T$ has a homogeneous spectrum of multiplicity 2.

3.3 Disjointness of convolutions

In this section we study the convolutions of the maximal spectral type σ. Therefore, we state the definition of a convolution of measures:

Definition 3.6. Let G be a topological group and μ, ν finite Borel measures on G. Then their convolution $\mu * \nu$ is defined by

$$(\mu * \nu)(A) = \int \int 1_A(x \cdot y)\; d\mu(x)\, d\nu(y)$$

for each measurable set A of G.

If all the convolutions $\sigma^k = \sigma * ... * \sigma$ for $k \in \mathbb{N}$ are pairwise mutually singular, one speaks about disjointness of convolutions. To guarantee this pairwise singularity of convolutions of the maximal spectral type of a measure-preserving transformation the following property is useful:

Definition 3.7. An automorphism T of a Lebesgue space (X, μ) is said to be κ-weak mixing, $\kappa \in [0,1]$, if there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that the weak convergence

$$U_T^{k_n} \longrightarrow_w (\kappa \cdot P_c + (1-\kappa) \cdot Id)$$

holds, where P_c is the projection on the subspace of constants.

Remark 3.8. By [St87], Proposition 3.1., we can characterise this property in geometric language: A transformation T is κ-weak mixing if and only if there is an increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that for all measurable sets A and B

$$\lim_{n\to\infty} \mu\left(A \cap T^{k_n} B\right) = \kappa \cdot \mu(A) \cdot \mu(B) + (1-\kappa) \cdot \mu(A \cap B).$$

We recognize that 0-weak mixing corresponds to rigidity and 1-weak mixing to the usual notion of weak mixing.

As announced this property has connections with certain properties of the maximal spectral type (see [St87], Theorem 1):

Lemma 3.9. *If the transformation T is κ-weak mixing for some $0 < \kappa < 1$ and σ is the maximal spectral type for $U_T|_{L_0^2(X,\mu)}$, then σ and all its convolutions $\sigma^k = \sigma * ... * \sigma$ are pairwise mutually singular.*

4 Construction of the conjugation maps

We fix an arbitrary Liouvillean number α and present step n in our inductive process of construction. Hence, we assume that we have already defined the rational numbers $\alpha_1, ..., \alpha_{n-1} \in \mathbb{S}^1$ and the conjugation map $H_{n-1} = \phi_1 \circ ... \circ \phi_{n-1} \in \text{Diff}^\infty(M, \mu)$. In order to construct the conjugation map ϕ_n we will need two types of maps which we will introduce in the subsequent subsections.

4.1 The map $\phi_{\lambda,\varepsilon}^{(i)}$

At first we apply "Moser's trick" to deduce the following result (similar to [FS05], Lemma 5.3.):

Lemma 4.1. *For every $\varepsilon \in \left(0, \frac{1}{4}\right)$ and every $i, j \in \{1, ..., m\}$ there exists a smooth measure-preserving diffeomorphism $\varphi_{\varepsilon,i,j}$ on $\mathbb{R}^m$, which is the rotation in the $x_i - x_j$-plane by $\pi/2$ about the point $\left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^m$ on $[\varepsilon, 1-\varepsilon]^m$ and coincides with the identity outside of $\left[\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}\right]^m$.*

Proof. W.l.o.g. we prove the statement in case of $i < j$. We denote $[0,1]^m$ by Δ and $[\varepsilon, 1-\varepsilon]^m$ by $\Delta(\varepsilon)$. Let ψ_ε be a smooth diffeomorphism satisfying

$$\psi_\varepsilon(x_1, .., x_m) = \begin{cases} (x_1, ..., x_m) & \text{on } \mathbb{R}^m \setminus \Delta\left(\frac{\varepsilon}{2}\right) \\ \left(\frac{1}{2} + \frac{1}{5\cdot\sqrt{m}} \cdot \left(x_1 - \frac{1}{2}\right), .., \frac{1}{2} + \frac{1}{5\cdot\sqrt{m}} \cdot \left(x_m - \frac{1}{2}\right)\right) & \text{on } \Delta(\varepsilon) \end{cases}$$

Furthermore, let τ_ε be a smooth diffeomorphism with the following properties

$$\tau_\varepsilon(x, y) = \begin{cases} (x_1, ..., x_m) & \text{on } \left\{\sum_{i=1}^m \left(x_i - \frac{1}{2}\right)^2 \geq \frac{1}{16}\right\} \\ (x_1, ..., x_{i-1}, 1-x_j, x_{i+1}, ..., x_{j-1}, x_i, x_{j+1}, ..., x_m) & \text{on } \left\{\sum_{i=1}^m \left(x_i - \frac{1}{2}\right)^2 \leq \frac{1}{50}\right\} \end{cases}$$

We define $\bar{\varphi}_\varepsilon := \psi_\varepsilon^{-1} \circ \tau_\varepsilon \circ \psi_\varepsilon$. Then the diffeomorphism $\bar{\varphi}_\varepsilon$ coincides with the identity on $\mathbb{R}^m \setminus \Delta\left(\frac{\varepsilon}{2}\right)$ and with the rotation in the $x_i - x_j$-plane on $\Delta(\varepsilon)$. From this we conclude that $\det\left(D\bar{\varphi}_\varepsilon\right) > 0$. Moreover, $\bar{\varphi}_\varepsilon$ is measure-preserving on $U_\varepsilon := \left(\mathbb{R}^m \setminus \Delta\left(\frac{\varepsilon}{2}\right)\right) \cup \Delta(\varepsilon)$.
With the aid of "Moser's trick" we want to construct a diffeomorphism φ_ε, that is measure-preserving on the whole $\mathbb{R}^m$ and agrees with $\bar{\varphi}_\varepsilon$ on U_ε. Therefore, we consider the canonical volume form Ω_0 on $\mathbb{R}^m$: $\Omega_0 = dx_1 \wedge ... \wedge dx_m$ respectively $\Omega_0 = d\omega_0$ using the $m-1$-form $\omega_0 = \frac{1}{m} \cdot \sum_{l=1}^{m} (-1)^{l-1} x_l \cdot dx_1 \wedge ... \wedge dx_{l-1} \wedge dx_{l+1} \wedge ... \wedge dx_m$. Additionally we introduce the volume form $\Omega_1 := \bar{\varphi}_\varepsilon^* \Omega_0$.
At first we note that $\bar{\varphi}_\varepsilon$ preserves the $m-1$-form ω_0 on U_ε: Clearly this holds on $\mathbb{R}^m \setminus \Delta\left(\frac{\varepsilon}{2}\right)$, where $\bar{\varphi}_\varepsilon$ is the identity. On $\Delta(\varepsilon)$ we have

$$\begin{aligned}
&\bar{\varphi}_\varepsilon^* \omega_0 (x_1, ..., x_m) = \omega_0 (x_1, ..., x_{i-1}, -x_j, x_{i+1}, ..., x_{j-1}, x_i, x_{j+1}, ..., x_m) \\
=& -\omega_0 (x_1, ..., x_{i-1}, x_j, x_{i+1}, ..., x_{j-1}, x_i, x_{j+1}, ..., x_m) = -(-1)^{2(j-i)-1} \omega_0 (x_1, ..., x_m) \\
=& \omega_0 (x_1, ..., x_m).
\end{aligned}$$

Furthermore, we introduce $\Omega' := \Omega_1 - \Omega_0$. Since the exterior derivative commutes with the pull-back, it holds $\Omega' = d\left(\bar{\varphi}_\varepsilon^* \omega_0 - \omega_0\right)$. In addition we consider the volume form $\Omega_t := \Omega_0 + t \cdot \Omega'$ and note that Ω_t is non-degenerate. Thus, we get a uniquely defined vectorfield X_t such that $\Omega_t (X_t, \cdot) = \left(\omega_0 - \bar{\varphi}_\varepsilon^* \omega_0\right)(\cdot)$. Since Δ is a compact manifold, the non-autonomous differential equation $\frac{d}{dt} u(t) = X_t(u(t))$ with initial values in Δ has a solution defined on $\mathbb{R}$. Hence, we get a one-parameter family of diffeomorphisms $\{\nu_t\}_{t \in [0,1]}$ on Δ satisfying $\dot{\nu}_t = X_t(\nu_t)$, $\nu_0 = id$.
Referring to [Ber98], Lemma 2.2., it holds

$$\frac{d}{dt} \nu_t^* \Omega_t = d\left(\nu_t^* \left(i(X_t) \Omega_t\right)\right) + \nu_t^* \left(\frac{d}{dt} \Omega_t + i(X_t) d\Omega_t\right).$$

Because of $d\left(\nu_t^* \left(i(X_t)\Omega_t\right)\right) = \nu_t^* \left(d\left(i(X_t)\Omega_t\right)\right)$ and $d\Omega_t = d\left(d\omega_0 + t \cdot \left(d\left(\bar{\varphi}_\varepsilon^* \omega_0\right) - d\omega_0\right)\right) = 0$ we compute:

$$\begin{aligned}
\frac{d}{dt} \nu_t^* \Omega_t &= \nu_t^* \left(d\left(i(X_t)\Omega_t\right)\right) + \nu_t^* \left(\frac{d}{dt}\Omega_t\right) = \nu_t^* d\left(\Omega_t(X_t, \cdot)\right) + \nu_t^* \Omega' \\
&= \nu_t^* d\left(\omega_0 - \bar{\varphi}_\varepsilon^* \omega_0\right) + \nu_t^* \Omega' = \nu_t^* \left(\Omega_0 - \Omega_1\right) + \nu_t^* \left(\Omega_1 - \Omega_0\right) = 0.
\end{aligned}$$

Consequently $\nu_1^* \Omega_1 = \nu_0^* \Omega_0 = \Omega_0$ using $\nu_0 = id$ in the last step. As we have seen $\bar{\varphi}_\varepsilon^* \omega_0 = \omega_0$ on U_ε. Therefore, on U_ε it holds: $\Omega_t(X_t, \cdot) = 0$. Since Ω_t is non-degenerate, we conclude $X_t = 0$ on U_ε and hence $\nu_1 = \nu_0 = id$ on $U_\varepsilon \cap \Delta$. Now we can extend ν_1 smoothly to $\mathbb{R}^m$ as the identity.
Denote $\varphi_\varepsilon := \bar{\varphi}_\varepsilon \circ \nu_1$. Because we have $\nu_1 = id$ on U_ε, φ_ε coincides with $\bar{\varphi}_\varepsilon$ on U_ε as announced. Furthermore, we have

$$\varphi_\varepsilon^* \Omega_0 = \left(\bar{\varphi}_\varepsilon \circ \nu_1\right)^* \Omega_0 = \nu_1^* \left(\bar{\varphi}_\varepsilon^* \Omega_0\right) = \nu_1^* \Omega_1 = \Omega_0.$$

Using the transformation formula we compute for an arbitrary measurable set $A \subseteq \mathbb{R}^m$:

$$\mu\left(\varphi_\varepsilon(A)\right) = \int_{\varphi_\varepsilon(A)} \Omega_0 = \int_A \left|\det\left(D\varphi_\varepsilon\right)\right| \cdot \Omega_0.$$

We have $\det\left(D\nu_1\right) > 0$ (because $\nu_0 = id$ and all the maps ν_t are diffeomorphisms) as well as $\det\left(D\bar{\varphi}_\varepsilon\right) > 0$ and thus $\left|\det\left(D\varphi_\varepsilon\right)\right| = \det\left(D\varphi_\varepsilon\right)$. Since $\varphi_\varepsilon^* \Omega_0 = \left(\det\left(D\varphi_\varepsilon\right)\right) \cdot \Omega_0$ (compare with [HK95], proposition 5.1.3.) we finally conclude:

$$\mu\left(\varphi_\varepsilon(A)\right) = \int_A \left(\det\left(D\varphi_\varepsilon\right)\right) \cdot \Omega_0 = \int_A \varphi_\varepsilon^* \Omega_0 = \int_A \Omega_0 = \mu(A).$$

Consequently φ_{ε} is a measure-preserving diffeomorphism on $\mathbb{R}^m$ satisfying the aimed properties. $\square$

Furthermore, for $\lambda \in \mathbb{N}$ we define the maps $C_{\lambda}(x_1, x_2, ..., x_m) = (\lambda \cdot x_1, x_2, ..., x_m)$. Hereby, we define a smooth measure-preserving diffeomorphism

$$\tilde{\phi}^{(i)}_{\lambda,\varepsilon} : \left[0, \frac{1}{\lambda}\right] \times [0,1]^{m-1} \to \left[0, \frac{1}{\lambda}\right] \times [0,1]^{m-1}, \qquad \tilde{\phi}^{(i)}_{\lambda,\varepsilon} := C_{\lambda}^{-1} \circ \varphi_{\varepsilon,1,i} \circ C_{\lambda}.$$

Since $\tilde{\phi}^{(i)}_{\lambda,\varepsilon}$ coincides with the identity on a neigbourhood of the boundary, we can proceed it using the description $\phi^{(i)}_{\lambda,\varepsilon}\left(x_1 + \frac{1}{\lambda}, x_2, ..., x_m\right) = \left(\frac{1}{\lambda}, 0, ..., 0\right) + \tilde{\phi}^{(i)}_{\lambda,\varepsilon}(x_1, ..., x_m)$ to a diffeomorphism on $\mathbb{S}^1 \times [0,1]^{m-1}$.

4.2 The map $\psi_{k,q,\vec{a},\varepsilon}$

The map $\psi_{k,q,\vec{a},\varepsilon}$ is constructed on the basis of [Be13], Lemma 4.3.:
Let $k, q \in \mathbb{Z}$, $\varepsilon > 0$ and $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function that equals 0 for $x \leq -1$ and 1 for $x \geq 0$. Moreover, for every $0 \leq i \leq k-1$ we have $a(i) \in \mathbb{Z}$ with $0 \leq a(i) \leq q-1$. This set of parameters is denoted by $\vec{a}$. With it we define the map $\tilde{\psi}_{k,q,\vec{a},\varepsilon} : [0,1] \to [0,1]$ by

$$\tilde{\psi}_{k,q,\vec{a},\varepsilon}(x) = \frac{a(0)}{q} + \frac{a(1)-a(0)}{q} \cdot \rho\left(\frac{x}{\varepsilon} - \frac{1}{k \cdot \varepsilon}\right) + ... + \frac{a(k-1)-a(k-2)}{q} \cdot \rho\left(\frac{x}{\varepsilon} - \frac{k-1}{k \cdot \varepsilon}\right).$$

Note that for every $0 \leq i \leq k-1$ we have $\tilde{\psi}_{k,q,\vec{a},\varepsilon}|_{\left[\frac{i}{k}, \frac{i+1}{k} - \varepsilon\right]} = \frac{a(i)}{q}$ and we can estimate $\left\| D^l \tilde{\psi}_{k,q,\vec{a},\varepsilon} \right\|_0 \leq \frac{1}{\varepsilon^l} \cdot \left\| D^l \rho \right\|_0$. In our constructions we will have $a(0) = 0$ as well as $a(k-1) = 0$. Besides this map $\tilde{\psi}_{k,q,\vec{a},\varepsilon}$ we use a smooth map $\sigma_{\varepsilon} : \mathbb{R} \to [0,1]$ satisfying $\sigma_{\varepsilon}(x) = 0$ for $x \leq \frac{\varepsilon}{2}$, $\sigma_{\varepsilon}(x) = 1$ for $\varepsilon \leq x \leq 1-\varepsilon$ and $\sigma_{\varepsilon}(x) = 0$ for $x \geq 1 - \frac{\varepsilon}{2}$. Then we define the measure-preserving diffeomorphism $\psi_{k,q,\vec{a},\varepsilon} : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ by

$$\psi_{k,q,\vec{a},\varepsilon}(\theta, r_1, ..., r_{m-1}) = \left(\theta - \tilde{\psi}_{k,q,\vec{a},\varepsilon}(r_1) \cdot \sigma_{\varepsilon}(r_2) \cdot ... \sigma_{\varepsilon}(r_{m-1}), r_1, ..., r_{m-1}\right).$$

We emphasize that the maps σ_{ε} are introduced to guarantee that $\psi_{k,q,\vec{a},\varepsilon}$ coincides with the identity in a neigbourhood of the boundary. Moreover, we observe $\psi_{k,q,\vec{a},\varepsilon} \circ R_{\frac{1}{q}} = R_{\frac{1}{q}} \circ \psi_{k,q,\vec{a},\varepsilon}$ and $|||\psi_{k,q,\vec{a},\varepsilon}|||_l \leq C(\varepsilon, l)$.

4.3 The conjugation map ϕ_n

Using the maps from the preceding subsections we construct the conjugation map ϕ_n:

$$\phi_n = \phi^{(m)}_{q_n, \frac{1}{4q_{n-1}}} \circ \phi^{(m-1)}_{q_n^2, \frac{1}{4q_{n-1}}} \circ ... \circ \phi^{(2)}_{q_n^{m-1}, \frac{1}{4q_{n-1}}} \circ \psi_{q_n, q_n, \vec{a}_n, \frac{1}{4q_{n-1}}} \circ \left(\phi^{(2)}_{q_n, \frac{1}{4q_{n-1}}}\right)^{-1} \circ \phi^{(2)}_{q_n^2, \frac{1}{4q_{n-1}}},$$

where the parameters $\vec{a}_n = (a_n(0), ..., a_n(q_n - 1))$ with $0 \leq a_n(i) \leq q_n - 1$ will be determined later (see the end of section 5).

Definition 4.2. By "good domain" G_n of ϕ_n and ϕ_n^{-1} we denote the domain, where all the occuring maps $\varphi_{\varepsilon,1,j}$ of $\phi^{(j)}_{\lambda_j,\varepsilon}$ act as the particular rotation and the map $\psi_{k,q,\vec{a},\varepsilon}$ acts as one of the translations by $\frac{a(i)}{q}$.

In order to guarantee that a strip of our partition element is contained in the "good domain" completely we choose $\tilde{\varepsilon}_n := \frac{1}{2q_{n-1}}$ slightly larger than $\varepsilon_n = \frac{1}{4q_{n-1}}$. Hereby, we observe

Lemma 4.3. *On an interval* $\left[\frac{l}{q_n}, \frac{l+1}{q_n}\right]$ *on the* θ*-axis the corresponding part of the "good domain" has length at least* $\left(1-2\cdot\tilde{\varepsilon}_n\right)^m \cdot \frac{1}{q_n} \geq \frac{1-m\cdot 2\cdot\tilde{\varepsilon}_n}{q_n}$.

Proof. By construction of the map φ_ε in Lemma 4.1 the "good domain" of the map $\phi^{(m)}_{q_n,\frac{1}{4q_{n-1}}}$ is $\left[\frac{l+\tilde{\varepsilon}_n}{q_n}, \frac{l+1-\tilde{\varepsilon}_n}{q_n}\right] \times \left[\tilde{\varepsilon}_n, 1-\tilde{\varepsilon}_n\right]^{m-1}$. Then there are $\frac{(1-2\tilde{\varepsilon}_n)\frac{1}{q_n}}{\frac{1}{q_n^2}} = (1-2\tilde{\varepsilon}_n)\cdot\frac{1}{q_n}$ domains of $\phi^{(m-1)}_{q_n^2,\frac{1}{4q_{n-1}}}$ contained in it, each one having a "good domain" with corresponding length of $(1-2\tilde{\varepsilon}_n)\frac{1}{q_n^2}$ on the θ-axis. Continuing in this way gives the statement. □

4.4 Sketch of the construction

As announced we would like to sketch the construction and its combinatorics.
In the subsequent section we will see that the constructed conjugation maps ϕ_n and $H_n = H_{n-1}\circ\phi_n$ allow explicit norm estimates (independent of the choice of the parameters $\vec{a}_n$ of our map $\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}$). Exploiting the fact that α is a Liouville number, this will enable us to prove convergence of the sequence $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ in $\mathcal{A}_\alpha$ in Lemma 5.8 and 5.9. This proof will give us a sequence $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n} \in \mathbb{Q}$ with $\tilde{p}_n, \tilde{q}_n$ relatively prime. In order to avoid technicalities in the positioning of the tower elements we want q_n of $\alpha_{n-1} = \frac{p_n}{q_n}$ to be a multiple of $2\cdot q_{n-2}\cdot q_{n-1}^m$. Hence, we put $q_n = 2\cdot q_{n-2}\cdot q_{n-1}^m\cdot\tilde{q}_n$, $p_n = 2\cdot q_{n-2}\cdot q_{n-1}^m\cdot\tilde{p}_n$ and note that $\alpha_{n-1} = \tilde{\alpha}_{n-1}$. Moreover, $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}}$ can be written in the form $\alpha_{n+1} = \alpha_n \pm \frac{\gamma_n}{q_{n+1}}$ with $\gamma_n \in \mathbb{N}$. Hereby, we define $m_n := \left\lfloor \frac{q_{n+1}}{\gamma_n\cdot q_n^2}\right\rfloor$. Additionally, we introduce two sets $\tilde{c}^{(n)}_{0,i}$, $i=1,2$, in section 6.1. With the aid of these we define the bases $c^{(n)}_{0,i} = H_{n-1}\left(\tilde{c}^{(n)}_{0,i}\right)$ as well as the heights m_n and m_n+1 of the towers which will be used to prove that $f = \lim_{n\to\infty} f_n$ admits a good linked approximation of type $(h,h+1)$. In particular, each set $\tilde{c}^{(n)}_{0,i}$ is contained in a cube of edge length $\frac{1}{\tilde{q}_n}$ such that the diameter of $c^{(n)}_{0,i}$ is small. Moreover, $\tilde{c}^{(n)}_{0,i}$ is built as a union of sets due to different reasons which we will explain in the following. First of all, the union over s_i and t as well as the corresponding "gaps" are used in order to position $\tilde{c}^{(n)}_{0,i}$ in the "good domain" of the map ϕ_{n+1}^{-1}. This will be very useful in the calculations of the speed of approximation, especially in the proof of Lemma 6.7.

In order to see that f admits an approximation of type $(h,h+1)$ the so-called "j-stripes", i.e. sets obtained by the union over s_i and t, are very important. The parameters λ_k in $\left(\phi^{(3)}_{\lambda_3,\varepsilon}\right)^{-1}\circ\cdots\circ\left(\phi^{(m)}_{\lambda_m,\varepsilon}\right)^{-1}$ are chosen in such a way that each "j-stripe" of $\tilde{c}^{(n)}_{0,i}$ is mapped to a set of almost full length in the $r_2,\ldots,r_{m-1}$-coordinates under this composition. Subsequently, $\left(\phi^{(2)}_{q_n^{m-1},\frac{1}{4q_{n-1}}}\right)^{-1}$ maps each of these images to a set with r_1-length about $\frac{1}{q_n}$ contained in one domain of definition of $\psi^{-1}_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}$ in $\left[\frac{1}{2},1\right]$. The application of this map $\psi^{-1}_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}$ yields a j-dependent translation by $\frac{a_n(j)}{q_n}$ in the θ-coordinate. Finally, $\left(\phi^{(2)}_{q_n^2,\frac{1}{4q_{n-1}}}\right)^{-1}\circ\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}$ maps each of these sets to a set of almost full length in the r_1-coordinate as well. Altogether, under ϕ_n^{-1} each "j-stripe" is mapped on a set of almost full length in the $r_2,\ldots,r_{m-1}$-coordinates and θ-width of approximately $\frac{\gamma_n\tilde{q}_n}{q_{n+1}}$ (which motivates the name "stripe"). In the schematic visualisation of the

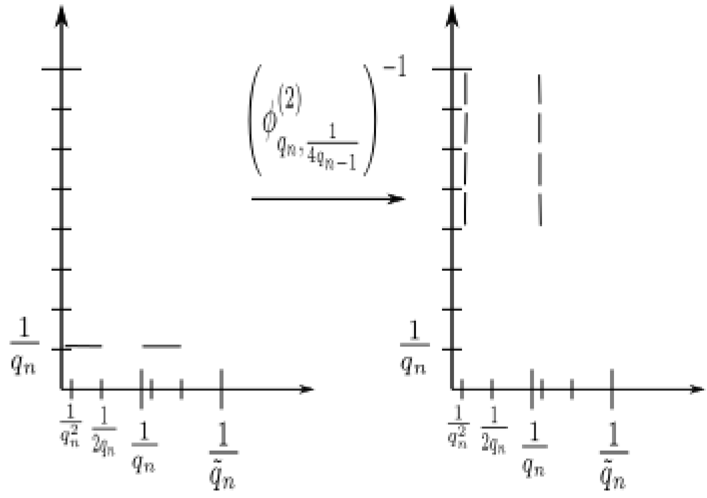

Figure 1: Qualitative shape of the set $\tilde{c}^{(n)}_{0,1}$ and the action of $\left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1}$ on it in case of dimension $m=2$, $\tilde{q}_n = 4$ and $q_n = 8$.

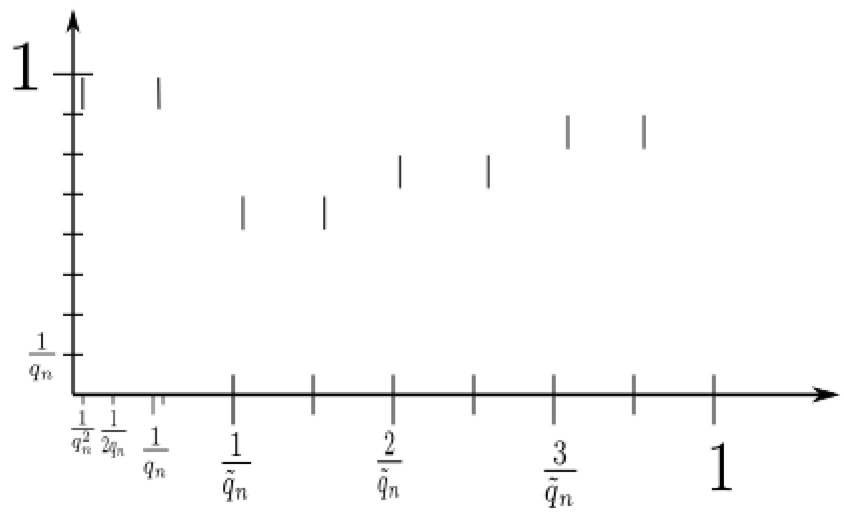

Figure 2: The action of $\psi^{-1}_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}$ on $\left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1}\left(\tilde{c}^{(n)}_{0,1}\right)$ in case of $m_n \cdot \tilde{\alpha}_n = \frac{3}{4}$.

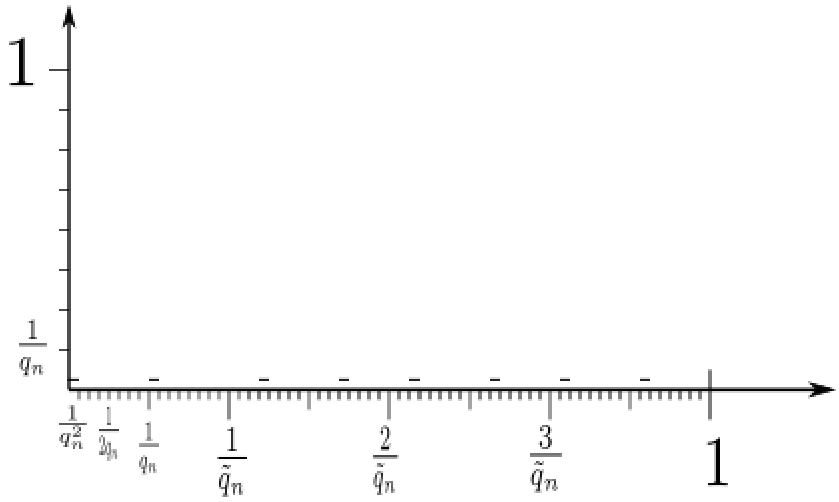

Figure 3: Visualization of the action of $\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}$ on $\psi^{-1}_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}} \circ \left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1}\left(\tilde{c}^{(n)}_{0,1}\right)$.

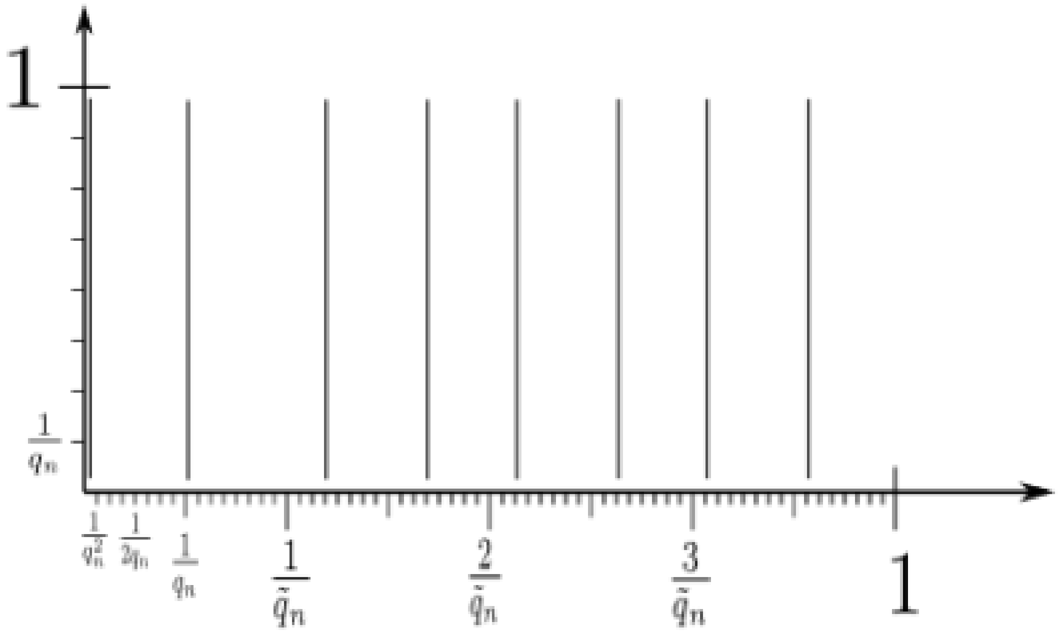

Figure 4: The action of $\left(\phi^{(2)}_{q_n^2,\frac{1}{4q_{n-1}}}\right)^{-1}$ on $\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}} \circ \psi^{-1}_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}} \circ \left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1}\left(\tilde{c}^{(n)}_{0,1}\right)$.

action of ϕ_n^{-1} in the figures the "j-stripes" are drawn without the "gaps". As seen this action of the conjugation map ϕ_n^{-1} depends on the parameters $a_n(j)$ in the construction of the map $\psi_{k,q,\vec{a},\varepsilon}$. In this connection we note that the number m_n is defined such that $m_n \cdot |\alpha_{n+1}-\alpha_n|$ is approximately $\frac{1}{q_n^2}$. Hence, under $R^{m_n}_{\alpha_{n+1}} \approx R_{\frac{m_n\cdot\tilde{p}_n}{\tilde{q}_n}\pm\frac{1}{q_n^2}}$ a "j-stripe" is mapped into another $\frac{1}{\tilde{q}_n}$-sector and is shifted by approximately $\frac{1}{q_n^2}$. In order to get recurrence in the base we need another stripe to be positioned there, which will be fulfilled by our choice of the parameters $a_n(j)$ at the end of subsection 5.2: These parameters will be determined in such a way that a great portion of $R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{c}^{(n)}_{0,1}\right)$ (resp. $R^{m_n+1}_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{c}^{(n)}_{0,2}\right)$) is mapped back into $\phi_n^{-1}\left(\tilde{c}^{(n)}_{0,1}\right)$ (resp. $\phi_n^{-1}\left(\tilde{c}^{(n)}_{0,2}\right)$) for the prescribed height m_n (resp. m_n+1) of the particular tower.
Since both towers have to be substantial, the measure of each base element has to be about $\frac{1}{2m_n}$. We ensure this by the union over k and l. Additionally, we require the union over k because p_n and q_n are not relatively prime as mentioned before and so our rotation R_{α_n} is an effective rotation by $\frac{\tilde{p}_n}{\tilde{q}_n}$.
The proof that f admits a good cyclic approximation is considerably easier. Since $f_n^{\tilde{q}_n} = \mathrm{id}$ we can use a subset with measure about $\frac{1}{\tilde{q}_n}$ of our tower base in the $(h,h+1)$-approximation.

5 Convergence of $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^{\infty}(M)$

5.1 Properties of the conjugation maps ϕ_n

In order to estimate the norm of the conjugating maps $\phi^{(m)}_{q_n,\frac{1}{4q_{n-1}}} \circ \phi^{(m-1)}_{q_n^2,\frac{1}{4q_{n-1}}} \circ \dots \circ \phi^{(2)}_{q_n^{m-1},\frac{1}{4q_{n-1}}}$ as well as $\left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1} \circ \phi^{(2)}_{q_n^2,\frac{1}{4q_{n-1}}}$ we will need the next technical result which is an application of the chain rule:

Lemma 5.1. *Let $\phi := \tilde{\phi}^{(m)}_{\lambda_m,\varepsilon} \circ \dots \circ \tilde{\phi}^{(2)}_{\lambda_2,\varepsilon}$, $j \in \{1,\dots,m\}$ and $k \in \mathbb{N}$. For any multiindex $\vec{a}$ with $|\vec{a}| = k$ the partial derivative $D_{\vec{a}}[\phi]_j$ consists of a sum of products of at most $(m-1)\cdot k$ terms of the following form*

$$D_{\vec{b}}\left(\left[\tilde{\phi}^{(i)}_{\lambda_i,\varepsilon}\right]_l\right) \circ \tilde{\phi}^{(i-1)}_{\lambda_{i-1},\varepsilon} \circ \dots \circ \tilde{\phi}^{(2)}_{\lambda_2,\varepsilon},$$

where $l \in \{1,\dots,m\}$, $i \in \{2,\dots,m\}$ and $\vec{b}$ is a multiindex with $\left|\vec{b}\right| \le k$.

In the same way we can show a similar statement holding for the inverses:

Lemma 5.2. *Let $\psi := \left(\tilde{\phi}^{(2)}_{\lambda_2,\varepsilon}\right)^{-1} \circ \dots \circ \left(\tilde{\phi}^{(m)}_{\lambda_m,\varepsilon}\right)^{-1}$, $j \in \{1,\dots,m\}$ and $k \in \mathbb{N}$. For any multiindex $\vec{a}$ with $|\vec{a}| = k$ the partial derivative $D_{\vec{a}}[\psi]_j$ consists of a sum of products of at most $(m-1)\cdot k$ terms of the following form*

$$D_{\vec{b}}\left(\left[\left(\tilde{\phi}^{(i)}_{\lambda_i,\varepsilon}\right)^{-1}\right]_l\right) \circ \left(\tilde{\phi}^{(i+1)}_{\lambda_{i+1},\varepsilon}\right)^{-1} \circ \dots \circ \left(\tilde{\phi}^{(m)}_{\lambda_m,\varepsilon}\right)^{-1},$$

where $l \in \{1,\dots,m\}$, $i \in \{2,\dots,m\}$ and $\vec{b}$ is a multiindex with $\left|\vec{b}\right| \le k$.

With these we can prove the following norm estimates:

Lemma 5.3. *For every $k \in \mathbb{N}$ we have*

$$|||\phi^{(m)}_{q_n,\frac{1}{4q_{n-1}}} \circ ... \circ \phi^{(2)}_{q_n^{m-1},\frac{1}{4q_{n-1}}}|||_k \leq C_1(m,k,q_{n-1}) \cdot q_n^{(m-1)^2\cdot k},$$

$$|||\left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1} \circ \phi^{(2)}_{q_n^2,\frac{1}{4q_{n-1}}}|||_k \leq C_2(m,k,q_{n-1}) \cdot q_n^{4\cdot k},$$

where $C_1(m,k,q_{n-1})$ as well as $C_2(m,k,q_{n-1})$ are constants depending on m, k and q_{n-1}, but are independent of q_n.

Proof. First of all, we consider the map $\tilde{\phi}^{(i)}_{\lambda,\varepsilon} = C_{\lambda}^{-1} \circ \varphi_{\varepsilon} \circ C_{\lambda}$ introduced in subsection 4.1:

$$\tilde{\phi}^{(i)}_{\lambda,\varepsilon}(x_1,...,x_m) = \left(\frac{1}{\lambda}[\varphi_{\varepsilon}]_1(\lambda x_1, x_2,...,x_m), [\varphi_{\varepsilon}]_2(\lambda x_1, x_2,...,x_m),...,[\varphi_{\varepsilon}]_m(\lambda x_1, x_2,...,x_m)\right).$$

Let $k \in \mathbb{N}$. We compute for a multiindex $\vec{a}$ with $0 \leq |\vec{a}| \leq k$: $\left\|D_{\vec{a}}\left[\tilde{\phi}^{(i)}_{\lambda,\varepsilon}\right]_1\right\|_0 \leq \lambda^{k-1} \cdot |||\varphi_{\varepsilon}|||_k$ and for $r \in \{2,...,m\}$: $\left\|D_{\vec{a}}\left[\tilde{\phi}^{(i)}_{\lambda,\varepsilon}\right]_r\right\|_0 \leq \lambda^k \cdot |||\varphi_{\varepsilon}|||_k$. Since $\left(\tilde{\phi}^{(i)}_{\lambda,\varepsilon}\right)^{-1}$ is of the same form, we have $|||\tilde{\phi}^{(i)}_{\lambda,\varepsilon}|||_k \leq C \cdot \lambda^k$.

In the next step we consider $\phi := \tilde{\phi}^{(m)}_{\lambda_m,\varepsilon} \circ ... \circ \tilde{\phi}^{(2)}_{\lambda_2,\varepsilon}$. Let $\lambda_{max} := \max\{\lambda_2,...,\lambda_m\}$. Inductively we will show $|||\phi|||_k \leq \tilde{C}(m,k,\varepsilon) \cdot \lambda_{max}^{(m-1)\cdot k}$ for every $k \in \mathbb{N}$, where $\tilde{C}(m,k,\varepsilon)$ is a constant depending on m, k and ε.
Start: $k = 1$
Let $l \in \{1,...,m\}$ be arbitrary. By Lemma 5.1 a partial derivative of $[\phi]_l$ of first order consists of a sum of products of at most $m-1$ first order partial derivatives of functions $\tilde{\phi}^{(j)}_{\lambda_j,\varepsilon}$. Then we obtain using $|||\tilde{\phi}^{(j)}_{\lambda_j,\varepsilon}|||_1 \leq C \cdot \lambda_{max}$ the estimate $\|D_i[\phi]_l\|_0 \leq C_1(m,\varepsilon) \cdot \lambda_{max}^{m-1}$ for every $i \in \{1,...,m\}$, where $C_1(m,\varepsilon)$ is a constant independent of λ.
With the aid of Lemma 5.2 we obtain the same statement for $\phi^{-1} = \left(\tilde{\phi}^{(2)}_{\lambda_2,\varepsilon}\right)^{-1} \circ ... \circ \left(\tilde{\phi}^{(m)}_{\lambda_m,\varepsilon}\right)^{-1}$.
Hence, we conclude: $|||\phi|||_1 \leq \tilde{C}(m,1,\varepsilon) \cdot \lambda_{max}^{m-1}$.
Assumption: The claim is true for $k \in \mathbb{N}$.
Induction step $k \to k+1$:
In the proof of Lemma 5.1 one observes that at the transition $k \to k+1$ in the product of at most $(m-1)\cdot k$ terms of the form $D_{\vec{b}}\left(\left[\tilde{\phi}^{(i)}_{\lambda_i,\varepsilon}\right]_l\right) \circ \tilde{\phi}^{(i-1)}_{\lambda_{i-1},\varepsilon} \circ ... \circ \tilde{\phi}^{(2)}_{\lambda_2,\varepsilon}$ one is replaced by a product of a term $\left(D_j D_{\vec{b}}\left[\tilde{\phi}^{(i)}_{\lambda_i,\varepsilon}\right]_l\right) \circ \tilde{\phi}^{(i-1)}_{\lambda_{i-1},\varepsilon} \circ ... \circ \tilde{\phi}^{(2)}_{\lambda_2,\varepsilon}$ with $j \in \{1,...,m\}$ and at most $m-2$ partial derivatives of first order. Because of $|||\tilde{\phi}^{(i)}_{\lambda_i,\varepsilon}|||_{k+1} \leq C \cdot \lambda_{max}^{k+1}$ and $|||\tilde{\phi}^{(j)}_{\lambda_j,\varepsilon}|||_1 \leq C \cdot \lambda_{max}$ the λ_{max}-exponent increases by at most $1 + (m-2)\cdot 1 = m-1$.
In the same spirit one uses the proof of Lemma 5.2 to show that also in case of ϕ^{-1} the λ_{max}-exponent increases by at most $m-1$.
Using the assumption we conclude

$$|||\phi|||_{k+1} \leq \hat{C}(m,k+1,\varepsilon) \cdot \lambda_{max}^{k\cdot(m-1)+m-1} = \hat{C}(m,k+1,\varepsilon) \cdot \lambda_{max}^{(k+1)\cdot(m-1)}.$$

So the proof by induction is completed.
In the setting of our explicit construction of the map $\phi^{(m)}_{q_n,\frac{1}{4q_{n-1}}} \circ ... \circ \phi^{(2)}_{q_n^{m-1},\frac{1}{4q_{n-1}}}$ we have $\varepsilon = \frac{1}{4q_{n-1}}$ and $\lambda_{max} = q_n^{m-1}$. Thus:

$$|||\phi^{(m)}_{q_n,\frac{1}{4q_{n-1}}} \circ ... \circ \phi^{(2)}_{q_n^{m-1},\frac{1}{4q_{n-1}}}|||_k \leq C(m,k,q_{n-1}) \cdot \left(q_n^{m-1}\right)^{(m-1)\cdot k} = C(m,k,q_{n-1}) \cdot q_n^{(m-1)^2\cdot k},$$

where $C(m,k,q_{n-1})$ is a constant independent of q_n.
In the same spirit we obtain the estimate on $\left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1}\circ\phi^{(2)}_{q_n^2,\frac{1}{4q_{n-1}}}$. □

Remark 5.4. In the proof of the following Lemmas we will use the formula of Faà di Bruno in several variables. It can be found in [CS96] for example.
For this purpose, we introduce an ordering on $\mathbb{N}_0^d$: For multiindices $\vec{\mu}=(\mu_1,...,\mu_d)$ and $\vec{\nu}=(\nu_1,...,\nu_d)$ in $\mathbb{N}_0^d$ we will write $\vec{\mu}\prec\vec{\nu}$, if one of the following properties is satisfied:

1. $|\vec{\mu}|<|\vec{\nu}|$, where $|\vec{\mu}|=\sum_{i=1}^{d}\mu_i$.
2. $|\vec{\mu}|=|\vec{\nu}|$ and $\mu_1<\nu_1$.
3. $|\vec{\mu}|=|\vec{\nu}|$, $\mu_i=\nu_i$ for $1\leq i\leq k$ and $\mu_{k+1}<\nu_{k+1}$ for a $1\leq k<d$.

Additionally we will use these notations:

- For $\vec{\nu}=(\nu_1,...,\nu_d)\in\mathbb{N}_0^d$:
$$\vec{\nu}!=\prod_{i=1}^{d}\nu_i!$$
- For $\vec{\nu}=(\nu_1,...,\nu_d)\in\mathbb{N}_0^d$ and $\vec{z}=(z_1,...,z_d)\in\mathbb{R}^d$:
$$\vec{z}^{\,\vec{\nu}}=\prod_{i=1}^{d}z_i^{\nu_i}$$

Then we get for the composition $h(x_1,...,x_d):=f\left(g^{(1)}(x_1,...,x_d),...,g^{(m)}(x_1,...,x_d)\right)$ with sufficiently differentiable functions $f:\mathbb{R}^m\to\mathbb{R}$, $g^{(i)}:\mathbb{R}^d\to\mathbb{R}$ and a multiindex $\vec{\nu}\in\mathbb{N}_0^d$ with $|\vec{\nu}|=n$:

$$D_{\vec{\nu}}h=\sum_{\vec{\lambda}\in\mathbb{N}_0^m\text{ with }1\leq|\vec{\lambda}|\leq n}D_{\vec{\lambda}}f\cdot\sum_{s=1}^{n}\sum_{p_s(\vec{\nu},\vec{\lambda})}\vec{\nu}!\cdot\prod_{j=1}^{s}\frac{\left[D_{\vec{l}_j}\vec{g}\right]^{\vec{k}_j}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}.$$

Here $\left[D_{\vec{l}_j}\vec{g}\right]$ denotes $\left(D_{\vec{l}_j}g^{(1)},...,D_{\vec{l}_j}g^{(m)}\right)$ and

$p_s\left(\vec{\nu},\vec{\lambda}\right):=$
$$\left\{\left(\vec{k}_1,...,\vec{k}_s,\vec{l}_1,...,\vec{l}_s\right):\vec{k}_i\in\mathbb{N}_0^m,\left|\vec{k}_i\right|>0,\vec{l}_i\in\mathbb{N}_0^d,0\prec\vec{l}_1\prec...\prec\vec{l}_s,\sum_{i=1}^{s}\vec{k}_i=\vec{\lambda}\text{ and }\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_i=\vec{\nu}\right\}.$$

With the aid of these technical results we can prove an estimate on the norms of the map ϕ_n:

Lemma 5.5. *For every $k\in\mathbb{N}$ it holds*

$$|||\phi_n|||_k\leq C\cdot q_n^{\left((m-1)^2+4\right)\cdot k},$$

where C is a constant depending on m, k, n and q_{n-1}, but is independent of q_n.

Proof. First of all, we consider the map $\hat{\phi}^{(1)} := \psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}} \circ \bar{\phi}^{(1)}$, at which we use the notation $\bar{\phi}^{(1)} := \left(\phi^{(2)}_{q_n,\frac{1}{4q_{n-1}}}\right)^{-1} \circ \phi^{(2)}_{q_n^2,\frac{1}{4q_{n-1}}}$. Let $k \in \mathbb{N}$. According to subsection 4.2 we have $|||\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}|||_k \leq \tilde{C}(m,k,q_{n-1})$, where $\tilde{C}(m,k,q_{n-1})$ is a constant independent of q_n.
Let $r \in \{1,...,m\}$ and $\vec{\nu}$ be any multiindex with $|\vec{\nu}| = k$. With the aid of the formula of Faà di Bruno mentioned in remark 5.4 we compute:

$$\left\|D_{\vec{\nu}}\left[\hat{\phi}^{(1)}\right]_r\right\|_0 = \left\|D_{\vec{\nu}}\left[\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}} \circ \bar{\phi}^{(1)}\right]_r\right\|_0$$

$$= \left\| \sum_{\vec{\lambda}\in\mathbb{N}_0^m, 1\leq|\vec{\lambda}|\leq k} D_{\vec{\lambda}}\left[\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}\right]_r \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\left[D_{\vec{l}_j}\bar{\phi}^{(1)}\right]^{\vec{k}_j}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}} \right\|_0$$

$$= \left\| \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} D_{\vec{\lambda}}\left[\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}\right]_r \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\prod_{t=1}^{m}\left(D_{\vec{l}_j}\left[\bar{\phi}^{(1)}\right]_t\right)^{\vec{k}_{j_t}}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}} \right\|_0$$

$$\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}\right]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\prod_{t=1}^{m}\left\|D_{\vec{l}_j}\left[\bar{\phi}^{(1)}\right]_t\right\|_0^{\vec{k}_{j_t}}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

$$\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}\right]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{|||\bar{\phi}^{(1)}|||_{|\vec{l}_j|}^{\sum_{t=1}^{m}\vec{k}_{j_t}}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

$$= \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\psi_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}\right]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{|||\bar{\phi}^{(1)}|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

As seen in Lemma 5.3 $|||\bar{\phi}^{(1)}|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq C \cdot q_n^{4\cdot|\vec{l}_j|\cdot|\vec{k}_j|}$. Hereby: $\prod_{j=1}^{s} |||\bar{\phi}^{(1)}|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C} \cdot q_n^{4\cdot\sum_{j=1}^{s}|\vec{l}_j|\cdot|\vec{k}_j|}$, where $\hat{C}$ is independent of q_n. By definition of the set $p_s\left(\vec{\nu},\vec{\lambda}\right)$ we have $\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_i = \vec{\nu}$. Hence:

$$k = |\vec{\nu}| = \left|\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_i\right| = \sum_{t=1}^{m}\left(\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_i\right)_t = \sum_{t=1}^{m}\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_{i_t} = \sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \left(\sum_{t=1}^{m}\vec{l}_{i_t}\right) = \sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \left|\vec{l}_i\right|$$

This shows $\prod_{j=1}^{s} |||\bar{\phi}^{(1)}|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C} \cdot q_n^{4\cdot k}$ and finally $\left\|D_{\vec{\nu}}\left[\hat{\phi}^{(1)}\right]_r\right\|_0 \leq C \cdot q_n^{4\cdot k}$. Analogously we compute $\left\|D_{\vec{\nu}}\left[\left(\bar{\phi}^{(1)}\right)^{-1} \circ \psi^{-1}_{q_n,q_n,\vec{a}_n,\frac{1}{4q_{n-1}}}\right]_r\right\|_0 \leq C \cdot |||\bar{\phi}^{(1)}|||_k \leq C \cdot q_n^{4k}$. Altogether, we obtain $|||\hat{\phi}^{(1)}|||_k \leq C \cdot q_n^{4k}$ with a constant C independent of q_n.
In the next step we denote $\bar{\phi}^{(2)} := \phi^{(m)}_{q_n,\frac{1}{4q_{n-1}}} \circ ... \circ \phi^{(2)}_{q_n^{m-1},\frac{1}{4q_{n-1}}}$ and consider $\phi_n = \bar{\phi}^{(2)} \circ \hat{\phi}^{(1)}$. By the same calculations as above we obtain for any multiindex $\vec{\nu}$ with $|\vec{\nu}| = k$:

$$\left\|D_{\vec{\nu}}\left[\bar{\phi}^{(2)} \circ \hat{\phi}^{(1)}\right]_r\right\|_0 \leq \tilde{C} \cdot |||\bar{\phi}^{(2)}|||_k \cdot q_n^{4\cdot k} \leq C \cdot q_n^{(m-1)^2\cdot k} \cdot q_n^{4\cdot k},$$

where we used Lemma 5.3 in the last step and the constants $C, \tilde{C}$ are independent of q_n. Analogously we show the same estimate on $\left(\hat{\phi}^{(1)}\right)^{-1} \circ \left(\bar{\phi}^{(2)}\right)^{-1}$.
Finally, we conclude:

$$|||\phi_n|||_k \leq C(m,k,q_{n-1}) \cdot q_n^{(m-1)^2\cdot k} \cdot q_n^{4\cdot k} = C(m,k,q_{n-1}) \cdot q_n^{\left((m-1)^2+4\right)\cdot k},$$

where $C(m,k,q_{n-1})$ is a constant independent of q_n. □

Again using the formula of Faà di Bruno we are able to prove an estimate on the norms of the map H_n:

Lemma 5.6. *For every $k \in \mathbb{N}$ we get:*

$$|||H_n|||_k \leq \check{C} \cdot q_n^{\left((m-1)^2+4\right)\cdot k},$$

where $\check{C}$ is a constant depending solely on m, k, q_{n-1} and H_{n-1}. Since H_{n-1} is independent of q_n in particular, the same is true for $\check{C}$.

Proof. Let $k \in \mathbb{N}$, $r \in \{1,...,m\}$ and $\vec{\nu} \in \mathbb{N}_0^m$ be a multiindex with $|\vec{\nu}| = k$. As above we estimate:

$$\begin{aligned}\|D_{\vec{\nu}}[H_n]_r\|_0 &= \|D_{\vec{\nu}}[H_{n-1}\circ\phi_n]_r\|_0 \\ &\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}[H_{n-1}]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{|||\phi_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}\end{aligned}$$

and compute using Lemma 5.5: $\prod_{j=1}^{s} |||\phi_n|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C} \cdot q_n^{\left((m-1)^2+4\right)\cdot k}$, where $\hat{C}$ is a constant independent of q_n. Since H_{n-1} was constructed independently of q_n, we conclude:

$$\|D_{\vec{\nu}}[H_n]_r\|_0 \leq \check{C} \cdot q_n^{\left((m-1)^2+4\right)\cdot k},$$

where $\check{C}$ is a constant independent of q_n.
In the same way we prove an analogous estimate on $\left\|D_{\vec{\nu}}\left[H_n^{-1}\right]_r\right\|_0$ and verify the claim. □

In particular, we see that this norm can be estimated by a power of q_n.

5.2 Proof of convergence

For the proof of the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in the Diff$^\infty(M)$-topology the next result, that can be found in [FSW07], Lemma 4, is very useful.

Lemma 5.7. *Let $k \in \mathbb{N}_0$ and h be a C^∞-diffeomorphism on M. Then we get for every $\alpha, \beta \in \mathbb{R}$:*

$$d_k\left(h\circ R_\alpha \circ h^{-1}, h\circ R_\beta\circ h^{-1}\right) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha-\beta|,$$

where the constant C_k depends solely on k and m. In particular $C_0 = 1$.

In the following Lemma (similar to [FS05], Lemma 5.7) we show that under some assumptions on the sequence $(\alpha_n)_{n\in\mathbb{N}}$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f \in \mathcal{A}_\alpha$ in the Diff$^\infty(M)$-topology. Afterwards, we will show that we can fulfil these conditions (see Lemma 5.9).

Lemma 5.8. *Let $\varepsilon > 0$ be arbitrary and $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^{\infty}\frac{1}{k_n} < \varepsilon$. Furthermore, we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha-\alpha_1| < \varepsilon \qquad \text{and} \qquad |\alpha-\alpha_n| \leq \frac{1}{2\cdot k_n\cdot C_{k_n}\cdot |||H_n|||_{k_n+1}^{k_n+1}} \text{ for every } n\in\mathbb{N},$$

where C_{k_n} are the constants from Lemma 5.7.

1. *Then the sequence of diffeomorphisms $f_n = H_n\circ R_{\alpha_{n+1}}\circ H_n^{-1}$ converges in the $\mathrm{Diff}^{\infty}(M)$-topology to a measure-preserving smooth diffeomorphism f, for which $d_{\infty}(f, R_{\alpha}) < 3\cdot\varepsilon$ holds.*
2. *Also the sequence of diffeomorphisms $\hat{f}_n = H_n\circ R_{\alpha}\circ H_n^{-1}\in\mathcal{A}_{\alpha}$ converges to f in the $\mathrm{Diff}^{\infty}(M)$-topology. Hence, $f\in\mathcal{A}_{\alpha}$.*

Proof. 1. According to our construction it holds $\phi_n\circ R_{\alpha_n} = R_{\alpha_n}\circ\phi_n$ and hence

$$\begin{aligned} f_{n-1} &= H_{n-1}\circ R_{\alpha_n}\circ H_{n-1}^{-1} = H_{n-1}\circ R_{\alpha_n}\circ\phi_n\circ\phi_n^{-1}\circ H_{n-1}^{-1} \\ &= H_{n-1}\circ\phi_n\circ R_{\alpha_n}\circ\phi_n^{-1}\circ H_{n-1}^{-1} = H_n\circ R_{\alpha_n}\circ H_n^{-1}. \end{aligned}$$

Applying Lemma 5.7 we obtain for every $k,n\in\mathbb{N}$:
(1)
$$d_k(f_n, f_{n-1}) = d_k\left(H_n\circ R_{\alpha_{n+1}}\circ H_n^{-1}, H_n\circ R_{\alpha_n}\circ H_n^{-1}\right) \leq C_k\cdot |||H_n|||_{k+1}^{k+1}\cdot|\alpha_{n+1}-\alpha_n|.$$

In equation B we will assume $|\alpha-\alpha_n| \stackrel{n\to\infty}{\longrightarrow} 0$ monotonically. Using the triangle inequality we obtain $|\alpha_{n+1}-\alpha_n| \leq |\alpha_{n+1}-\alpha| + |\alpha-\alpha_n| \leq 2\cdot|\alpha-\alpha_n|$ and therefore equation 1 becomes:

$$d_k(f_n, f_{n-1}) \leq C_k\cdot |||H_n|||_{k+1}^{k+1}\cdot 2\cdot|\alpha-\alpha_n|.$$

By the assumptions of this Lemma it follows for every $k\leq k_n$:

$$(2)\quad d_k(f_n, f_{n-1}) \leq d_{k_n}(f_n, f_{n-1}) \leq C_{k_n}\cdot |||H_n|||_{k_n+1}^{k_n+1}\cdot 2\cdot\frac{1}{2\cdot k_n\cdot C_{k_n}\cdot |||H_n|||_{k_n+1}^{k_n+1}} \leq \frac{1}{k_n}.$$

In the next step we show, that for arbitrary $k\in\mathbb{N}$ $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Diff}^k(M)$, i.e. $\lim_{n,m\to\infty} d_k(f_n, f_m) = 0$. For this purpose, we calculate:

$$(3)\qquad \lim_{n\to\infty} d_k(f_n, f_m) \leq \lim_{n\to\infty}\sum_{i=m+1}^{n} d_k(f_i, f_{i-1}) = \sum_{i=m+1}^{\infty} d_k(f_i, f_{i-1}).$$

We consider the limit process $m\to\infty$, i.e. we can assume $k\leq k_m$ and obtain from equations 2 and 3:

$$\lim_{n,m\to\infty} d_k(f_n, f_m) \leq \lim_{m\to\infty}\sum_{i=m+1}^{\infty}\frac{1}{k_i} = 0.$$

Since $\mathrm{Diff}^k(M)$ is complete, the sequence $(f_n)_{n\in\mathbb{N}}$ converges consequently in $\mathrm{Diff}^k(M)$ for every $k\in\mathbb{N}$. Thus, the sequence converges in $\mathrm{Diff}^{\infty}(M)$ by definition.

Furthermore, we estimate:

$$(4)\qquad d_{\infty}(R_{\alpha}, f) = d_{\infty}\left(R_{\alpha}, \lim_{n\to\infty} f_n\right) \leq d_{\infty}(R_{\alpha}, R_{\alpha_1}) + \sum_{n=1}^{\infty} d_{\infty}(f_n, f_{n-1}),$$

where we used the notation $f_0 = R_{\alpha_1}$.
By explicit calculations we obtain $d_k(R_\alpha, R_{\alpha_1}) = d_0(R_\alpha, R_{\alpha_1}) = |\alpha - \alpha_1|$ for every $k \in \mathbb{N}$, hence

$$d_\infty(R_\alpha, R_{\alpha_1}) = \sum_{k=1}^{\infty} \frac{|\alpha-\alpha_1|}{2^k \cdot (1 + d_k(R_\alpha, R_{\alpha_1}))} \leq |\alpha - \alpha_1| \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = |\alpha - \alpha_1|.$$

Additionally it holds:

$$\begin{aligned}\sum_{n=1}^{\infty} d_\infty(f_n, f_{n-1}) &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{d_k(f_n, f_{n-1})}{2^k \cdot (1 + d_k(f_n, f_{n-1}))}\\ &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} \frac{d_k(f_n, f_{n-1})}{2^k \cdot (1 + d_k(f_n, f_{n-1}))} + \sum_{k=k_n+1}^{\infty} \frac{d_k(f_n, f_{n-1})}{2^k \cdot (1 + d_k(f_n, f_{n-1}))}\right).\end{aligned}$$

As seen above $d_k(f_n, f_{n-1}) \leq \frac{1}{k_n}$ for every $k \leq k_n$. Hereby, it follows further:

$$\begin{aligned}\sum_{n=1}^{\infty} d_\infty(f_n, f_{n-1}) &\leq \sum_{n=1}^{\infty}\left(\frac{1}{k_n} \cdot \sum_{k=1}^{k_n} \frac{1}{2^k} + \sum_{k=k_n+1}^{\infty} \frac{d_k(f_n, f_{n-1})}{2^k \cdot (1 + d_k(f_n, f_{n-1}))}\right)\\ &\leq \sum_{n=1}^{\infty} \frac{1}{k_n} + \sum_{n=1}^{\infty}\sum_{k=k_n+1}^{\infty} \frac{1}{2^k}.\end{aligned}$$

Because of $\sum_{k=k_n+1}^{\infty} \frac{1}{2^k} = 2 - \sum_{k=0}^{k_n} \frac{1}{2^k} = \left(\frac{1}{2}\right)^{k_n} \leq \frac{1}{k_n}$ we conclude:

$$\sum_{n=1}^{\infty} d_\infty(f_n, f_{n-1}) \leq \sum_{n=1}^{\infty} \frac{1}{k_n} + \sum_{n=1}^{\infty} \frac{1}{k_n} < 2 \cdot \varepsilon.$$

Hence, using equation 4 we obtain the aimed estimate $d_\infty(f, R_\alpha) < 3 \cdot \varepsilon$.

2. We have to show: $\hat{f}_n \to f$ in $\mathrm{Diff}^\infty(M)$.
For it we compute with the aid of Lemma 5.7 for every $n \in \mathbb{N}$ and $k \leq k_n$:

$$\begin{aligned} d_k\left(f_n, \hat{f}_n\right) &\leq d_{k_n}\left(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_\alpha \circ H_n^{-1}\right)\\ &\leq C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1} \cdot |\alpha_{n+1} - \alpha| \leq C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1} \cdot |\alpha_n - \alpha|\\ &\leq C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1} \cdot \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}} = \frac{1}{2 \cdot k_n} \leq \frac{1}{k_n}.\end{aligned}$$

Fix some $k \in \mathbb{N}$.
Claim: $\forall \delta > 0 \;\; \exists N \;\; \forall n \geq N: \;\; d_k\left(f, \hat{f}_n\right) < \delta$, i.e. $\hat{f}_n \to f$ in $\mathrm{Diff}^k(M)$.
Proof: Let $\delta > 0$ be given. Since $f_n \to f$ in $\mathrm{Diff}^\infty(M)$ we have $f_n \to f$ in $\mathrm{Diff}^k(M)$ in particular. Hence, there is $n_1 \in \mathbb{N}$, such that $d_k(f, f_n) < \frac{\delta}{2}$ for every $n \geq n_1$. Because of $k_n \to \infty$ we conclude the existence of $n_2 \in \mathbb{N}$, such that $\frac{1}{k_n} < \frac{\delta}{2}$ for every $n \geq n_2$, as well as the existence of $n_3 \in \mathbb{N}$, such that $k_n \geq k$ for every $n \geq n_3$. Then we obtain for every $n \geq \max\{n_1, n_2, n_3\}$:

$$d_k\left(f, \hat{f}_n\right) \leq d_k(f, f_n) + d_k\left(f_n, \hat{f}_n\right) < \frac{\delta}{2} + d_{k_n}\left(f_n, \hat{f}_n\right) \leq \frac{\delta}{2} + \frac{1}{k_n} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, the claim is proven.

In the next step we show: $\lim_{n\to\infty} d_{\infty}\left(\hat{f}_n, f\right) = 0$. For this purpose, we examine:

$$\begin{aligned} d_{\infty}\left(f_n, \hat{f}_n\right) &= \sum_{k=1}^{k_n} \frac{d_k\left(f_n, \hat{f}_n\right)}{2^k \cdot \left(1 + d_k\left(f_n, \hat{f}_n\right)\right)} + \sum_{k=k_n+1}^{\infty} \frac{d_k\left(f_n, \hat{f}_n\right)}{2^k \cdot \left(1 + d_k\left(f_n, \hat{f}_n\right)\right)} \\ &\leq \frac{1}{k_n} \cdot \sum_{k=1}^{k_n} \frac{1}{2^k} + \sum_{k=k_n+1}^{\infty} \frac{1}{2^k} \leq \frac{1}{k_n} + \left(\frac{1}{2}\right)^{k_n}. \end{aligned}$$

Consequently $\lim_{n\to\infty} d_{\infty}\left(f_n, \hat{f}_n\right) = 0$. Hereby, we compute:

$$\begin{aligned} \lim_{n\to\infty} d_{\infty}\left(f, \hat{f}_n\right) &= \lim_{n\to\infty} d_{\infty}\left(\lim_{m\to\infty} f_m, \hat{f}_n\right) = \lim_{n\to\infty}\lim_{m\to\infty} d_{\infty}\left(f_m, \hat{f}_n\right) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\sum_{i=n+1}^{m} d_{\infty}\left(f_i, f_{i-1}\right) + d_{\infty}\left(f_n, \hat{f}_n\right)\right) \\ &= \lim_{n\to\infty}\sum_{i=n+1}^{\infty} d_{\infty}\left(f_i, f_{i-1}\right) + \lim_{n\to\infty} d_{\infty}\left(f_n, \hat{f}_n\right) = 0. \end{aligned}$$

As asserted we obtain: $\lim_{n\to\infty} d_{\infty}\left(\hat{f}_n, f\right) = 0$.

□

As announced we show that we can satisfy the conditions from Lemma 5.8 in our constructions:

Lemma 5.9. *Let $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty}\frac{1}{k_n} < \infty$ and C_{k_n} be the constants from Lemma 5.7. For any Liouvillean number α there exists a sequence $\alpha_n = \frac{p_n}{q_n}$ of rational numbers with*

(A) $$2 \cdot q_{n-2} \cdot q_{n-1}^{m} \text{ divides } q_n$$

(B) $$\left(\alpha_n\right)_{n\in\mathbb{N}} \text{ converges to } \alpha \text{ monotonically}$$

such that our conjugation maps H_n constructed in section 4 fulfil the following conditions:

1. *For every $n \in \mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}}.$$
2. *For every $n \in \mathbb{N}$*
$$\left\|DH_{n-1}\right\|_0 \cdot \frac{16 \cdot q_{n-2} \cdot q_{n-1}^{m} \cdot \sqrt{m}}{q_n} < \frac{1}{n}.$$

In particular, we have $q_n > n \cdot 16 \cdot q_{n-2} \cdot q_{n-1}^{m}$.

Proof. The sequence of rational numbers $\alpha_n = \frac{p_n}{q_n}$ will be created out of $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$ with $\tilde{p}_n \leq p_n$ and $\tilde{q}_n \leq q_n$ relatively prime.

In Lemma 5.6 we saw $|||H_n|||_{k_n+1} \leq \breve{C}_n \cdot q_n^{\left((m-1)^2+4\right)\cdot(k_n+1)}$, where the constant $\breve{C}_n$ was independent of q_n. Thus, we can require $\tilde{q}_n \geq \breve{C}_n$ for every $n \in \mathbb{N}$. Hereby, we get the estimate $|||H_n|||_{k_n+1} \leq q_n^{\left((m-1)^2+5\right)\cdot(k_n+1)}$. Furthermore, we can demand

$$\tilde{q}_n > 8 \cdot n \cdot \breve{C}_{n-1} \cdot q_{n-1}^{(m-1)^2+5} \cdot \sqrt{m} \geq 8 \cdot n \cdot \|DH_{n-1}\|_0 \cdot \sqrt{m}. \tag{5}$$

Since α is a Liouvillean number, under the above restrictions we find a sequence of rational numbers $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$ with $\tilde{p}_n, \tilde{q}_n$ relatively prime satisfying:

$$|\alpha - \tilde{\alpha}_n| = \left|\alpha - \frac{\tilde{p}_n}{\tilde{q}_n}\right| < \frac{|\alpha - \alpha_{n-1}|}{2 \cdot k_n \cdot C_{k_n} \cdot \left(2q_{n-2}q_{n-1}^m\right)^{\left((m-1)^2+5\right)\cdot(k_n+1)^2} \cdot \tilde{q}_n^{\left((m-1)^2+5\right)\cdot(k_n+1)^2}}.$$

Put $q_n := 2 \cdot q_{n-2} \cdot q_{n-1}^m \cdot \tilde{q}_n$ and $p_n := 2 \cdot q_{n-2} \cdot q_{n-1}^m \cdot \tilde{p}_n$. Then we obtain:

$$|\alpha - \alpha_n| < \frac{|\alpha - \alpha_{n-1}|}{2 \cdot k_n \cdot C_{k_n} \cdot q_n^{\left((m-1)^2+5\right)\cdot(k_n+1)^2}}.$$

Thus, we have $|\alpha - \alpha_n| \to 0$ as $n \to \infty$ monotonically.

Because of $|||H_n|||_{k_n+1}^{k_n+1} \leq q_n^{\left((m-1)^2+5\right)\cdot(k_n+1)^2}$ this yields: $|\alpha - \alpha_n| < \frac{1}{2\cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}}$. Thus, the first property of this Lemma is fulfilled.

Equation 5 implies the second property, because

$$q_n = 2 \cdot q_{n-2} \cdot q_{n-1}^m \cdot \tilde{q}_n > 16 \cdot q_{n-2} \cdot q_{n-1}^m \cdot n \cdot \|DH_{n-1}\|_0 \cdot \sqrt{m}.$$

□

Remark 5.10. Lemma 5.9 shows that the conditions of Lemma 5.8 are satisfied. Therefore, our sequence of constructed diffeomorphisms f_n converges in the $\mathrm{Diff}^{\infty}(M)$-topology to a diffeomorphism $f \in \mathcal{A}_{\alpha}$.

The numbers $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}}$ can be written in the following form:

$$\alpha_{n+1} = \alpha_n \pm \frac{\gamma_n}{q_{n+1}} = \tilde{\alpha}_n \pm \frac{\gamma_n}{q_{n+1}},$$

where $\gamma_n \in \mathbb{N}$ and $\frac{\gamma_n}{q_{n+1}} = |\alpha_{n+1} - \alpha_n| \leq 2 \cdot |\alpha - \alpha_n|$. In particular, we have

$$\frac{\gamma_n}{q_{n+1}} \leq 2 \cdot |\alpha - \alpha_n| \leq 2 \cdot \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot q_n^{\left((m-1)^2+5\right)\cdot(k_n+1)^2}} \leq \frac{1}{k_n \cdot C_{k_n} \cdot q_n^{m+2}}. \tag{6}$$

Remark 5.11. We point out that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is independent of the choices of the parameters $a_n(i)$. So we can determine these parameters afterwards, which will depend on either "case +" (i.e. $\alpha_{n+1} = \alpha_n + \frac{\gamma_n}{q_{n+1}}$) or "case -" (i.e. $\alpha_{n+1} = \alpha_n - \frac{\gamma_n}{q_{n+1}}$).

In the proof of the $(h, h+1)$-property the number $m_n = \left\lfloor \frac{q_{n+1}}{\gamma_n \cdot q_n^2} \right\rfloor$ (see equation 7) will play a decisive role. This number m_n is known, when $\alpha_{n+1} = \tilde{\alpha}_n \pm \frac{\gamma_n}{q_{n+1}}$ is determined guaranteeing the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\text{Diff}^{\infty}(M)$ with the help of Lemma 5.9. Then we can compute $m_n \cdot \tilde{\alpha}_n$. Let $\frac{r}{\tilde{q}_n} := m_n \cdot \tilde{\alpha}_n \mod 1$. Hereby, in "case +" we define $\tilde{a}_n(l \cdot \tilde{q}_n + i)$ to be:

$$\begin{cases} 0 & \text{if } 0 \leq l \leq 2q_{n-2}q_{n-1}^{m-1} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \\ (\tilde{q}_n - i - 1) \cdot (r + \tilde{p}_n) \mod \tilde{q}_n & \text{if } 2q_{n-2}q_{n-1}^{m-1} \leq l \leq q_{n-2}q_{n-1}^{m} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \\ (\tilde{q}_n - i - 1) \cdot r \mod \tilde{q}_n & \text{if } q_{n-2}q_{n-1}^{m} \leq l \leq 2q_{n-2}q_{n-1}^{m} - 2q_{n-2}q_{n-1}^{m-1} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \\ 0 & \text{if } 2q_{n-2}q_{n-1}^{m} - 2q_{n-2}q_{n-1}^{m-1} \leq l \leq 2q_{n-2}q_{n-1}^{m} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \end{cases}$$

In "case -" the parameter $\tilde{a}_n(l \cdot \tilde{q}_n + i)$ is chosen as follows:

$$\begin{cases} 0 & \text{if } 0 \leq l \leq 2q_{n-2}q_{n-1}^{m-1} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \\ (i+1) \cdot (r + \tilde{p}_n) \mod \tilde{q}_n & \text{if } 2q_{n-2}q_{n-1}^{m-1} \leq l \leq q_{n-2}q_{n-1}^{m} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \\ (i+1) \cdot r \mod \tilde{q}_n & \text{if } q_{n-2}q_{n-1}^{m} \leq l \leq 2q_{n-2}q_{n-1}^{m} - 2q_{n-2}q_{n-1}^{m-1} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \\ 0 & \text{if } 2q_{n-2}q_{n-1}^{m} - 2q_{n-2}q_{n-1}^{m-1} \leq l \leq 2q_{n-2}q_{n-1}^{m} - 1 \text{ and } 0 \leq i \leq \tilde{q}_n - 1 \end{cases}$$

In both cases we define $a_n(l \cdot \tilde{q}_n + i) = 2 \cdot q_{n-2} \cdot q_{n-1}^m \cdot \tilde{a}_n(l \cdot \tilde{q}_n + i)$.

6 Proof of $(h, h+1)$-property

6.1 Towers for approximation of type $(h, h+1)$

Let $\tilde{c}_{0,1}^{(n)}$ be the set

- **in case of dimension $m = 2$:**

$$\bigcup \left[\frac{k}{q_n} + \frac{l}{2q_{n-2}q_{n-1}^2 \cdot q_n} + \frac{j}{q_n^2} + \frac{1}{q_{n-1} \cdot q_n^2}, \frac{k}{q_n} + \frac{l}{2q_{n-2}q_{n-1}^2 \cdot q_n} + \frac{j+1}{q_n^2} - \frac{1}{q_{n-1} \cdot q_n^2}\right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^2}{q_{n+1}} + \frac{t \cdot q_n^2}{q_{n+1}^2} + \frac{q_n^2}{q_n \cdot q_{n+1}^2}, \frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^2}{q_{n+1}} + \frac{(t+1) \cdot q_n^2}{q_{n+1}^2} - \frac{q_n^2}{q_n \cdot q_{n+1}^2}\right],$$

where the union is taken over all $j, k, l, s_2, t \in \mathbb{Z}$ satisfying $\left\lceil \frac{q_{n+1}}{q_n} \right\rceil \leq t \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_n} \right\rceil - 1$, $0 \leq j \leq \tilde{q}_n - 1 = \frac{q_n}{2q_{n-2}q_{n-1}^2} - 1$, $0 \leq k \leq 2q_{n-2}q_{n-1}^2 - 1$, $2q_{n-2}q_{n-1} \leq l \leq q_{n-2}q_{n-1}^2 - 2$ and $0 \leq s_2 \leq \gamma_n \tilde{q}_n - 1$.

- **in case of dimension $m = 3$:**

$$\bigcup \left[\frac{k}{q_n} + \frac{s_1}{q_n \cdot q_{n+1}} + \frac{t}{q_n \cdot q_{n+1}^2} + \frac{1}{q_n^2 \cdot q_{n+1}^2}, \frac{k}{q_n} + \frac{s_1}{q_n \cdot q_{n+1}} + \frac{t+1}{q_n \cdot q_{n+1}^2} - \frac{1}{q_n^2 \cdot q_{n+1}^2}\right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^3}{q_{n+1}} + \frac{q_n^3}{q_n \cdot q_{n+1}}, \frac{1}{q_{n-1}} + \frac{(s_2+1) \cdot q_n^3}{q_{n+1}} - \frac{q_n^3}{q_n \cdot q_{n+1}}\right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{l}{2q_{n-2}q_{n-1}^3 \cdot q_n} + \frac{j}{q_n^2} + \frac{1}{q_{n-1} \cdot q_n^2}, \frac{1}{q_{n-1}} + \frac{l}{2q_{n-2}q_{n-1}^3 \cdot q_n} + \frac{j+1}{q_n^2} - \frac{1}{q_{n-1} \cdot q_n^2}\right],$$

where the union is taken over all $j, k, l, s_1, t, s_2 \in \mathbb{Z}$ satisfying $0 \leq k \leq 2q_{n-2}q_{n-1}^3 - 1$, $\left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil \leq s_1 \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil - 1$, $\left\lceil \frac{q_{n+1}}{q_n} \right\rceil \leq t \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_n} \right\rceil - 1$, $0 \leq s_2 \leq \gamma_n \tilde{q}_n - 1$, $0 \leq j \leq \tilde{q}_n - 1 =$

$\frac{q_n}{2q_{n-2}q_{n-1}^3} - 1$ and $2q_{n-2}q_{n-1}^2 \leq l \leq q_{n-2}q_{n-1}^3 - 2$.

- in case of dimension $m \geq 4$:

$$\bigcup \left[\frac{k}{q_n} + \frac{s_1}{q_n \cdot q_{n+1}} + \frac{1}{q_n^2 \cdot q_{n+1}}, \frac{k}{q_n} + \frac{s_1+1}{q_n \cdot q_{n+1}} - \frac{1}{q_n^2 \cdot q_{n+1}}\right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^m}{q_{n+1}} + \frac{q_n^m}{q_n \cdot q_{n+1}}, \frac{1}{q_{n-1}} + \frac{(s_2+1) \cdot q_n^m}{q_{n+1}} - \frac{q_n^m}{q_n \cdot q_{n+1}}\right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{l}{2q_{n-2}q_{n-1}^m \cdot q_n} + \frac{j}{q_n^2} + \frac{1}{q_{n-1} \cdot q_n^2}, \frac{1}{q_{n-1}} + \frac{l}{2q_{n-2}q_{n-1}^m \cdot q_n} + \frac{j+1}{q_n^2} - \frac{1}{q_{n-1} \cdot q_n^2}\right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_4}{q_n \cdot q_{n+1}} + \frac{t}{q_n \cdot q_{n+1}^2} + \frac{1}{q_n^2 \cdot q_{n+1}^2}, \frac{1}{q_{n-1}} + \frac{s_4}{q_n \cdot q_{n+1}} + \frac{t+1}{q_n \cdot q_{n+1}^2} - \frac{1}{q_n^2 \cdot q_{n+1}^2}\right] \times$$
$$\prod_{i=5}^{m} \left[\frac{1}{q_{n-1}} + \frac{s_i}{q_n \cdot q_{n+1}} + \frac{1}{q_n^2 \cdot q_{n+1}}, \frac{1}{q_{n-1}} + \frac{s_i+1}{q_n \cdot q_{n+1}} - \frac{1}{q_n^2 \cdot q_{n+1}}\right],$$

where the union is taken over all $j, k, l, s_2, s_i, t \in \mathbb{Z}$ satisfying $\left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil \leq s_i \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil - 1$ for $i = 1, 4, 5, ..., m$, $0 \leq j \leq \tilde{q}_n - 1 = \frac{q_n}{2q_{n-2}q_{n-1}^m} - 1$, $0 \leq k \leq 2q_{n-2}q_{n-1}^m - 1$, $0 \leq s_2 \leq \gamma_n \tilde{q}_n - 1$, $2q_{n-2}q_{n-1}^{m-1} \leq l \leq q_{n-2}q_{n-1}^m - 2$ and $\left\lceil \frac{q_{n+1}}{q_n} \right\rceil \leq t \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_n} \right\rceil - 1$.
With these the base $c_{0,1}^{(n)}$ of the first tower is defined to be the set $c_{0,1}^{(n)} := H_{n-1}\left(\tilde{c}_{0,1}^{(n)}\right)$.

Moreover, we observe that $\phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right)$ is always contained in the left half of domains of the form $\left[\frac{u}{q_n}, \frac{u+1}{q_n}\right] \times [0,1]^{m-1}$. The sets $\tilde{c}_{0,2}^{(n)}$ are defined similarly to $\tilde{c}_{0,1}^{(n)}$. This time, $\phi_n^{-1}\left(\tilde{c}_{0,2}^{(n)}\right)$ is supposed to be contained in the right half of the aforementioned domains. For example, in case of dimension $m = 2$ let $\tilde{c}_{0,2}^{(n)}$ be the set

$$\bigcup \left[\frac{k}{q_n} + \frac{1}{2q_n} + \frac{l}{2q_{n-2}q_{n-1}^2 \cdot q_n} + \frac{j}{q_n^2} + \frac{1}{q_{n-1} \cdot q_n^2}, \frac{k}{q_n} + \frac{1}{2q_n} + \frac{l}{2q_{n-2}q_{n-1}^2 \cdot q_n} + \frac{j+1}{q_n^2} - \frac{1}{q_{n-1} \cdot q_n^2}\right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^2}{q_{n+1}} + \frac{t \cdot q_n^2}{q_{n+1}^2} + \frac{q_n^2}{q_n \cdot q_{n+1}^2}, \frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^2}{q_{n+1}} + \frac{(t+1) \cdot q_n^2}{q_{n+1}^2} - \frac{q_n^2}{q_n \cdot q_{n+1}^2}\right],$$

where the union is taken over all $j, k, l, s_2, t \in \mathbb{Z}$ satisfying $\left\lceil \frac{q_{n+1}}{q_n} \right\rceil \leq t \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_n} \right\rceil - 1$, $0 \leq j \leq \tilde{q}_n - 1 = \frac{q_n}{2q_{n-2}q_{n-1}^2} - 1$, $0 \leq k \leq 2q_{n-2}q_{n-1}^2 - 1$, $1 \leq l \leq q_{n-2}q_{n-1}^2 - 2q_{n-2}q_{n-1} - 1$ and $0 \leq s_2 \leq \gamma_n \tilde{q}_n - 1$.

Analogously, we define $\tilde{c}_{0,2}^{(n)}$ in higher dimensions. Then $c_{0,2}^{(n)} := H_{n-1}\left(\tilde{c}_{0,2}^{(n)}\right)$ is the base of the second tower.

Remark 6.1. We note that $\left(1 - \frac{3 \cdot 2m}{q_{n-1}}\right) \cdot \frac{\gamma_n \cdot q_n^2}{2q_{n+1}} \leq \mu\left(\tilde{c}_{0,i}^{(n)}\right) \leq \frac{\gamma_n \cdot q_n^2}{2q_{n+1}}$ for $i = 1, 2$. Moreover, we have in both cases:

$$\operatorname{diam}\left(c_{0,i}^{(n)}\right) \leq \|DH_{n-1}\|_0 \cdot \operatorname{diam}\left(\tilde{c}_{0,i}^{(n)}\right) \leq \|DH_{n-1}\|_0 \cdot \frac{\sqrt{m}}{\tilde{q}_n},$$

which is smaller than $\frac{1}{n}$ because of Lemma 5.9, 2..

In the next step we will construct a sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers in such a way that $m_n \cdot (\alpha_{n+1} - \alpha_n) = \pm m_n \cdot \frac{\gamma_n}{q_{n+1}}$ is approximately $\pm \frac{1}{q_n^2}$:

$$m_n := \left\lfloor \frac{q_{n+1}}{\gamma_n \cdot q_n^2} \right\rfloor . \tag{7}$$

With this sequence we define the following sets:

$$\begin{aligned} c_{i,1}^{(n)} &:= f_n^i \left(c_{0,1}^{(n)} \right) \text{ for } i = 0, 1, ..., m_n - 1, \\ c_{i,2}^{(n)} &:= f_n^i \left(c_{0,2}^{(n)} \right) \text{ for } i = 0, 1, ..., m_n. \end{aligned}$$

It should be noted that these sets are disjoint by construction. Hence, we are able to define these two towers:

Definition 6.2. The first tower $B_1^{(n)}$ with base $c_{0,1}^{(n)}$ and height m_n consists of the sets $c_{i,1}^{(n)}$ for $i = 0, 1, ..., m_n - 1$. The second tower $B_2^{(n)}$ with base $c_{0,2}^{(n)}$ and height $m_n + 1$ consists of the sets $c_{i,2}^{(n)}$ for $i = 0, 1, ..., m_n$.

In the rest of this subsection we check that these towers satisfy the requirements of the definition of a $(h, h+1)$-approximation. First of all, we notice that both towers are substantial because we have:

$$\mu\left(B_1^{(n)}\right) = m_n \cdot \mu\left(c_{0,1}^{(n)}\right) \geq \left(\frac{q_{n+1}}{\gamma_n \cdot q_n^2} - 1\right) \cdot \left(1 - \frac{3 \cdot 2m}{q_{n-1}}\right) \cdot \frac{\gamma_n \cdot q_n^2}{2q_{n+1}} \geq \left(1 - \frac{3 \cdot 2m}{q_{n-1}}\right) \frac{1}{2} - \frac{\gamma_n \cdot q_n^2}{2q_{n+1}},$$
$$\mu\left(B_2^{(n)}\right) = (m_n + 1) \cdot \mu\left(c_{0,2}^{(n)}\right) \geq \frac{q_{n+1}}{\gamma_n \cdot q_n^2} \cdot \left(1 - \frac{3 \cdot 2m}{q_{n-1}}\right) \cdot \frac{\gamma_n \cdot q_n^2}{2q_{n+1}} = \left(1 - \frac{3 \cdot 2m}{q_{n-1}}\right) \cdot \frac{1}{2}.$$

Using the notation from section 2 we consider the partial partition

$$\xi_n := \left\{ c_{i,1}^{(n)}, c_{k,2}^{(n)} \; : \; i = 0, 1, ..., m_n - 1; \; k = 0, 1, ..., m_n \right\}$$

and σ_n the associated permutation satisfying $\sigma_n \left(c_{i,1}^{(n)} \right) = c_{i+1,1}^{(n)}$ for $i = 0, 1, ..., m_n - 2$, $\sigma_n \left(c_{m_n-1,1}^{(n)} \right) = c_{0,1}^{(n)}$ as well as $\sigma_n \left(c_{k,2}^{(n)} \right) = c_{k+1,2}^{(n)}$ for $k = 0, 1, ..., m_n - 1$, $\sigma_n \left(c_{m_n,2}^{(n)} \right) = c_{0,2}^{(n)}$. We have to show: $\xi_n \to \varepsilon$ as $n \to \infty$. This property is fulfilled by the next lemma.

Lemma 6.3. *We have*

$$\xi_n \to \varepsilon \text{ as } n \to \infty.$$

Proof. The lemma is proven if we show that the partial partitions $\tilde{\xi}_n := \left\{ c \in \xi_n \; : \; \operatorname{diam}(c) < \frac{1}{n} \right\}$ satisfy $\mu\left(\bigcup_{c \in \tilde{\xi}_n} c\right) \to 1$ as $n \to \infty$. For this purpose, we examine which tower elements satisfy the condition on their diameter. Since

$$c_{i,j}^{(n)} = f_n^i \left(c_{0,j}^{(n)} \right) = H_{n-1} \circ \phi_n \circ R_{\alpha_{n+1}}^i \circ \phi_n^{-1} \circ H_{n-1}^{-1} \left(H_{n-1} \left(\tilde{c}_{0,j}^{(n)} \right) \right) = H_{n-1} \circ \phi_n \circ R_{\alpha_{n+1}}^i \circ \phi_n^{-1} \left(\tilde{c}_{0,j}^{(n)} \right)$$

we have to check for how many iterates i the set $R_{\alpha_{n+1}}^i \circ \phi_n^{-1} \left(\tilde{c}_{0,j}^{(n)} \right)$ is contained in the "good domain" of ϕ_n. Note that the bases of both towers are positioned in this "good domain". By the $\frac{1}{q_n}$-equivariance of the map ϕ_n and $i \cdot \alpha_{n+1} = \frac{i \cdot p_n}{q_n} \pm \frac{i \cdot \gamma_n}{q_{n+1}}$ we consider the displacement $\frac{i \cdot \gamma_n}{q_{n+1}}$, which is at most $\frac{1}{q_n^2}$ because of $i \leq m_n \leq \frac{q_{n+1}}{\gamma_n \cdot q_n^2}$. So the restrictions will come from the maps $\phi^{(2)}_{q_n^2, \frac{1}{4q_{n-1}}}$, $\phi^{(2)}_{q_n^{m-1}, \frac{1}{4q_{n-1}}}$,...,$\phi^{(m-1)}_{q_n^2, \frac{1}{4q_{n-1}}}$. By the same observations as stated in Lemma 4.3 we can estimate the number of allowed iterates $i \in \{0, 1, ..., m_n - 1\}$ by $\left(1 - \frac{1}{q_{n-1}}\right)^{m-1} \cdot m_n$. We conclude that there are at least $2 \cdot \left(1 - \frac{1}{q_{n-1}}\right)^{m-1} \cdot m_n$ partition elements $c^{(n)}_{i,j}$ in $\tilde{\xi}_n$ and this corresponds to a measure

$$\begin{aligned}\mu\left(\bigcup_{c \in \tilde{\xi}_n} c\right) &\geq 2 \cdot \left(1 - \frac{1}{q_{n-1}}\right)^{m-1} \cdot m_n \cdot \mu\left(c^{(n)}_{i,j}\right) \\ &\geq 2 \cdot \left(1 - \frac{1}{q_{n-1}}\right)^{m-1} \cdot \left(\frac{q_{n+1}}{\gamma_n \cdot q_n^2} - 1\right) \cdot \left(1 - \frac{3 \cdot 2m}{q_{n-1}}\right) \cdot \frac{\gamma_n \cdot q_n^2}{2q_{n+1}},\end{aligned}$$

which converges to 1 as $n \to \infty$. □

Actually we have a linked approximation of type $(h, h+1)$: Once again using the notation of section 2 we consider the partial partition

$$\eta_n := \left\{c^{(n)}_{i,1} \cup c^{(n)}_{i,2},\, c^{(n)}_{m_n,2} \;:\; 0 \leq i \leq m_n - 1\right\}$$

and prove $\eta_n \to \varepsilon$ as $n \to \infty$.

Lemma 6.4. *We have*

$$\eta_n \to \varepsilon \text{ as } n \to \infty.$$

Proof. To see this we examine the partition

$$\tilde{\eta}_n := \left\{\tilde{c}_i := c^{(n)}_{i,1} \cup c^{(n)}_{i,2} \;:\; 0 \leq i \leq m_n - 1,\ \operatorname{diam}(\tilde{c}_i) < \frac{1}{n}\right\}$$

and show $\lim_{n\to\infty} \mu\left(\bigcup_{\tilde{c} \in \tilde{\eta}_n} \tilde{c}\right) = 1$. For this purpose, we note that $c^{(n)}_{i,1} \cup c^{(n)}_{i,2}$ is contained in one $\frac{1}{\tilde{q}_n}$-cube, if $R^i_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{c}^{(n)}_{0,1} \cup \tilde{c}^{(n)}_{0,2}\right)$ belongs to the "good domain" of the map ϕ_n and the deviation $i \cdot |\alpha_{n+1} - \alpha_n|$ is less than $\frac{1}{q_n^2}$. Hence, the calculations from the previous lemma apply. □

6.2 Speed of approximation

In this section we want we prove that f admits a good linked approximation of type $(h.h+1)$:

Proposition 6.5. *The constructed diffeomorphism $f \in \mathcal{A}_\alpha\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ admits a good linked approximation of type $(h.h+1)$.*

Since f admits a linked approximation by Lemma 6.4, we have to compute the speed of the approximation in order to prove this statement. First of all, we observe

(8)
$$\sum_{c \in \xi_n} \mu\left(f(c) \triangle \sigma_n(c)\right) \leq \sum_{c \in \xi_n} \left(\mu\left(f(c) \triangle f_{n+1}(c)\right) + \mu\left(f_{n+1}(c) \triangle f_n(c)\right) + \mu\left(f_n(c) \triangle \sigma_n(c)\right)\right)$$

recalling that σ_n is the associated permutation satisfying $\sigma_n\left(c^{(n)}_{i,1}\right) = c^{(n)}_{i+1,1}$ for $i = 0, 1, ..., m_n - 2$, $\sigma_n\left(c^{(n)}_{m_n-1,1}\right) = c^{(n)}_{0,1}$ as well as $\sigma_n\left(c^{(n)}_{k,2}\right) = c^{(n)}_{k+1,2}$ for $k = 0, 1, ..., m_n - 1$, $\sigma_n\left(c^{(n)}_{m_n,2}\right) = c^{(n)}_{0,2}$. In the subsequent lemmas we examine each summand.

Lemma 6.6. *We have*

$$\sum_{c \in \xi_n} \mu\left(f_n(c) \triangle \sigma_n(c)\right) \leq \frac{8 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}}.$$

Proof. We note $\sigma_n|_{c_{i,1}^{(n)}} = f_n|_{c_{i,1}^{(n)}}$ for $i = 0, ..., m_n - 2$ and $\sigma_n\left(c_{m_n-1,1}^{(n)}\right) = c_{0,1}^{(n)}$ as well as $\sigma_n|_{c_{i,2}^{(n)}} = f_n|_{c_{i,2}^{(n)}}$ for $i = 0, ..., m_n - 1$, $\sigma_n\left(c_{m_n,2}^{(n)}\right) = c_{0,2}^{(n)}$.
To estimate the expression $\sum_{c \in \xi_n} \mu\left(f_n(c) \triangle \sigma_n(c)\right)$ we observe that $f_n\left(c_{m_n-1,1}^{(n)}\right)$ and $c_{0,1}^{(n)}$ differ in the deviation of $m_n \cdot |\alpha_{n+1} - \alpha_n|$ from $\frac{1}{q_n^2}$ and in that the part of $c_{0,1}^{(n)}$ corresponding to the both values $l = q_{n-2}q_{n-1}^m - 2$ as well as $j = \tilde{q}_n - 1$ in "case +" (resp. $l = 2q_{n-2}q_{n-1}^{m-1}$ and $j = 0$ in "case -") is not mapped back to $c_{0,1}^{(n)}$ under $f_n^{m_n}$. This second discrepancy yields a measure difference of at most $2 \cdot \frac{\gamma_n \cdot q_n}{q_{n+1}} \cdot \left(1 - \frac{2}{q_n}\right)^{2m-1} \leq \frac{2 \cdot \gamma_n \cdot q_n}{q_{n+1}}$.
Examining the first one we recall that q_n^2 divides q_{n+1} by requirement A. So $q_{n+1} = \bar{q}_{n+1} \cdot q_n^2$, where $\bar{q}_{n+1} \in \mathbb{Z}$. This implies $m_n = \left\lfloor \frac{q_{n+1}}{\gamma_n \cdot q_n^2} \right\rfloor = \left\lfloor \frac{\bar{q}_{n+1}}{\gamma_n} \right\rfloor$ and we can write $\bar{q}_{n+1} = m_n \cdot \gamma_n + t$ with $t \in \mathbb{Z}$, $0 \leq t \leq \gamma_n - 1$. Then we have

$$m_n \cdot |\alpha_{n+1} - \alpha_n| = m_n \cdot \frac{\gamma_n}{q_{n+1}} = \frac{m_n \cdot \gamma_n + t}{q_{n+1}} - \frac{t}{q_{n+1}} = \frac{\bar{q}_{n+1}}{q_{n+1}} - \frac{t}{q_{n+1}} = \frac{1}{q_n^2} - \frac{t}{q_{n+1}}.$$

Hence, the deviation can take the values 0, $\frac{1}{q_{n+1}}$,... or $\frac{\gamma_n - 1}{q_{n+1}}$. Because of this deviation some $\frac{1}{q_{n+1}}$-stripes of $\phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right)$ (i.e. parts corresponding to a fixed value of k, l, j, s_2 and that have a measure of at most $\left(1 - \frac{2}{q_n}\right)^{2m-1} \cdot \frac{1}{q_{n+1}}$) are shifted out of $\phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right)$. The caused measure difference is at most $2 \cdot \frac{\gamma_n - 1}{q_{n+1}} \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n \cdot \left(1 - \frac{2}{q_n}\right)^{2m}$ (because for each possible value of k, l, j these stripes are shifted out). Then we have:

$$\mu\left(f_n^{m_n}\left(c_{0,1}^{(n)}\right) \triangle c_{0,1}^{(n)}\right) \leq \frac{2 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}} + \frac{2 \cdot \gamma_n \cdot q_n}{q_{n+1}} \leq \frac{4 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}}.$$

Analogously $f_n\left(c_{m_n,2}^{(n)}\right)$ and $c_{0,2}^{(n)}$ differ in the displacing of at most $\gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n$ such $\frac{1}{q_{n+1}}$-stripes caused by the deviation of $(m_n + 1) \cdot |\alpha_{n+1} - \alpha_n|$ from $\frac{1}{q_n^2}$ and in that the part of $c_{0,2}^{(n)}$ corresponding to $l = q_{n-2}q_{n-1}^m - 2q_{n-2}q_{n-1}^{m-1} - 1$ as well as $j = \tilde{q}_n - 1$ in "case +" (resp. $l = 1$ and $j = 0$ in the "case -") is not mapped back to $c_{0,2}^{(n)}$. Thus, we get

$$\mu\left(f_n^{m_n+1}\left(c_{0,2}^{(n)}\right) \triangle c_{0,2}^{(n)}\right) \leq \frac{4 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}}.$$

This yields

$$\sum_{c \in \xi_n} \mu\left(f_n(c) \triangle \sigma_n(c)\right) \leq \frac{8 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}}.$$

□

In the next step we consider $\sum_{c \in \xi_n} \mu\left(f_{n+1}(c) \triangle f_n(c)\right)$:

Lemma 6.7. *We have*

$$\sum_{c \in \xi_n} \mu\left(f_{n+1}(c) \triangle f_n(c)\right) \leq \frac{2 \cdot q_{n+1}^m \cdot \gamma_{n+1}}{q_{n+2}}.$$

Proof. We have to compare the sets $\phi_{n+1} \circ R_{\alpha_{n+2}} \circ \phi_{n+1}^{-1} \circ R_{\alpha_{n+1}}^{i} \circ \phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right) = \phi_{n+1} \circ R_{\alpha_{n+2}} \circ R_{\alpha_{n+1}}^{i} \circ \phi_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right)$ and $R_{\alpha_{n+1}}^{i+1} \circ \phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right) = \phi_{n+1} \circ R_{\alpha_{n+1}}^{i+1} \circ \phi_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right)$ (recall that we have $\phi_{n+1} \circ R_{\frac{1}{q_{n+1}}} = R_{\frac{1}{q_{n+1}}} \circ \phi_{n+1}$ by construction). Since $R_{\alpha_{n+1}}^{i+1} \circ \phi_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right)$ as well as $R_{\alpha_{n+2}} \circ R_{\alpha_{n+1}}^{i} \circ \phi_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_{0,1}^{(n)}\right)$ are positioned in the "good domain" of the map ϕ_{n+1} for $i < m_n$ by definition of $\tilde{c}_{0,1}^{(n)}$, the deviation $|\alpha_{n+2} - \alpha_{n+1}| = \frac{\gamma_{n+1}}{q_{n+2}}$ on the θ-axis for every of the at most $\frac{1}{2} \cdot \gamma_n \cdot q_n^2 \cdot q_{n+1}^{m-1} \cdot \left(1 - \frac{2}{q_n}\right)^m$ stripes of θ-width $\left(1 - \frac{2}{q_{n-1}}\right) \frac{1}{q_{n+1}^m}$ causes the following measure difference:

$$\mu\left(f_{n+1}\left(c_{i,1}^{(n)}\right) \triangle f_n\left(c_{i,1}^{(n)}\right)\right) \leq 2 \cdot \frac{\gamma_{n+1}}{q_{n+2}} \cdot \frac{1}{2} \cdot \gamma_n \cdot q_n^2 \cdot q_{n+1}^{m-1} \cdot \left(1 - \frac{2}{q_n}\right)^m \cdot \left(1 - \frac{2}{q_n}\right)^{m-1}.$$

This difference occurs for every $i \in \{0, ..., m_n - 1\}$ and allows us to estimate:

$$\sum_{i=0}^{m_n - 1} \mu\left(f_{n+1}\left(c_{i,1}^{(n)}\right) \triangle f_n\left(c_{i,1}^{(n)}\right)\right) \leq 2 \cdot \frac{1}{2} \cdot \gamma_n \cdot q_n^2 \cdot q_{n+1}^{m-1} \cdot m_n \cdot \left(1 - \frac{2}{q_n}\right)^{2m-1} \cdot \frac{\gamma_{n+1}}{q_{n+2}} \leq q_{n+1}^m \cdot \frac{\gamma_{n+1}}{q_{n+2}}.$$

Similarly we estimate

$$\sum_{i=0}^{m_n} \mu\left(f_{n+1}\left(c_{i,2}^{(n)}\right) \triangle f_n\left(c_{i,2}^{(n)}\right)\right) \leq 2 \cdot \frac{1}{2} \cdot \gamma_n \cdot q_n^2 \cdot q_{n+1}^{m-1} \cdot (m_n + 1) \cdot \left(1 - \frac{2}{q_n}\right)^{2m-1} \cdot \frac{\gamma_{n+1}}{q_{n+2}} \leq q_{n+1}^m \cdot \frac{\gamma_{n+1}}{q_{n+2}}.$$

Thus, we obtain

$$\sum_{c \in \xi_n} \mu\left(f_{n+1}(c) \triangle f_n(c)\right) \leq \frac{2 \cdot q_{n+1}^m \cdot \gamma_{n+1}}{q_{n+2}}.$$

□

Lastly we consider

Lemma 6.8. *We have*

$$\sum_{c \in \xi_n} \mu\left(f(c) \triangle f_{n+1}(c)\right) \leq \frac{10 \cdot m}{q_{n+1}}.$$

Proof. We compute for every $c \in \xi_n$ using the notation $\bar{c} := H_{n+1}^{-1}(c)$:

$$\begin{aligned}
\mu\left(f_{n+2}(c) \triangle f_{n+1}(c)\right) &= \mu\left(H_{n+2} \circ R_{\alpha_{n+3}} \circ \phi_{n+2}^{-1}(\bar{c}) \triangle H_{n+1} \circ R_{\alpha_{n+2}}(\bar{c})\right) \\
&= \mu\left(H_{n+2} \circ R_{\alpha_{n+3}} \circ \phi_{n+2}^{-1}(\bar{c}) \triangle H_{n+1} \circ \phi_{n+2} \circ R_{\alpha_{n+2}} \circ \phi_{n+2}^{-1}(\bar{c})\right) \\
&= \mu\left(R_{\alpha_{n+3}} \circ \phi_{n+2}^{-1}(\bar{c}) \triangle R_{\alpha_{n+2}} \circ \phi_{n+2}^{-1}(\bar{c})\right).
\end{aligned}$$

Since we have no control on $\phi_{n+2}^{-1}(\bar{c})$ for these areas of $\bar{c}$ that do not belong to the "good domain" of the map ϕ_{n+2}, they will be part of the measure difference in our estimates. On the other hand, for the part of $\bar{c}$ belonging to the "good domain" of the map ϕ_{n+2} the difference is caused by the deviation $|\alpha_{n+3} - \alpha_{n+2}|$. Using Lemma 4.3 the "good domain" of the map ϕ_{n+2} on an θ-interval of the form $\left[\frac{l}{q_{n+2}}, \frac{l+1}{q_{n+2}}\right]$ has length at least $\left(1 - \frac{m}{q_{n+1}}\right) \cdot \frac{1}{q_{n+2}}$. It follows that the measure difference of $R_{\alpha_{n+3}} \circ \phi_{n+2}^{-1}(\bar{c})$ and $R_{\alpha_{n+2}} \circ \phi_{n+2}^{-1}(\bar{c})$ on a section of the form $\left[\frac{l}{q_{n+2}}, \frac{l+1}{q_{n+2}}\right] \times [0,1]^{m-1}$ is at most

$$2 \cdot \left(\frac{m}{q_{n+1}} \cdot \frac{1}{q_{n+2}} + |\alpha_{n+3} - \alpha_{n+2}| \cdot \left(1 - \frac{2}{q_{n+1}}\right)^{m-1} \right) \leq \frac{4m}{q_{n+1}} \cdot \frac{1}{q_{n+2}}.$$

Moreover, we recall that $H_{n+1}^{-1}(c)$ consists of at most $\frac{1}{2} \cdot \gamma_n \cdot q_n^2 \cdot q_{n+1}^{m-1} \cdot \left(1 - \frac{2}{q_n}\right)^m$ stripes of θ-width $\left(1 - \frac{2}{q_{n-1}}\right) \frac{1}{q_{n+1}^m}$. Since the θ-width $\left(1 - \frac{2}{q_{n-1}}\right) \frac{1}{q_{n+1}^m}$ accords to at most $\left\lceil \left(1 - \frac{2}{q_{n-1}}\right) \frac{q_{n+2}}{q_{n+1}^m} \right\rceil + 2$ intervals of type $\left[\frac{l}{q_{n+2}}, \frac{l+1}{q_{n+2}}\right]$, we have

$$\mu\left(f_{n+2}(c) \triangle f_{n+1}(c)\right) \leq \frac{1}{2} \cdot \gamma_n \cdot q_n^2 \cdot q_{n+1}^{m-1} \cdot \left(1 - \frac{2}{q_n}\right)^m \cdot \left(\left\lceil \left(1 - \frac{2}{q_{n-1}}\right) \frac{q_{n+2}}{q_{n+1}^m} \right\rceil + 2 \right) \cdot \frac{4m}{q_{n+1}} \cdot \frac{1}{q_{n+2}}$$

Each of the $(2m_n + 1)$ elements $c \in \xi_n$ contributes and so we obtain

$$\sum_{c \in \xi_n} \mu\left(f_{n+2}(c) \triangle f_{n+1}(c)\right) \leq \frac{5 \cdot m}{q_{n+1}}.$$

Analogously estimating the other summands we observe

$$\begin{aligned}
&\sum_{c \in \xi_n} \mu\left(f(c) \triangle f_{n+1}(c)\right) \\
\leq &\sum_{k=1}^{\infty} \sum_{c \in \xi_n} \mu \left(f_{n+k+1}\left(c \cap \bigcap_{j=1}^{k-1} H_{n+j}(G_{n+j+1}) \right) \triangle f_{n+k}\left(c \cap \bigcap_{j=1}^{k-1} H_{n+j}(G_{n+j+1}) \right) \right) \\
\leq &\sum_{k=n+1}^{\infty} \frac{5 \cdot m}{q_k} \leq \frac{10 \cdot m}{q_{n+1}}.
\end{aligned}$$

□

Proof of Proposition 6.5. Using equation 8 and the precedent three lemmas we conclude

$$\sum_{c \in \xi_n} \mu\left(f(c) \triangle \sigma_n(c)\right) \leq \frac{8 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}} + \frac{2 \cdot q_{n+1}^m \cdot \gamma_{n+1}}{q_{n+2}} + \frac{10 \cdot m}{q_{n+1}}.$$

In order to prove that this speed of approximation is of order $o\left(\frac{1}{m_n}\right)$ we compute

$$\begin{aligned}
&\frac{\frac{8 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}} + \frac{2 \cdot q_{n+1}^m \cdot \gamma_{n+1}}{q_{n+2}} + \frac{10 \cdot m}{q_{n+1}}}{\frac{1}{m_n}} \\
\leq &\frac{q_{n+1}}{\gamma_n \cdot q_n^2} \cdot \left(\frac{8 \cdot \gamma_n \cdot q_{n-2} \cdot q_{n-1}^m \cdot q_n}{q_{n+1}} + \frac{2 \cdot q_{n+1}^m \cdot \gamma_{n+1}}{q_{n+2}} + \frac{10 \cdot m}{q_{n+1}} \right) \\
= &\frac{8 \cdot q_{n-2} \cdot q_{n-1}^m}{q_n} + \frac{2 \cdot q_{n+1}^{m+1} \cdot \gamma_{n+1}}{\gamma_n \cdot q_n^2 \cdot q_{n+2}} + \frac{10 \cdot m}{\gamma_n \cdot q_n^2}.
\end{aligned}$$

Since this converges to 0 as $n \to \infty$ (in particular because of equation 6), we have a good linked approximation of type $(h, h+1)$. □

7 Proof of good cyclic approximation

7.1 Tower for good cyclic approximation

Let $\tilde{d}_0^{(n)}$ be the set

- in case of dimension $m = 2$:

$$\bigcup \left[\frac{1}{q_{n-1} \cdot q_n} + \frac{1}{q_{n-1} \cdot q_n^2}, \frac{1}{q_{n-1} \cdot q_n} + \frac{1}{q_n^2} - \frac{1}{q_{n-1} \cdot q_n^2} \right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^2}{q_{n+1}} + \frac{s_2^{(2)} \cdot q_n^2}{q_{n+1}^2} + \frac{q_n^2}{q_n \cdot q_{n+1}^2}, \frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^2}{q_{n+1}} + \frac{\left(s_2^{(2)} + 1\right) \cdot q_n^2}{q_{n+1}^2} - \frac{q_n^2}{q_n \cdot q_{n+1}^2} \right],$$

where the union is taken over all $s_2, s_2^{(2)} \in \mathbb{Z}$ satisfying $\left\lceil \frac{q_{n+1}}{q_n} \right\rceil \leq s_2^{(2)} \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_n} \right\rceil - 1$, $0 \leq s_2 \leq 2q_{n-1}q_n^2 - 1$.

- in case of dimension $m = 3$:

$$\bigcup \left[\frac{s_1^{(1)}}{q_n \cdot q_{n+1}} + \frac{s_1^{(2)}}{q_n \cdot q_{n+1}^2} + \frac{1}{q_n^2 \cdot q_{n+1}^2}, \frac{s_1^{(1)}}{q_n \cdot q_{n+1}} + \frac{s_1^{(2)} + 1}{q_n \cdot q_{n+1}^2} - \frac{1}{q_n^2 \cdot q_{n+1}^2} \right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^3}{q_{n+1}} + \frac{q_n^3}{q_n \cdot q_{n+1}}, \frac{1}{q_{n-1}} + \frac{(s_2 + 1) \cdot q_n^3}{q_{n+1}} - \frac{q_n^3}{q_n \cdot q_{n+1}} \right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{1}{q_{n-1} \cdot q_n} + \frac{1}{q_{n-1} \cdot q_n^2}, \frac{1}{q_{n-1}} + \frac{1}{q_{n-1} \cdot q_n} + \frac{1}{q_n^2} - \frac{1}{q_{n-1} \cdot q_n^2} \right],$$

where the union is taken over all $s_1^{(1)}, s_1^{(2)}, s_2 \in \mathbb{Z}$ satisfying $\left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil \leq s_1^{(1)} \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil - 1$, $\left\lceil \frac{q_{n+1}}{q_n} \right\rceil \leq s_1^{(2)} \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_n} \right\rceil - 1$ and $0 \leq s_2 \leq 2q_{n-1}q_n^3 - 1$.

- in case of dimension $m \geq 4$:

$$\bigcup \left[\frac{s_1}{q_n \cdot q_{n+1}} + \frac{1}{q_n^2 \cdot q_{n+1}}, \frac{s_1 + 1}{q_n \cdot q_{n+1}} - \frac{1}{q_n^2 \cdot q_{n+1}} \right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_2 \cdot q_n^m}{q_{n+1}} + \frac{q_n^m}{q_n \cdot q_{n+1}}, \frac{1}{q_{n-1}} + \frac{(s_2 + 1) \cdot q_n^m}{q_{n+1}} - \frac{q_n^m}{q_n \cdot q_{n+1}} \right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{1}{q_{n-1} \cdot q_n} + \frac{1}{q_{n-1} \cdot q_n^2}, \frac{1}{q_{n-1}} + \frac{1}{q_{n-1} \cdot q_n} + \frac{1}{q_n^2} - \frac{1}{q_{n-1} \cdot q_n^2} \right] \times$$
$$\left[\frac{1}{q_{n-1}} + \frac{s_4}{q_n \cdot q_{n+1}} + \frac{s_4^{(2)}}{q_n \cdot q_{n+1}^2} + \frac{1}{q_n^2 \cdot q_{n+1}^2}, \frac{1}{q_{n-1}} + \frac{s_4}{q_n \cdot q_{n+1}} + \frac{s_4^{(2)} + 1}{q_n \cdot q_{n+1}^2} - \frac{1}{q_n^2 \cdot q_{n+1}^2} \right] \times$$
$$\prod_{i=5}^{m} \left[\frac{1}{q_{n-1}} + \frac{s_i}{q_n \cdot q_{n+1}} + \frac{1}{q_n^2 \cdot q_{n+1}}, \frac{1}{q_{n-1}} + \frac{s_i + 1}{q_n \cdot q_{n+1}} - \frac{1}{q_n^2 \cdot q_{n+1}} \right],$$

where the union is taken over all $s_i, s_4^{(2)} \in \mathbb{Z}$ satisfying $\left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil \leq s_i \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_{n-1}} \right\rceil - 1$ for $i = 1, 4, 5, ..., m$, $\left\lceil \frac{q_{n+1}}{q_n} \right\rceil \leq s_4^{(2)} \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{q_n} \right\rceil - 1$ and $0 \leq s_2 \leq 2q_{n-1}q_n^m - 1$.

With these the base $d_0^{(n)}$ of the tower is defined to be the set $d_0^{(n)} := H_{n-1}\left(\tilde{d}_0^{(n)}\right)$ and the tower levels are the sets

$$d_i^{(n)} := f_n^i\left(d_0^{(n)}\right) \quad \text{for } i = 0, ..., \tilde{q}_{n+1} - 1$$

Recall the relations $q_{n+1} = 2 \cdot q_{n-1} \cdot q_n^m \cdot \tilde{q}_{n+1}$ as well as $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \frac{\tilde{p}_{n+1}}{\tilde{q}_{n+1}}$, where $\tilde{p}_{n+1}$ and $\tilde{q}_{n+1}$ are relatively prime. Hence, the tower levels are disjoint sets of equal measure not less than $\frac{2q_{n-1}q_n^m}{q_{n+1}} \cdot \left(1 - \frac{3}{q_{n-1}}\right)^{2m-1}$. Moreover, the associated cyclic permutation $\tilde{\sigma}_n$ is given by the description $\tilde{\sigma}_n|_{d_i^{(n)}} = f_n|_{d_i^{(n)}}$ for $i = 0, ..., \tilde{q}_{n+1} - 2$ and $\tilde{\sigma}_n\left(d^{(n)}_{\tilde{q}_{n+1}-1}\right) = d_0^{(n)}$. Since $f_n^{\tilde{q}_{n+1}} = \mathrm{id}$ we also have $\tilde{\sigma}_n|_{d^{(n)}_{\tilde{q}_{n+1}-1}} = f_n|_{d^{(n)}_{\tilde{q}_{n+1}-1}}$.
In order to see that this provides a cyclic approximation of the constructed map f we show that the partial partition $\Gamma_n := \left\{d_i^{(n)} \ : \ i = 0, ..., \tilde{q}_{n+1} - 1\right\}$ converges to the decomposition into points.

Lemma 7.1. *We have $\Gamma_n \to \varepsilon$ as $n \to \infty$.*

Proof. It suffices to show that the partial partition $\tilde{\Gamma}_n := \left\{d_i^{(n)} \in \Gamma_n \ : \ \operatorname{diam}\left(d_i^{(n)}\right) < \frac{1}{n}\right\}$ satisfies $\lim_{n\to\infty} \mu\left(\bigcup_{d\in\tilde{\Gamma}_n} d\right) = 1$. As in the previous chapter we have to check for how many iterates $i \in \{0, 1, ..., \tilde{q}_{n+1} - 1\}$ the set $R^i_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{d}_0^{(n)}\right)$ is contained in the "good domain" of the map ϕ_n whose corresponding length on the θ-axis is at least $\left(1 - \frac{1}{q_{n-1}}\right)^m$ by Lemma 4.3. Since $R^i_{\tilde{\alpha}_{n+1}}$ is equally distributed on $\mathbb{S}^1$, there are at least $\left(1 - \frac{2}{q_{n-1}}\right)^m \cdot \tilde{q}_{n+1}$ such iterates i. Then we conclude

$$\begin{aligned}
\mu\left(\bigcup_{d\in\tilde{\Gamma}_n} d\right) &\geq \left(1 - \frac{2}{q_{n-1}}\right)^m \cdot \tilde{q}_{n+1} \cdot \mu\left(d_0^{(n)}\right) \\
&\geq \left(1 - \frac{2}{q_{n-1}}\right)^m \cdot \frac{q_{n+1}}{2 \cdot q_{n-1} \cdot q_n^m} \cdot \frac{2q_{n-1}q_n^m}{q_{n+1}} \cdot \left(1 - \frac{3}{q_{n-1}}\right)^{2m-1} \\
&\geq \left(1 - \frac{3}{q_{n-1}}\right)^{3m-1},
\end{aligned}$$

which converges to 1 as $n \to \infty$. □

7.2 Speed of approximation

In this subsection we show that f admits a good cyclic approximation.

Proposition 7.2. *The constructed diffeomorphism $f \in \mathcal{A}_\alpha\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ admits a good cyclic approximation.*

As observed in the previous subsection $\tilde{\sigma}_n|_d = f_n|_d$ for every $d \in \Gamma_n$. Thus, for the speed of approximation it holds:

$$\sum_{d\in\Gamma_n} \mu\left(f(d) \triangle \tilde{\sigma}_n(d)\right) \leq \sum_{d\in\Gamma_n} \left(\mu\left(f(d) \triangle f_{n+1}(d)\right) + \mu\left(f_{n+1}(d) \triangle f_n(d)\right)\right). \tag{9}$$

First of all, we aim for estimating the error $\sum_{d\in\Gamma_n} \mu\left(f_{n+1}(d) \triangle f_n(d)\right)$.

Lemma 7.3. *We have*

$$\sum_{d\in\Gamma_n}\mu\left(f_{n+1}\left(d\right)\triangle f_n\left(d\right)\right)\leq\frac{2\cdot q_{n+1}^m\cdot\gamma_{n+1}}{q_{n+2}}.$$

Proof. We have to compare the sets $\phi_{n+1}\circ R_{\alpha_{n+2}}\circ\phi_{n+1}^{-1}\circ R^i_{\alpha_{n+1}}\circ\phi_n^{-1}\left(\tilde{d}_0^{(n)}\right)=\phi_{n+1}\circ R_{\alpha_{n+2}}\circ R^i_{\alpha_{n+1}}\circ\phi_{n+1}^{-1}\circ\phi_n^{-1}\left(\tilde{d}_0^{(n)}\right)$ and $\phi_{n+1}\circ R^{i+1}_{\alpha_{n+1}}\circ\phi_{n+1}^{-1}\circ\phi_n^{-1}\left(\tilde{d}_0^{(n)}\right)$ for every $i\in\{0,...,\tilde{q}_{n+1}-1\}$. Since $R_{\alpha_{n+2}}\circ R^i_{\alpha_{n+1}}\circ\phi_{n+1}^{-1}\circ\phi_n^{-1}\left(\tilde{d}_0^{(n)}\right)$ as well as $R^{i+1}_{\alpha_{n+1}}\circ\phi_{n+1}^{-1}\circ\phi_n^{-1}\left(\tilde{d}_0^{(n)}\right)$ are positioned in the "good domain" of the map ϕ_{n+1}, the discrepancy $|\alpha_{n+2}-\alpha_{n+1}|=\frac{\gamma_{n+1}}{q_{n+2}}$ on the θ-axis for every of the at most $2q_{n-1}q_n^mq_{n+1}^{m-1}\cdot\left(1-\frac{2}{q_n}\right)^{m-1}$ stripes causes the following measure difference

$$\mu\left(f_{n+1}\left(d_i^{(n)}\right)\triangle f_n\left(d_i^{(n)}\right)\right)\leq 2\cdot 2q_{n-1}q_n^m\cdot q_{n+1}^{m-1}\cdot\left(1-\frac{2}{q_n}\right)^{m-1}\cdot\frac{\gamma_{n+1}}{q_{n+2}}.$$

This difference occurs for every $i\in\left\{0,...,\frac{q_{n+1}}{2q_{n-1}q_n^m}-1\right\}$ and thus we can estimate

$$\sum_{d\in\Gamma_n}\mu\left(f_{n+1}\left(d\right)\triangle f_n\left(d\right)\right)\leq\frac{2\cdot q_{n+1}^m\cdot\gamma_{n+1}}{q_{n+2}}.$$

□

In the next step we prove

Lemma 7.4. *We have*

$$\sum_{d\in\Gamma_n}\mu\left(f\left(d\right)\triangle f_{n+1}\left(d\right)\right)\leq\frac{16\cdot m}{q_{n+1}}.$$

Proof. In order to estimate $\mu\left(f_{n+2}\left(d\right)\Delta f_{n+1}\left(d\right)\right)$ we argue as in the previous chapter using that $H_{n+1}^{-1}\left(d\right)$ consists of at most $2q_{n-1}q_n^mq_{n+1}^{m-1}\cdot\left(1-\frac{2}{q_n}\right)^{m-1}$ stripes of θ-width $\left(1-\frac{2}{q_{n-1}}\right)\frac{1}{q_{n+1}^m}$. Therewith, we obtain

$$\begin{aligned}&\mu\left(f_{n+2}\left(d\right)\triangle f_{n+1}\left(d\right)\right)\\&\leq 2\cdot 2q_{n-1}q_n^m\cdot q_{n+1}^{m-1}\cdot\left(1-\frac{2}{q_n}\right)^{m-1}\cdot\left(\left\lceil\left(1-\frac{2}{q_{n-1}}\right)\frac{q_{n+2}}{q_{n+1}^m}\right\rceil+2\right)\cdot\frac{4m}{q_{n+1}}\cdot\frac{1}{q_{n+2}}.\end{aligned}$$

Each of the $\tilde{q}_{n+1}=\frac{q_{n+1}}{2q_{n-1}q_n^m}$ elements $d\in\Gamma_n$ contributes and so we obtain

$$\sum_{d\in\Gamma_n}\mu\left(f_{n+2}\left(d\right)\triangle f_{n+1}\left(d\right)\right)\leq\frac{8\cdot m}{q_{n+1}}.$$

Analogously we estimate the other summands and obtain

$$\begin{aligned}&\sum_{d\in\Gamma_n}\mu\left(f\left(d\right)\triangle f_{n+1}\left(d\right)\right)\\&\leq\sum_{k=1}^{\infty}\sum_{d\in\Gamma_n}\mu\left(f_{n+k+1}\left(d\cap\bigcap_{j=1}^{k-1}H_{n+j}\left(G_{n+j+1}\right)\right)\triangle f_{n+k}\left(d\cap\bigcap_{j=1}^{k-1}H_{n+j}\left(G_{n+j+1}\right)\right)\right)\\&\leq\sum_{k=n+1}^{\infty}\frac{8\cdot m}{q_k}\leq\frac{16\cdot m}{q_{n+1}}.\end{aligned}$$

□

Proof of Proposition 7.2. Using equation 9 as well as lemmas 7.3 and 7.4 we conclude

$$\sum_{d\in\Gamma_n} \mu\left(f\left(d\right)\triangle\tilde{\sigma}_n\left(d\right)\right) \leq \frac{2\cdot q_{n+1}^m\cdot\gamma_{n+1}}{q_{n+2}} + \frac{16\cdot m}{q_{n+1}}.$$

In order to prove that this speed of approximation is of order $o\left(\frac{1}{\tilde{q}_{n+1}}\right)$ we compute

$$\begin{aligned}\frac{\frac{2\cdot q_{n+1}^m\cdot\gamma_{n+1}}{q_{n+2}}+\frac{16\cdot m}{q_{n+1}}}{\frac{1}{\tilde{q}_{n+1}}} &\leq \frac{q_{n+1}}{2\cdot q_{n-1}\cdot q_n^m}\cdot\left(\frac{2\cdot q_{n+1}^m\cdot\gamma_{n+1}}{q_{n+2}}+\frac{16\cdot m}{q_{n+1}}\right)\\ &= \frac{q_{n+1}^{m+1}\cdot\gamma_{n+1}}{q_{n-1}\cdot q_n^m\cdot q_{n+2}}+\frac{8\cdot m}{q_{n-1}\cdot q_n^m}.\end{aligned}$$

Since this converges to 0 as $n\to\infty$ (again by equation 6), we have a good cyclic approximation. □

8 Proof of Proposition 1.9

In the setting of our explicit constructions Lemma 5.9 shows that we can satisfy the requirements of Lemma 5.8. Hence, we obtain convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in the Diff$^\infty$-topology to a volume-preserving diffeomorphism $f\in\mathcal{A}_\alpha$. Moreover, Lemma 5.8 proves that for every $\varepsilon>0$ we can do the constructions in such a way that the limit diffeomorphism f satisfies $d_\infty(f,R_\alpha)<\varepsilon$. By Proposition 6.5 and Proposition 7.2 f admits a good linked approximation of type $(h,h+1)$ as well as a good cyclic approximation.

9 Reduction to Proposition 1.9

Using the concepts and results from the previous sections we can reduce the proof of the Theorem to Proposition 1.9. Indeed, such a constructed diffeomorphism has the following properties:

- Since f allows a good cylic approximation, it has simple spectrum by Lemma 3.4. Furthermore, f admits a good linked approximation of type $(h,h+1)$ and so we can use Remark 2.5 to obtain the weak convergence $U_f^{h+1}\longrightarrow_w r\cdot U_f+(1-r)\cdot Id$ for some $r\in(0,1)$. Hence, we can apply Lemma 3.5 and conclude that $f\times f$ has homogeneous spectrum of multiplicity 2.

- With the aid of Lemma 2.6 and the good cyclic approximation of f we have that f is ergodic. Then we can exploit the good approximation of type $(h,h+1)$ and Lemma 2.7 to see that f is even weak mixing. Due to Remark 3.8 there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that the convergence $U_f^{k_n}\to_w P_c$ holds in the weak operator topology as $n\to\infty$. Along this sequence we have using Remark 2.5

  $$U_f^{k_n\cdot(h+1)}\longrightarrow_w \left(r\cdot P_c+(1-r)\cdot Id\right)$$

  for some $r\in(0,1)$. Thus, f is κ-weak mixing (with $\kappa=r\in(0,1)$). Then the maximal spectral type σ of f is disjoint with its convolutions by Lemma 3.9.

By Proposition 1.9 the set of diffeomorphisms having the aimed properties is dense in $\mathcal{A}_\alpha$ with respect to the Diff$^\infty$-topology: Because of $\mathcal{A}_\alpha=\overline{\left\{h\circ R_\alpha\circ h^{-1}:h\in\mathrm{Diff}^\infty\left(M,\mu\right)\right\}}^{C^\infty}$ it suffices

to show that for every diffeomorphism $h \in \mathrm{Diff}^{\infty}(M,\mu)$ and every $\epsilon > 0$ there is a diffeomorphism $\tilde{f}$ with the aimed three properties such that $d_{\infty}\left(\tilde{f}, h \circ R_{\alpha} \circ h^{-1}\right) < \epsilon$. For this purpose, let $h \in \mathrm{Diff}^{\infty}(M,\mu)$ and $\epsilon > 0$ be arbitrary. By [Om74], p. 3, resp. [KM97], Theorem 43.1., $\mathrm{Diff}^{\infty}(M)$ is a Lie group. In particular, the conjugating map $g \mapsto h \circ g \circ h^{-1}$ is continuous with respect to the metric d_{∞}. Continuity in the point R_{α} yields the existence of $\delta > 0$, such that $d_{\infty}(g, R_{\alpha}) < \delta$ implies $d_{\infty}\left(h \circ g \circ h^{-1}, h \circ R_{\alpha} \circ h^{-1}\right) < \epsilon$. By Proposition 1.9 we can find a diffeomorphism f with the three properties and $d_{\infty}(f, R_{\alpha}) < \delta$. Hence, $\tilde{f} := h \circ f \circ h^{-1}$ satisfies $d_{\infty}\left(\tilde{f}, h \circ R_{\alpha} \circ h^{-1}\right) < \epsilon$ and has the aimed qualities because the properties are invariant under isomorphisms.
In order to prove the genericity of the aimed properties we use the same approach as in [AK70], section 7. For this purpose, we consider all possible sequences of diffeomorphisms $(f_n)_{n\in\mathbb{N}}$ satisfying the requirements of the inductive hypothesis and our construction from the previous sections. Let $U_n(f_n)$ be the subsequent neighbourhood of the diffeomorphism f_n:

$$U_n(f_n) :=$$
$$\left\{ g \in \mathrm{Diff}^{\infty}_{\mu}(M) : d_{k_n}(f_n, g) < \frac{2}{k_n}, \sum_{c \in \xi_n} \mu\left(g(c) \triangle f_n(c)\right) < \frac{1}{n m_n}, \sum_{d \in \Gamma_n} \mu\left(g(d) \triangle f_n(d)\right) < \frac{1}{n \tilde{q}_{n+1}} \right\}$$

By Θ_n we denote the union of all neighbourhoods $U_n(f_n)$ over all the diffeomorphisms f_n in the above mentioned sequences. Since the neighbourhoods $U_n(f_n)$ are open, the sets Θ_n are open as well. Then

$$\Theta := \bigcap_{n\in\mathbb{N}} \bigcup_{s \geq n} \Theta_s$$

is a G_{δ}-set as the countable intersection of open sets.

- For all the sequences $(f_n)_{n\in\mathbb{N}}$ built by our constructions from the previous sections the respective limit diffeomorphism $f \in \mathcal{A}_{\alpha}$ belongs to Θ because it belongs to $U_n(f_n)$ for every $n \in \mathbb{N}$ by construction. So Θ contains all the constructed diffeomorphisms with the aimed properties. Hence, it is dense in $\mathcal{A}_{\alpha}$ due to the above considerations.

- In the next step we want to show that $f \in \Theta$ admits a good linked approximation of type $(h, h+1)$ as well as a good cyclic approximation:
For any $f \in \bigcap_{n\in\mathbb{N}} \bigcup_{s\geq n} \Theta_s$ there is a sequence $(n_k)_{k\in\mathbb{N}}$ with $n_k \to \infty$ as $k \to \infty$, such that $f \in \Theta_{n_k}$. So there is a sequence $(f_{n_k})_{k\in\mathbb{N}}$ of diffeomorphisms, at which f_{n_k} is the n_k-th element of one of the above mentioned sequences of constructed diffeomorphisms, such that $f \in U_{n_k}(f_{n_k})$. We observe that $\xi_{n_k} \to \varepsilon$ as well as $\Gamma_{n_k} \to \varepsilon$ as $k \to \infty$, where ξ_{n_k} and Γ_{n_k} are the partitions belonging to the diffeomorphism f_{n_k}. Then f admits a good linked approximation of type $(h, h+1)$ as well as a good cyclic approximation by the definition of the neighbourhoods $U_{n_k}(f_{n_k})$.

Thus, the set of diffeomorphisms in $\mathcal{A}_{\alpha}$ admitting a good linked approximation of type $(h, h+1)$ as well as a good cyclic approximation contains a dense G_{δ}-set. Since these types of approximation imply the aimed properties, we conclude that the set of diffeomorphisms $f \in \mathcal{A}_{\alpha}$ with the following properties

- a good approximation of type $(h, h+1)$;

- a maximal spectral type disjoint with its convolutions;

- a homogeneous spectrum of multiplicity two for the Cartesian square $f \times f$

is a residual subset in the C^∞-topology. So the Theorem is deduced.

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