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DEFINABLE MAXIMAL INDEPENDENT FAMILIES

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ABSTRACT. We study maximal independent families (m.i.f.) in the projective hierarchy. We show that (a) the existence of a $\mathbf{\Sigma}^1_2$ m.i.f. is equivalent to the existence of a $\mathbf{\Pi}^1_1$ m.i.f., (b) in the Cohen model, there are no projective maximal independent families, and (c) in the Sacks model, there is a $\mathbf{\Pi}^1_1$ m.i.f. We also consider a new cardinal invariant related to the question of destroying or preserving maximal independent families.

1. INTRODUCTION

In descriptive set theory, a recurrent line of inquiry is whether objects defined in a non-constructive ways can exist on given levels of the projective hierarchy. For example, an ultrafilter cannot be $\mathbf{\Sigma}^1_1$. Mathias [7] proved that there are no $\mathbf{\Sigma}^1_1$ maximal almost disjoint (mad) families. If $V = L$ then there are $\mathbf{\Delta}^1_2$ ultrafilters and $\mathbf{\Pi}^1_1$ mad families.

In this paper, we look at *maximal independent families* from the definable point of view. Our main results are Theorem 2.1, Theorem 3.1 and Theorem 4.1, stating that the existence of a $\mathbf{\Sigma}^1_2$ m.i.f. is equivalent to the existence of a $\mathbf{\Pi}^1_1$ m.i.f., that in the Cohen model there are no projective m.i.f.'s, and that in the Sacks model there is a $\mathbf{\Pi}^1_1$ m.i.f., respectively.

Definition 1.1. A family $\mathcal{I} \subseteq [\omega]^\omega$ is *independent* if for all $a_1, \dots, a_n \in \mathcal{I}$ and different $b_1, \dots, b_\ell \in \mathcal{I}$

$$a_1 \cap \dots a_n \cap (\omega \setminus b_1) \cap \cdots \cap (\omega \setminus b_\ell) \text{ is infinite.}$$

A family $\mathcal{I} \subseteq [\omega]^\omega$ is called a *maximal independent family* (*m.i.f.*) if it is independent and maximal with regard to this property.

We will typically use the following abbreviation:

$$\sigma(\bar{a};\bar{b}) := a_1 \cap \dots a_n \cap (\omega \setminus b_1) \cap \cdots \cap (\omega \setminus b_\ell)$$

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where it will be assumed that all of the a_i are different from all of the b_j. Note that maximality of $\mathcal{I}$ is equivalent to:

$$\forall X \in [\omega]^\omega \; \exists a_1, \dots, a_n, b_1, \dots, b_\ell \in \mathcal{I} \text{ s.t. } \; \sigma(\bar{a}; \bar{b}) \subseteq^* X \; \text{ or } \; \sigma(\bar{a}; \bar{b}) \cap X =^* \varnothing.$$

By identifying the space $[\omega]^\omega$ with 2^ω via characteristic functions, one can consider independent families as subsets of the reals and study their complexity in the projective hieararchy.

Lemma 1.2. *If $\mathcal{I}$ is a $\mathbf{\Sigma}^1_n$ m.i.f. then it is $\mathbf{\Delta}^1_n$.*

Proof. Suppose $\mathcal{I}$ is a $\mathbf{\Sigma}^1_n$ m.i.f. Then $X \notin \mathcal{I}$ iff:

$$\exists a_1, \dots, a_n, b_1, \dots, b_\ell \in \mathcal{I} \; \text{ s.t. } \; X \notin \{a_1, \dots, a_n, b_1, \dots, b_\ell\} \; \text{ and}$$
$$\sigma(\bar{a}, \bar{b}) \subseteq^* X \; \text{ or } \; \sigma(\bar{a}, \bar{b}) \cap X =^* \varnothing.$$

This statement is easily seen to be $\mathbf{\Sigma}^1_n$. □

Theorem 1.3 (Miller; [8])**.** *There is no analytic m.i.f.*

An analysis of Miller's proof shows that it really only uses the Baire property of analytic sets. In particular, if we use $\mathbf{\Sigma}^1_n(\mathbb{C})$ to denote the statement "all $\mathbf{\Sigma}^1_n$ sets have the Baire property", then Miller's proof shows that for any n, $\mathbf{\Sigma}^1_n(\mathbb{C})$ implies that there are no $\mathbf{\Sigma}^1_n$ m.i.f. It follows that

(1) In the Hechler model, as well as the Amoeba or Amoeba-for-category model, there is no $\mathbf{\Sigma}^1_2$ m.i.f.,
(2) In the Solovay model (the Lévy-collapse of an inaccessible), as well as Shelah's model for the Baire Property without inaccessibles [9], there is no projective m.i.f.,
(3) In $L(\mathbb{R})$ of the above two models, there is no m.i.f. at all, and
(4) $\mathsf{AD} \Rightarrow$ there is no m.i.f.

In this paper, we prove a stronger result, namely, that in the Cohen model there is no projective m.i.f., and in the $L(\mathbb{R})$ of the Cohen model there is no m.i.f. at all. Notice that since $\mathbf{\Sigma}^1_2(\mathbb{C})$ is false in the Cohen model, this shows that the converse implication "$\mathbf{\Sigma}^1_n(\mathbb{C}) \Leftarrow \nexists \mathbf{\Sigma}^1_n$-m.i.f." consistently fails.

On the other hand, it is easy to construct a m.i.f. by induction using a wellorder of the reals, and thus, it is not hard to see that in L there exists a $\mathbf{\Sigma}^1_2$ m.i.f. In [8] Miller used sophisticated coding techniques to show that, in fact, this m.i.f. can be constructed in a $\mathbf{\Pi}^1_1$ fashion Building on an idea of Asger Törnquist [11], we show that in fact this proof is unnecessary since one can derive, directly in ZFC, that if there exists a $\mathbf{\Sigma}^1_2$ m.i.f. then there exists a $\mathbf{\Pi}^1_1$ m.i.f.

A construction originally attributed to Eisworth and Shelah (see [2]), implicitly appearing in Shelah's proof of $\mathfrak{i} < \mathfrak{u}$ [10] and elaborated in [5], yields a forcing for generically adding a Sacks-indestructible m.i.f. In this paper we show that this family can be defined in a $\mathbf{\Sigma}^1_2$ way in L. Therefore, in the countable support iteration of Sacks forcing, as well as the product of Sacks forcing, starting from L, there exists a $\mathbf{\Sigma}^1_2$ m.i.f., and hence a $\mathbf{\Pi}^1_1$ m.i.f. In fact, a slight modification produces a family iwhich is indestructible by the poset used in [10], which shows that the consistency of $\mathfrak{i} < \mathfrak{u}$ can be witnessed by a $\mathbf{\Pi}^1_1$ m.i.f.

2. $\mathbf{\Sigma}^1_2$ AND $\mathbf{\Pi}^1_1$ M.I.F'S

Theorem 2.1. *If there exists a $\mathbf{\Sigma}^1_2$ m.i.f. then there exists a $\mathbf{\Pi}^1_1$ m.i.f.*

Proof. Suppose $\mathcal{I}_0$ is a $\mathbf{\Sigma}^1_2$ maximal independent family. Let $F_0 \subseteq ([\omega]^\omega)^2$ be a $\mathbf{\Pi}^1_1$ set such that $\mathcal{I}_0$ is the projection of F_0. Consider the space $\omega \mathbin{\dot\cup} 2^{<\omega}$ as a disjoint union, and consider the mapping

$$g: \begin{array}{l} ([\omega]^\omega)^2 \longrightarrow \mathscr{P}(\omega \mathbin{\dot\cup} 2^{<\omega}) \\ (x,y) \longmapsto x \cup \{\chi_y \restriction n \mid n < \omega\} \end{array}$$

where χ_y is the characteristic function of y. It is not hard to see that g is a continuous function (in the sense of the space $\mathscr{P}(\omega \mathbin{\dot\cup} 2^{<\omega})$).

By $\mathbf{\Pi}^1_1$-uniformization, there exists a $\mathbf{\Pi}^1_1$ set $F \subseteq F_0$ which is the graph of a function, i.e., $\forall x \in \mathcal{I}_0 \; \exists! y \; ((x,y) \in F)$. We let $\mathcal{I} := g[F]$ and claim that $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ m.i.f.

To see that $\mathcal{I}$ is $\mathbf{\Pi}^1_1$, note that for $z \in [\omega \mathbin{\dot\cup} 2^{<\omega}]^\omega$, there is an explicit way to recover x and y such that $g(x,y) = z$, if such x and y exist. More precisely: if $B \subseteq 2^{<\omega}$, let $\lim(B) := \{y \in 2^\omega \mid \forall n \; (y \restriction n \in B)\}$. Note that if $B \cap 2^n \neq \varnothing$ for every n, then also $\lim(B) \neq \varnothing$. So we can say the following: $z \in \mathcal{I}$ if and only if

(1) $\forall n \; \exists s \in (z \cap 2^{<\omega}) \; (|s| = n)$,
(2) $\forall y, y' \; (y \in \lim(z \cap 2^{<\omega}) \wedge y' \in \lim(z \cap 2^{<\omega}) \; \rightarrow \; y = y')$,
(3) $\forall y \; (y \in \lim(z \cap 2^{<\omega}) \; \rightarrow \; (z \cap \omega, y) \in F)$.

This gives a $\mathbf{\Pi}^1_1$ definition of $\mathcal{I}$.

To see that $\mathcal{I}$ is independent, suppose we have $z_1, \dots z_n$ and $w_1, \dots w_\ell \in \mathcal{I}$, the z's being different from the w's. Write $a_i := z_i \cap \omega$ and $b_j := w_j \cap \omega$. Then all a_i and b_j are in $\mathrm{dom}(F) = \mathcal{I}_0$, and moreover, since F is a function, the a_i's are different from the b_j's. But then we have that $\sigma(z_1, \dots, z_n; w_1, \dots, w_\ell) \supseteq \sigma(a_1, \dots, a_n; b_1, \dots, b_\ell)$ is infinite, since the latter set is infinite by the independence of $\mathcal{I}_0$.

To show maximality of $\mathcal{I}$, suppose $W \in [\omega \mathbin{\dot\cup} 2^{<\omega}]^\omega$ and $W \notin \mathcal{I}$. Let $A := W \cap \omega$. By maximality of $\mathcal{I}_0$, there are $a_1, \dots, a_n \in \mathcal{I}_0$ and different $b_1, \dots, b_\ell \in \mathcal{I}_0$ such that $\sigma(a_1, \dots, a_n, A; \; b_1, \dots, b_\ell)$ is finite or $\sigma(a_1, \dots, a_n; \; b_1, \dots, b_\ell, A)$ is finite, w.l.o.g. the former. Then there are $z_1, \dots, z_n$ and different $w_1, \dots, w_\ell$ such that $a_i = z_i \cap \omega$ and $b_j = w_j \cap \omega$. To make sure that the "$2^{<\omega}$-part" of the z_i's and the w_j's does not make the intersection infinite, we pick two additional $t_0 \neq t_1 \in \mathcal{I}$, different from the z_i's and the w_j's. Let $t_0 = g(x_0, y_0)$ and $t_1 = g(x_1, y_1)$. If $y_0 = y_1$, then $(t_0 \setminus t_1) \cap 2^{<\omega} = \varnothing$, hence $\sigma(x_1, \dots, x_n, W, t_0; \; b_1, \dots, b_\ell, t_1)$ is finite. If, on the other hand, $y_0 \neq y_1$, then the sets $\{\chi_{y_0} \restriction n \mid n < \omega\}$ and $\{\chi_{y_1} \restriction n \mid n < \omega\}$ are almost disjoint, so $(t_0 \cap t_1) \cap 2^{<\omega}$ is finite. In that case, $\sigma(x_1, \dots, x_n, W, t_0, t_1; \; b_1, \dots b_\ell)$ is finite. So in any case, $\mathcal{I} \cap \{W\}$ is not independent, completing the proof. ☐

Clearly the above proof also holds pointwise for every parameter, i.e., if there is a $\Sigma^1_2(a)$ m.i.f. then there is a $\Pi^1_1(a)$ m.i.f. However, since $\mathbf{\Pi}^1_1$-uniformisation is essential, the following natural question remains open:

Question 2.2. Does the existence of a $\mathbf{\Sigma}^1_n$ m.i.f. imply the existence of a $\mathbf{\Pi}^1_{n-1}$ m.i.f., for $n > 2$?

3. Projective M.I.F.'s

Our main result is the following:

Theorem 3.1. *In the Cohen model there are no projective m.i.f.'s*

The theorem is proved in three steps. First, we isolate a new regularity property which was implicit in Miller's original proof.

Definition 3.2. A tree $T \subseteq 2^{<\omega}$ is called *perfect almost disjoint* (*perfect a.d.*) if it is a perfect tree and $\forall x \neq y \in [T]$ the set $\{n \mid x(n) = y(n) = 1\}$ is finite. A tree $S \subseteq 2^{<\omega}$ is called *perfect almost covering* (*perfect a.c.*) if it is a perfect tree and $\forall x \neq y \in [T]$, the set $\{n \mid x(n) = y(n) = 0\}$ is finite.

Definition 3.3. A set $X \subseteq 2^\omega$ satisfies the *perfect-a.d.-a.c. property*, abbreviated by $\mathbb{S}_{ad\text{-}ac}$, if there exists a perfect a.d. tree T with $[T] \subseteq X$, or there exists a perfect a.c. tree S with $[S] \cap X = \varnothing$.

Remark 3.4. Note that one could also define the symmetric property: $X \subseteq 2^\omega$ satisfies the *perfect-a.c.-a.d. property* if there exists a perfect a.c. tree S with $[S] \subseteq X$, or there exists a perfect a.d. tree T with $[T] \cap X = \varnothing$. A curious aspect of our proof is that either of these two properties yields the proof in an analogous fashion, but since one of them is sufficient we pick the former.

Lemma 3.5. $\boldsymbol{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac}) \Rightarrow \nexists \boldsymbol{\Sigma}^1_n$*-m.i.f.*

Proof. The maximality of $\mathcal{I}$ now implies that $[\omega]^\omega = H \cup K$. Moreover, H is a $\boldsymbol{\Sigma}^1_n$ set (K also is, but this turns out to be irrelevant).

From $\boldsymbol{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac})$ we then obtain a perfect almost disjoint tree T with $[T] \subseteq H$, or a perfect almost covering tree S with $[S] \cap H = \varnothing$, hence $[S] \subseteq K$. Assume the former.

For each $X \in [T]$ let $a_1^X, \dots, a_{n_X}^X$ and $b_1^X, \dots, b_{\ell_X}^X$ witness the fact that $X \in H$. Applying the Δ-systems lemma to the family $\{\{a_1^X, \dots, a_{n_X}^X, b_1^X, \dots, b_{\ell_X}^X\} \mid X \in [T]\}$, find an uncountable subset of $[T]$ with a fixed root R. Moreover, an uncountable sub-family of this family has the property that the elements of the root R have the same function in the sense of "being an a_i^X" or "being a b_j^X". It follows that, for distinct X, Y from this family, we have:

$$\{a_1^X, \dots, a_{n_X}^X, a_1^Y, \dots, a_{n_Y}^T\} \cap \{b_1^X, \dots, b_{\ell_X}^X, b_1^Y, \dots, b_{\ell_Y}^T\} = \varnothing.$$

But then the boolean combination $\sigma(\bar{a}^X \cup \bar{a}^Y; \bar{b}^X \cup \bar{b}^Y) \subseteq^* X \cap Y =^* \varnothing$, contradicting the independence of $\mathcal{I}$. $\square$

The next step is to show that Cohen forcing adds a perfect almost disjoint and perfect almost covering trees of Cohen reals.

Lemma 3.6. *Let $[s]$ be a basic open set and $c \in [s]$ a Cohen real over the ground model V. Then in $V[c]$ there exists a perfect almost disjoint set and a perfect almost covering set of Cohen reals over V, contained inside $[s]$.*

Proof. For simplicity assume that $[s] = 2^\omega$, and we only prove the case with almost disjoint trees since the other case is similar. Let $\mathbb{P}$ denote the partial order consisting of finite trees $T \subseteq 2^{<\omega}$ with the property that $\exists 0 = k_0 < k_1 < \cdots < k_\ell$ such that

$T \subseteq 2^{\leq k_\ell}$ and for every $i < \ell$, there is *at most one* $t \in T$ where $t\restriction[k_i, k_{i+1})$ is not constantly 0. The trees are ordered by end-extension.

If G is $\mathbb{P}$-generic, let T_G denote the naturally defined limit of the trees in G. By a standard genericity argument, $[T_G]$ must be perfect. Given any two branches $x, y \in [T_G]$, the construction ensures that for all n after the point where x and y split, either $x(n) = 0$ or $y(n) = 0$, therefore x and y are almost disjoint. To show that every $x \in [T_G]$ is Cohen over V, let D be Cohen-dense and $T \in \mathbb{P}$ fixed. Enumerate all terminal nodes of T by $\{t_1, \dots, t_k\}$. Extend

Since $\mathbb{P}$ is countable, it is isomorphic to Cohen forcing. Therefore, if $V[c]$ is a Cohen extension of V, it is also a $\mathbb{P}$-generic extension of V, so there exists a perfect almost disjoint set $[T_G]$ of Cohen reals. □

With these two components we can complete the proof of the theorem.

Proof of Theorem 3.1. Let $W := V^{\mathbb{C}_\kappa}$ (for any $\kappa > \omega$), and let A be a set in W defined by a formula $\Phi(x)$ with real or ordinal parameters, w.l.o.g. all of which are in V. In W, let c be Cohen over V, and assume w.l.o.g. that $\Phi(c)$. Then $V[c] \models$ "$p \Vdash_{\mathbb{Q}} \Phi(\check{c})$", where $\mathbb{Q}$ is the remainder forcing leading from $V[c]$ to W and p is some $\mathbb{Q}$-condition. However, since $\mathbb{C}_\kappa$ is the product forcing, $\mathbb{Q}$ is isomorphic to $\mathbb{C}_\kappa$. Moreover, since $\mathbb{C}_\kappa$ is homogeneous we can assume that p is the trivial condition, hence we really have:

$$V[c] \models \text{“} \Vdash_{\mathbb{C}_\kappa} \Phi(\check{c})\text{”}$$

Let $[s]$ be a Cohen condition with $c \in [s]$ forcing this statement in V. By Lemma 3.6, first we find a perfect a.d. tree T with $T \in V[c]$, $[T] \subseteq [s]$ and such that all $x \in [T]$ are Cohen over V. Note that this fact remains true in W, since "being a perfect set of Cohen reals" is upwards absolute. Now, for any such $x \in [T]$ (in W), we have that $x \in [s]$, and therefore $V[x]$ satisfies whatever $[s]$ forces, in particular

$$V[x] \models \text{“} \Vdash_{\mathbb{C}_\kappa} \Phi(\check{x})\text{”}$$

But, again, the remainder forcing leading from $V[x]$ to W is isomorphic to $\mathbb{C}_\kappa$, and it follows that $W \models \Phi(x)$.

Similarly, we also find a perfect a.c. tree S with exactly the same properties. Thus A satisfies both $\mathbb{S}_{\textit{ad-ac}}$ and $\mathbb{S}_{\textit{ac-ad}}$, and the rest follows by Lemma 3.5. □

4. $\mathbf{\Pi}^1_1$ m.i.f. in the Sacks model

In contrast to the above, this section is devoted to the following result:

Theorem 4.1. *In the countable-support iteration of Sacks forcing, as well as the countable-support product of Sacks forcing, starting from L, there exists a* $\mathbf{\Pi}^1_1$ *m.i.f.*

As a consequence, we obtain Con($\exists\mathbf{\Pi}^1_1$-m.i.f. of size $< 2^{\aleph_0}$), and in fact even Con($\exists\mathbf{\Pi}^1_1$-m.i.f. + $\mathfrak{i} < 2^{\aleph_0}$). Another consequence is the consistency of $\exists\mathbf{\Pi}^1_1$-m.i.f. together with "all $\mathbf{\Sigma}^1_2$ sets have the Marczewski-property" (where $X \subseteq 2^\omega$ has the *Marczewski-property* if every perfect set P contains is a perfect subset P' with $P' \cap X = \varnothing$), see [4, Theorem 7.1].

The construction we use appeared implicitly in [10] where, among other things, a forcing notion $\mathbb{P}$ for generically adding a Sacks-indestructible m.i.f. was isolated. These ideas were elaborated and studied further in [5]. Here we show that the combinatorics of this forcing can also be used to explicitly define a Sacks-indestructible m.i.f. in a model of CH, and that in L, such a Sacks-indestructible m.i.f. can be defined in a $\mathbf{\Sigma}^1_2$-fashion. We start by recalling some technical definitions from [5].

To reduce cumbersome notation, in this section the following will be useful:

Notation 4.2. If $\mathcal{I} \subseteq [\omega]^\omega$ then

- $\mathrm{FF}(\mathcal{I}) := \{h : \mathcal{I} \to 2 \mid |\mathrm{dom}(h)| < \omega\}$, and
- For $h \in \mathrm{FF}(\mathcal{I})$ we write

$$\sigma(h) := \bigcap\{A \mid A \in \mathrm{dom}(h) \ \wedge \ h(A) = 1\} \cap \bigcap\{\omega \setminus A \ \mid \ A \in \mathrm{dom}(h) \ \wedge \ h(A) = 0\}.$$

Definition 4.3. An independent family $\mathcal{I}$ is called a *densely maximal independent family* if for all $X \subseteq \omega$, for all $h \in \mathrm{FF}(\mathcal{I})$ there exists $h' \in \mathrm{FF}(\mathcal{I})$ with $h' \supseteq h$ such that $\sigma(h') \subseteq^* X$ or $\sigma(h') \cap X =^* \varnothing$.

Definition 4.4. Let $\mathcal{I}$ be an independent family. The *density ideal of* $\mathcal{I}$ is

$$\mathrm{id}(\mathcal{I}) := \{X \subseteq \omega \ \mid \ \forall h \in \mathrm{FF}(\mathcal{I}) \, \exists h' \in \mathrm{FF}(\mathcal{I}) \, (h' \supseteq h \ \wedge \ \sigma(h') \cap X =^* \varnothing)\}.$$

The dual filter is denoted by $\mathrm{id}^*(\mathcal{I})$.

Lemma 4.5. *If* $\mathcal{I} \subseteq \mathcal{I}'$ *then* $\mathrm{id}(\mathcal{I}) \subseteq \mathrm{id}(\mathcal{I}')$*, and if* $\mathcal{I} = \bigcup_{\alpha<\kappa} \mathcal{I}_\alpha$ *for a regular* κ*, then* $\mathrm{id}(\mathcal{I}) = \bigcup_{\alpha<\kappa} \mathrm{id}(\mathcal{I}_\alpha)$.

Proof. The first statement is straightforward, and for the second statement, if $X \in \mathrm{id}(\mathcal{I})$ then we can let $\alpha < \kappa$ be the last ordinal closed under the $h \mapsto h'$ operation given by the definition of $\mathrm{id}(\mathcal{I})$. □

Recall that a filter $\mathcal{F}$ on ω is a *p-filter* iff for every $\{X_n \mid n < \omega\} \subseteq \mathcal{F}$ there exists $X \in \mathcal{F}$ with $X \subseteq^* X_n$ for all n ("X is a pseudointersection of the X_n's"). A filter $\mathcal{F}$ on ω is a *q-filter* if for every partition of ω into finite sets $\mathcal{E} = \{E_n \mid n < \omega\}$, there is $X \in \mathcal{F}$ such that $|X \cap E_n| \leq 1$ for all n ("X is a semiselector for $\mathcal{E}$"). A filter $\mathcal{F}$ is a *Ramsey filter* if it is both a p-filter and a q-filter (cf. [1, Section 4.5.A]). The main ingredient in our proof is the following result:

Theorem 4.6 ([10], [5, Corollary 37])**.** *Let* $\mathcal{I}$ *be a densely maximal independent family, such that the dual filter* $\mathrm{id}^*(\mathcal{I})$ *is generated by a Ramsey filter and the filter of cofinite sets (Fréchet filter). Then* $\mathcal{I}$ *remains maximal after a countable-support iteration of Sacks forcing, as well as a countable-support product of Sacks forcing.*

Definition 4.7. Let $\mathbb{P}$ be the forcing poset of all pairs $(\mathcal{A}, A)$ where $\mathcal{A}$ is a countable independent family, $A \in [\omega]^\omega$, and for all $h \in \mathrm{FF}(\mathcal{I})$, $\sigma(h) \cap A$ is infinite. The ordering is given by $(\mathcal{A}', A') \leq (\mathcal{A}, A)$ iff $\mathcal{A}' \supseteq \mathcal{A}$ and $A' \subseteq^* A$.

In [5, 10] this forcing was used to generically add a Sacks-indestructible m.i.f. Here, rather than forcing with $\mathbb{P}$ we will be using it in a purely combinatorial fashion to construct a Sacks-indestructible m.i.f. in a model of CH, and, in particular, a $\mathbf{\Sigma}^1_2$ m.i.f. in L.

The following properties of $\mathbb{P}$ were proved in [10, 5]:

Lemma 4.8.

(1) $\mathbb{P}$ *is* σ*-closed.*
(2) *If* $(\mathcal{A}, A) \in \mathbb{P}$ *then there exists* $B \subseteq A$ *such that* $B \notin \mathcal{A}$ *and* $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$.

(3) *If $Y \subseteq \omega$ is an arbitrary set, then for every $(\mathcal{A}, A) \in \mathbb{P}$ there exists $(\mathcal{B}, B) \leq (\mathcal{A}, A)$ such that*

$$\forall h \in \mathrm{FF}(\mathcal{B})\ \exists h' \in \mathrm{FF}(\mathcal{B}) \text{ s.t. } h' \supseteq h \text{ and } \sigma(h') \subseteq^* Y \text{ or } \sigma(h') \cap Y =^* \varnothing.$$

(4) *Let $\mathcal{E} := \{E_n \mid n < \omega\}$ be a partition of ω into finite sets. Then for every $(\mathcal{A}, A) \in \mathbb{P}$ there is $B \subseteq A$ such that $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and $|B \cap E_n| \leq 1$ for all n ("B is a semiselector for $\mathcal{E}$.").*

(5) *For all $(\mathcal{A}, A) \in \mathbb{P}$, if $X \in \mathrm{id}(\mathcal{A})$ then there is B such that $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and $B \cap X = \varnothing$.*

Proof. See Proposition 15, Lemma 17, Corollary 19 and Lemma 14 from [5], respectively. □

Definition 4.9. We call $\{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}$ an *indestructibility tower*, if it is a strictly decreasing sequence of $\mathbb{P}$-conditions and, letting $\mathcal{A} := \bigcup_{\alpha\in\omega_1} \mathcal{A}_\alpha$, the following four requirements are satisfied:

(1) For every $Y \subseteq \omega$, for every $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \in \mathrm{FF}(\mathcal{A})$ with $h' \supseteq h$ such that $\sigma(h') \subseteq^* Y$ or $\sigma(h') \cap Y =^* \varnothing$).

(2) For every partition $\mathcal{E} := \{E_n \mid n < \omega\}$ of ω into finite sets, there is $\alpha < \omega_1$ such that $|A_\alpha \cap E_n| \leq 1$ for all n (A_α is a semiselector for $\mathcal{E}$).

(3) For each $\alpha < \omega_1$ there is an infinite $A \subseteq^* A_\alpha$ such that $A \in \mathcal{A}_{\alpha+1} \setminus \mathcal{A}_\alpha$.

(4) For every $X \in \mathrm{id}(\mathcal{A})$ there is an $\alpha < \omega_1$ such that $X \cap A_\alpha =^* \varnothing$.

Lemma 4.10. *If $\{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}$ is an indestructibility tower, then $\mathcal{A} := \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ is a m.i.f. which remains maximal after a countable-support iteration and a countable-support product of Sacks forcing.*

Proof. In light of Theorem 4.6 it suffices to show that $\mathcal{A}$ is a densely maximal family and that $\mathrm{id}^*(\mathcal{I})$ is generated by a Ramsey filter and the filter of cofinite sets. Dense maximality follows immediately from condition (1). For (2), we show the following:

Claim. $\mathrm{id}(\mathcal{A})$ is generated by $\{\omega\backslash A_\alpha \mid \alpha < \omega_1\}$ and $[\omega]^{<\omega}$.

Proof of claim. Since the A_α's are wellordered by $\supseteq^*$, it suffices to show that $X \in \mathrm{id}(\mathcal{A})$ iff $X \cap A_\alpha =^* \varnothing$ for some $\alpha < \omega_1$. The forward direction holds by condition (4). For the other direction, suppose $X \cap A_\alpha =^* \varnothing$ and let $h \in \mathrm{FF}(\mathcal{A})$ be arbitrary. Let $\beta \geq \alpha$ be such that $h \in \mathrm{FF}(\mathcal{A}_\beta)$. By (3) there is an infinite $B \subseteq^* A_\beta$ such that $B \in \mathcal{A}_{\beta+1} \setminus \mathcal{A}_\beta$. In particular, $B \notin \mathrm{dom}(h)$, so we can extend h to form $h' := h \cup \{(B, 1)\}$. Then $h' \in \mathrm{FF}(\mathcal{A}_{\beta+1})$, and moreover $\sigma(h') \subseteq B \subseteq^* A_\beta \subseteq^* A_\alpha$. Therefore $\sigma(h') \cap X \subseteq^* A_\alpha \cap X =^* \varnothing$. This shows that $X \in \mathrm{id}(\mathcal{A})$ and completes the proof. □(Claim)

Notice that since $\{A_\alpha \mid \alpha < \omega_1\}$ is a tower, the filter it generates is a p-filter. Moreover, by condition (2), it is a q-filter, and thus a Ramsey filter, as we had to show. □

Theorem 4.11.

(1) *If CH holds then there exists an indestructibility tower.*

(2) *If $V = L$ then there exists a $\mathbf{\Sigma}^1_2$-definable indestructibility tower.*

Proof. We give a detailed proof of the first assertion and then show how to adapt it to get a $\mathbf{\Sigma}^1_2$ construction in L.

(1) Let $\{X_\alpha \mid \alpha < \omega_1\}$ enumerate all subsets of ω and let $\{\mathcal{E}_\alpha \mid \alpha < \omega_1\}$ enumerate all partitions of ω into finite sets.

Let $(\mathcal{A}_0, A_0) \in \mathbb{P}$ be any condition. At stage α, suppose $(\mathcal{A}_\beta, A_\beta)$ for all $\beta \leq \alpha$ has been constructed. The new condition is designed in four steps:

- Consider the sets $\{X_\beta \mid \beta \leq \alpha\}$. By repeatedly applying Lemma 4.8 (3) in countably many steps, followed by σ-closure which holds due to Lemma 4.8 (1), we find an extension $(\mathcal{A}'_\alpha, A'_\alpha) \leq (\mathcal{A}_\alpha, A_\alpha)$ such that, for all $\beta \leq \alpha$, for all $h \in \mathrm{FF}(\mathcal{A}_\alpha)$ (not necessarily for all $h \in \mathrm{FF}(\mathcal{A}'_\alpha)$) there exists $h' \in \mathrm{FF}(\mathcal{A}'_\alpha)$, such that $h' \supseteq h$ and $\sigma(h') \subseteq^* X_\beta$ or $\sigma(h') \cap X_\beta =^* \varnothing$.
- Consider the partition $\mathcal{E}_\alpha = \{E^n_\alpha \mid n < \omega\}$. By Lemma 4.8 (4) we find an extension $(\mathcal{A}''_\alpha, A''_\alpha) \leq (\mathcal{A}'_\alpha, A'_\alpha)$ such that $|A''_\alpha \cap E^n_\alpha| \leq 1$ for all n (A''_α is a semi-selector for $\mathcal{E}_\alpha$).
- Consider (again) the sets $\{X_\beta \mid \beta \leq \alpha\}$. By repeatedly applying Lemma 4.8 (5) in countably many steps, followed by σ-closure, we find a further extension $(\mathcal{A}'''_\alpha, A'''_\alpha) \leq (\mathcal{A}''_\alpha, A''_\alpha)$ such that, for every β, if $X_\beta \in \mathrm{id}(\mathcal{A}_\alpha)$ (which implies that $X \in \mathrm{id}(\tilde{A})$ for any $\tilde{A}$ extending $\mathcal{A}_\alpha$), then $A'''_\alpha \cap X_\beta =^* \varnothing$.
- Finally, use Lemma 4.8 (2) to find a $B \subseteq^* A'''_\alpha$, such that $B \notin \mathcal{A}'''_\alpha$, and $(\mathcal{A}'''_\alpha \cup \{B\}, A'''_\alpha)$ is a condition. We let $(\mathcal{A}_{\alpha+1}, A_{\alpha+1})$ be that condition.

This completes the construction of the induction step (in steps 2 and 3 we could in fact have taken $\mathcal{A}'''_\alpha = \mathcal{A}''_\alpha = \mathcal{A}'_\alpha$ but that is not relevant). At limit stages λ, use σ-closure to again find a condition $(\mathcal{A}_\lambda, A_\lambda)$ which extends all $(\mathcal{A}_\alpha, A_\alpha)$ for $\alpha < \lambda$.

It is now easy to verify that $\{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}$ satisfies conditions (1)–(4), where for (4) we use the fact that if $X \in \mathrm{id}(\mathcal{A})$ then $X \in \mathrm{id}(\mathcal{A}_\alpha)$ for some $\alpha < \omega_1$, see Lemma 4.5.

(2) If $V = L$ then repeat the same proof, but additionally, pick the canonical well-order $<_L$ of the reals of L to well-order the sequences $\{X_\alpha \mid \alpha < \omega_1\}$ and $\{\mathcal{E}_\alpha \mid \alpha < \omega_1\}$. At each step α of the construction, the preceding proof shows how to find an $(\mathcal{A}_{\alpha+1}, A_{\alpha+1})$ satisfying certain requirements. Now, we make sure to always pick the $<_L$*-least* condition $(\mathcal{A}_{\alpha+1}, A_{\alpha+1})$ satisfying the same requirements.

This way, it follows that the construction at each step α only depends on the preceding $\beta \leq \alpha$ and is thus absolute between L and an L_δ for some appropriate $\delta < \omega_1$. More precisely, if $\mathfrak{A} = \{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}$, then there is a formula Φ defining $\mathfrak{A}$ in an absolute way, i.e., $(\mathcal{A}, A) \in \mathfrak{A}$ iff $\Phi(\mathcal{A}, A)$ iff there exists $\delta < \omega_1$ such that $L_\delta \models \Phi(\mathcal{A}, A)$.

Let ZFC^* be a sufficiently large fragment of ZFC such that if a transitive model $M \models \mathsf{ZFC}^* + V = L$ then $M = L_\xi$ for some ξ. Now we can write $\Phi(\mathcal{A}, A)$ iff $\exists E \subseteq \omega \times \omega$ such that

- E is well-founded,
- $(\omega, E) \models \mathsf{ZFC}^* + V = L$,
- $(\omega, E) \models \Phi(\pi^{-1}(\mathcal{A}, A))$, where $\pi : (\omega, E) \cong (M, \epsilon)$ is the transitive collapse of (ω, E).

By standard methods (cf. [6, Proposition 13.8 ff.]) the two latter statements are arithmetic and well-foundedness is $\mathbf{\Pi}^1_1$. Thus $\Phi(\mathcal{A}, A)$ is equivalent to a $\mathbf{\Sigma}^1_2$ statement. □

Proof of Theorem 4.1. Let $\mathfrak{A} = \{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}$ be a $\mathbf{\Sigma}^1_2$-definable indestructibility tower in L. If V is the extension in the iteration/product of Sacks forcing, then $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ is still a maximal independent family with a $\mathbf{\Sigma}^1_2$ definition. By Theorem 2.1, there exists a $\mathbf{\Pi}^1_1$ m.i.f. as well. □

Remark 4.12. In Shelah's proof of the consistency of $\mathfrak{i} < \mathfrak{u}$ [10] a forcing closely related to Sacks was used, which increases $\mathfrak{u}$ as well as the continuum (note that in the Sacks model $\mathfrak{u} < 2^{\aleph_0}$). By a slight modification of the method in this section, it is easy to construct a $\mathbf{\Sigma}^1_2$ m.i.f. which is not only Sacks-indestructible, but indestructible by the poset from [10]. This shows that the witness for the m.i.f. in the proof of the consistency of $\mathfrak{i} < \mathfrak{u}$ can in fact be $\mathbf{\Pi}^1_1$-definable.

5. $\aleph_1$-Borel and $\aleph_1$-closed m.i.f's

The question of definable m.i.f's is closely related to questions concerning certain cardinal invariants (compare with [3]).

Definition 5.1.

(1) $\mathfrak{i}$ is the least size of a m.i.f.
(2) $\mathfrak{i}_{\text{cl}}$ is the least κ such that there exists a collection $\{C_\alpha \mid \alpha < \kappa\}$, where each C_α is a *closed* independent family, and $\bigcup_{\alpha<\kappa} C_\alpha$ is a m.i.f.
(3) $\mathfrak{i}_B$ is the least κ such that there exists a collection $\{B_\alpha \mid \alpha < \kappa\}$, where each B_α is a *Borel* independent family, and $\bigcup_{\alpha<\kappa} B_\alpha$ is a m.i.f.

It is clear that $\mathfrak{i}_B \leq \mathfrak{i}_{\text{cl}} \leq \mathfrak{i}$. It is also known that $\mathfrak{r} \leq \mathfrak{i}$ and $\mathfrak{d} \leq \mathfrak{i}$, where $\mathfrak{d}$ and $\mathfrak{r}$ denote the dominating and reaping numbers, respectively. Notice that if $\mathfrak{i}_B > \aleph_1$, then there are no $\mathbf{\Sigma}^1_2$ m.i.f.'s (since $\mathbf{\Sigma}^1_2$-sets are $\aleph_1$-unions of Borel sets).

Theorem 5.1. $\text{cov}(\mathcal{M}) \leq \mathfrak{i}_B$.

Proof. Let $\kappa < \text{cov}(\mathcal{M})$ and let $\{B_\alpha \mid \alpha < \kappa\}$ be a collection of Borel independent families. We need to show that $\mathcal{I} := \bigcup_{\alpha<\kappa} B_\alpha$ is not maximal.

Suppose otherwise, and for every finite $E \subseteq \kappa$ define

Let $\mathcal{I}$ be $\mathbf{\Sigma}^1_n$, and assume, towards contradiction, that $\mathcal{I}$ is a m.i.f. Let

$$H_E := \{X \mid \exists \bar{a}, \bar{b} \in \bigcup_{\alpha \in E} B_\alpha \text{ s.t. } \sigma(\bar{a}; \bar{b}) \subseteq^* X\}$$

$$K_E := \{X \mid \exists \bar{a}, \bar{b} \in \bigcup_{\alpha \in E} B_\alpha \text{ s.t. } \sigma(\bar{a}; \bar{b}) \cap X =^* \varnothing\}.$$

Notice that by maximality of $\mathcal{I} = \bigcup_{\alpha<\kappa} B_\alpha$, we have

$$\bigcup \{H_E \cup K_E \mid E \in [\kappa]^{<\omega}\} = [\omega]^\omega.$$

Since $\kappa < \mathfrak{d} = \text{cov}(K_\sigma)$, there must exist a finite $E \subseteq \kappa$ such that $H_E \cup K_E \notin \mathcal{M}$. Suppose $H_E \notin \mathcal{M}$: since H_E is analytic, there exists a basic open $[s]$ with $[s] \subseteq^* H_E$ (where $\subseteq^*$ means "modulo meager"). By an argument similar to the one in Lemma 3.6, we can construct a perfect a.d. tree T with $[T] \subseteq H_E$. But then, by the argument from Lemma 3.5, it follows that $\bigcup_{\alpha \in E} B_\alpha$ is not independent, contrary to the assumption. Likewise, if $K_E \notin \mathcal{M}$ then using the argument from Lemma 3.6, there exists a perfect a.c. tree S with $[S] \subseteq H_K$, and the rest is the same. □

We end this paper with the following open questions:

Question 5.2.

(1) Does the existence of a $\mathbf{\Sigma}^1_{n+1}$ m.i.f. imply the existence of a $\mathbf{\Pi}^1_n$ m.i.f. for $n > 2$?
(2) Is the existence of a $\mathbf{\Pi}^1_1$ m.i.f. consistent with $\mathfrak{i} > \aleph_1$? Is it consistent with $\mathfrak{d} > \aleph_1$ or $\mathfrak{r} > \aleph_1$?
(3) What about a $\mathbf{\Pi}^1_2$ m.i.f.?
(4) Is it consistent that $\mathfrak{i}_{\rm cl} < \mathfrak{d}$ or $\mathfrak{i}_B < \mathfrak{d}$?
(5) Is it consistent that $\mathfrak{i}_{\rm cl} < \mathfrak{r}$ or $\mathfrak{i}_B < \mathfrak{r}$?
(6) Is it consistent that $\mathfrak{i}_{\rm cl} < \mathfrak{i}$ or $\mathfrak{i}_B < \mathfrak{i}$?

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