Regularity Properties and Definability in the Real Number Continuum

Idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy

Yurii Khomskii

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Contents

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Yurii Khomskii

Chapter 1

Introduction

1.1 History and motivation

The subject of this dissertation is the interplay between regularity properties and definability in the real number continuum. By “regularity properties”, we are referring to certain desirable properties of sets of real numbers, something that makes them well-behaved, conforming to our intuition, or easy to study; by “definability”, we are referring to the logical description that determines the composition of such sets, given in some specified formal or informal language.

Real numbers are ubiquitous in nearly all areas of science and mathematics. They are indispensable for calculations involving the physical world, and we use them to describe reality and to model our three-dimensional Euclidean space.

Although real numbers were known since antiquity, their exact nature was not clearly understood and their use was limited to concrete cases, such as the number π in geometry. The first use of the concept of a real number in its full and unbridled form was the development of calculus in the late 17th century, which required an abstract treatment of convergent sequences and limit processes rather than individual numbers. Even then, the precise nature of these numbers was left undefined, and it was not until the late 19th century that the *real number continuum* was given a proper mathematical definition and recognized as a unique and extremely important concept.

As 19th century mathematics progressed and its methods became more advanced, increasingly complex and counter-intuitive aspects of the continuum were being discovered. It became commonplace for mathematicians to construct objects that were highly irregular, paradoxical or otherwise bizarre. In this light, regularity properties provided a counterweight, a way of saying when a certain object was not too unnatural. At the turn of the century, three particular properties were isolated, and in the decades that followed the study of these properties would become crucial in the search for structure in the real number continuum.

The first such property arose from the need for a more rigorous definition of the integral. As the formerly used Riemann integral did not suffice to deal with the complexity of late 19th century analysis, Henri Lebesgue, in his thesis [Leb02] from 1902, introduced the Lebesgue integral which is still widely in use today. His definition depended crucially on what we now call *Lebesgue measure*, a mathematical concept capturing the intuition of "size" or "volume" of an object. Accordingly, a set is called *Lebesgue measurable* if it admits such a notion of size.

The second property was isolated by René-Louis Baire [Bai99] in his study of topological properties of the continuum. With the notions of *open* and *closed* sets having already been established as central to analysis and topology, Baire looked at sets which are "almost" open, that is, open except for a (in a topologically precise sense) negligible component. Such sets are now said to have the *property of Baire*.

The last notion of regularity came out of foundational concerns. After Georg Cantor establishing that the set of real numbers was uncountable in 1874, a considerable amount of effort was devoted to trying to prove the Continuum Hypothesis, the statement that any uncountable set has at least the cardinality of the continuum. One approach involved *perfect sets*, i.e., sets of reals that are closed and contain no isolated points, which always have the cardinality of the continuum. If one could prove that every uncountable subset of the continuum contains a perfect set, then one would have established the Continuum Hypothesis at least within the realm of the real numbers. This dichotomy (which, of course, no one was able to prove) became known as the *perfect set property*.

Not long after these three properties were isolated, counterexamples were produced. The earliest example was due to Giuseppe Vitali [Vit05] in 1905, who constructed an object that could not be Lebesgue measurable or have the property of Baire (the *Vitali set*). Shortly afterwards, Felix Bernstein [Ber08] produced an object which, additionally, did not satisfy the perfect set property either (the *Bernstein set*). Other, more involved constructions were soon discovered, some of them leading to outright bizarre results such as the Banach-Tarski paradox, which draws crucially on the existence of non-measurable sets.

Of course, it was becoming clear that all such proofs were non-constructive, in the sense that they did not provide a concrete example of the irregular object whose existence they proved. Rather, this existence was established indirectly, using an evocation of the Axiom of Choice, a fundamental principle of set theory. It was, in part, due to these paradoxical consequences that this axiom was considered problematic and viewed with a great deal of skepticism at the time.

But while for some mathematicians it was the Axiom of Choice that was the main culprit, with the existence of irregularities and the corresponding paradoxes providing sufficient evidence for the eviction of this particular axiom from the domain of mathematics, others preferred to focus on the *definability* of subsets of the continuum, to make precise what "non-constructive" or "non-definable" meant, and to pay attention to the definable sets while admitting that non-definable ones existed, too. A clear case for definability was provided by Émile Borel, who defined a natural algebra of sets, the *Borel sets*, as those obtained from the open ones using the operations of countable union, countable intersection and complementation. Each Borel set comes along with a description, a definition, a recipe for its construction, so to say. Thus, Borel sets are quite the opposite of the non-constructive objects given to us by an abstract principle like the Axiom of Choice. Indeed, as was already implicit in the original definitions, all Borel sets are Lebesgue measurable and satisfy the property of Baire. In 1916, it was proved by Felix Hausdorff [Hau16] and independently by Pavel Aleksandrov [Ale16] that all Borel sets satisfy the perfect set property as well. This was a satisfactory situation as it meant that irregularities, even if they did exist, would not show up on the level of the Borel sets. Since the Borel algebra is closed under certain set-theoretic operations, one might attempt to squeeze all mathematical practice into the realm of Borel sets, thus avoiding any anomalies or irregularities.

But while the Borel algebra is fairly rich, there are natural mathematical operations that transcend its boundaries. This was first noticed by Mikhail Suslin in 1917, who, while studying [Leb02], found that Lebesgue had made a remarkable mistake: he had claimed that the projection of a Borel set (in a higher dimension) was itself Borel. Suslin constructed a counterexample to Lebesgue's claim, and, motivated by this discovery, introduced the class of *analytic* sets as those obtained from Borel sets by the operation of projection. As the analytic sets were still easily definable but lay beyond the Borel level, Suslin proceeded to investigate their properties, particularly in relation to their regularity. In [Sus17] he was able to prove that all analytic sets are Lebesgue measurable, have the property of Baire and the perfect set property. So no irregularity could occur on the analytic level, either.

As the class of analytic sets is not closed under complements, one may consider, as a separate definability class, the *co-analytic* sets, i.e., those sets whose complement is analytic. Suslin's result implied that co-analytic sets are also Lebesgue measurable and have the property of Baire, although no such conclusion could be drawn regarding the perfect set property. Of course, there is no reason to stop here, either, and if one considers the projections of co-analytic sets one gets to a strictly higher definability level, the $\boldsymbol{\Sigma}^1_2$ sets in contemporary terminology. The sets whose complement is a $\boldsymbol{\Sigma}^1_2$ set are called $\boldsymbol{\Pi}^1_2$, and the projections of $\boldsymbol{\Pi}^1_2$ sets lead to an even higher definability level, the $\boldsymbol{\Sigma}^1_3$ sets, etc. In this fashion one obtains the *projective hierarchy*, and a set is called *projective* if it appears at some finite level in it, i.e., if it is $\boldsymbol{\Sigma}^1_n$ or $\boldsymbol{\Pi}^1_n$ for some n.

The projective hierarchy was understood to be a very natural measure of definability, and the investigation of it led to a distinct area of mathematics now called *descriptive set theory* (the study of "descriptions", or "definitions", of sets). The next challenge was to show that the projective sets satisfied all the regularity properties. However, efforts in this direction already grounded on the first level beyond Suslin's result: it was impossible to determine whether all $\mathbf{\Sigma}^1_2$ sets were Lebesgue measurable or had the property of Baire, and whether all co-analytic sets satisfied the perfect set property. The obstacles encountered in this problem were so severe that some mathematicians were prompted to speculate on its potential "unsolvability". Nikolai Luzin, an early proponent of descriptive set theory, described the state of affairs in 1925 thus:

> "The theory of analytic sets presents a perfect harmony: any analytic set is either countable or of the cardinality of the continuum; an analytic set is never a set of the third category [satisfies the Baire property] ... finally, an analytic set is always measurable.
>
> There remains but one significant gap: one does not know whether every complementary analytic (that is, the complement of an analytic) uncountable set has the cardinality of the continuum.
>
> The efforts that I exerted on the resolution of this question led me to the following totally unexpected discovery: there exists a family ... consisting of effective [definable] sets, such that one does not know *and one will never know* whether every set from this family, if uncountable, has the cardinality of the continuum, nor whether it is of the third category, nor whether it is measurable. ... This is the family of the *projective sets* of Mr. H. Lebesgue. It remains but to recognize the nature of this new development." [Luz25, p 1572]

At the time, people were not yet aware of the "incompleteness phenomenon" in mathematics, so it is unlikely that Luzin had any rigorous notion in mind when saying "one will never know". Nevertheless, his predictions turned out to be correct, and the next step towards a clarification of this problem came in 1938 from Kurt Gödel's foundational work. By then, the axiomatization of mathematics using ZFC (Zermelo-Fraenkel with Choice) in first order logic had become standard and Gödel's incompleteness theorem had already been proved. In [Göd38], Gödel defined the *constructible universe* L, a so-called "inner model", a "sub-universe" within the universe of all sets, which was itself a model of all the axioms of ZFC as well as additional axioms, most notably the Continuum Hypothesis. In [Göd38] Gödel announced that in L there is a $\mathbf{\Sigma}^1_2$ non-Lebesgue-measurable set of reals and a co-analytic set without the perfect set property (a $\mathbf{\Sigma}^1_2$ set without the property of Baire can be derived from the same proof). In meta-mathematical terms, it meant that these assertions were *consistent* with the axioms of set theory, i.e., one would never be able to prove that all $\mathbf{\Sigma}^1_2$ sets (and therefore, all projective sets) are Lebesgue measurable and satisfy the property of Baire, nor that all co-analytic sets have the perfect set property, at least, assuming only the basic axioms of set theory. So at least one half of Luzin's conjecture turned out to be correct.

Was it, perhaps, possible to prove the opposite statement, namely that there *are* irregularities on some given level of the projective hierarchy? That this could not be done, either, was shown by a celebrated result of Robert Solovay, but had to wait until 1970. Using a then recently discovered method called *forcing*, Solovay [Sol70] constructed a model of set theory in which all projective sets were Lebesgue measurable, had the property of Baire and the perfect set property. Thus, if one interprets Luzin's "one will never know" as "it is not provable from ZFC", then this is indeed correct: the regularity of all projective sets is undecidable by the fundamental axioms of set theory.

An even more exciting consequence of [Sol70] was the existence of a model of ZF (without Choice) in which *every* subset of the continuum was Lebesgue measurable, had the property of Baire and the perfect set property. This meant that the original use of the Axiom of Choice in constructing counterexamples to regularity properties was perfectly justified, as it would simply have been impossible to construct such counterexamples without it.

Using the method of forcing, models of set theory could be extended to produce larger models, in which the truth of a certain mathematical statements could be controlled to some degree. While the forcing used to construct the Solovay model was rather strong, requiring an inaccessible cardinal to work properly and yielding an extension very much larger than Gödel's constructible universe L, this was not the case for the $\mathbf{\Sigma}^1_2$ level. To obtain a model where all $\mathbf{\Sigma}^1_2$ sets were measurable, it sufficed to use a relatively mild extension. In fact, a characterization result was proved in [Sol70] stating that all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable *if and only if* the set-theoretic universe is at least as large as a certain forcing extension of L (for a precise statement, see Theorem 1.3.9). In other words, there was a *minimal* way to extend L in order to obtain a model in which every $\mathbf{\Sigma}^1_2$ set was Lebesgue measurable. A similar characterization was shown for the Baire property. With this at hand, one could have direct control over the truth of the statements "all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable" and "all $\mathbf{\Sigma}^1_2$ sets satisfy the Baire property" in different models of set theory.

Until the 1980s, the Lebesgue measure and Baire property were considered virtually analogous, and results proved for one could be translated to yield the same result for the other. The first change to this was brought on by Saharon Shelah's [She84], which showed that the Baire property of all projective sets could also be established by a forcing not requiring an inaccessible cardinal to begin with, whereas this was *not* true for Lebesgue measure. In terms of consistency strength, the statement "all projective sets are Lebesgue measurable" was stronger than the statement "all projective sets satisfy the property of Baire". Shortly afterwards, Jean Raisonnier and Jacques Stern [RS85], and independently Tomek Bartoszyński [Bar84], uncovered an asymmetry already inherent at the $\mathbf{\Sigma}^1_2$ level—namely, *if* all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable *then* all $\mathbf{\Sigma}^1_2$ sets satisfy the property of Baire. The converse implication, on the other hand, does not hold (by [Iho88]). Here, too, measurability turned out to be stronger than the Baire property.

By this time, the concept of a "regularity property" had been extended far beyond the three classical cases we have been discussing so far. For one, the Baire property could be generalized to other topological spaces, and even to partial orders in general. A number of statements in infinitary combinatorics gave rise to natural notions of regularity, from which the *Ramsey property* is probably the most well-known. Even the perfect set property turned out to be one in a line of similar "dichotomy-style" properties. In each case, the same pattern emerged: with the Axiom of Choice one can construct counterexamples, but this axiom is provably necessary; all Borel and analytic sets are regular; there are $\mathbf{\Sigma}^1_2$ or even co-analytic counterexamples in L, but not if sufficient forcing over L has been done, etc.

One of the more interesting aspects of these properties is their asymmetry, and the low levels of the projective hierarchy are very well suited to study it. Just like the asymmetry between Lebesgue measure and the Baire property, a similar phenomenon tends to appear with other notions of regularity. Thus, the hypothesis that, say, all $\mathbf{\Sigma}^1_2$ or all co-analytic sets are regular in one sense may directly imply that the analogous hypothesis concerning another notion of regularity holds. In other cases, this direct implication is (consistently) false. For proving such results, a characterization theorem relating regularity to some forcing-theoretic statement, like the one proved by Solovay in [Sol70], is always very useful. Research in this direction has been done by Haim Judah, Saharon Shelah, Jörg Brendle, Benedikt Löwe and Lorenz Halbeisen among others (see [IS89, Iho88, BL99, BHL05, BL11]), and a general theorem unifying many kinds of regularity properties was proved by Daisuke Ikegami in [Ike10a, Ike10b].

All the questions studied in this dissertation concern this basic relationship between regularity and definability, which has been established throughout the course of the 20th century. Of particular importance are the asymmetry, the implications and non-implications between the various notions of regularity and the characterization of it using transcendence over L. Another, relatively distinct, interest of ours is the study of special kinds of *irregular objects*. The Vitali set and the Bernstein set were already mentioned as counterexamples to regularity; another one would be a *non-principal ultrafilter* on the natural numbers, which, if considered from a topological point of view, gives rise to a set that is both non-measurable and doesn't satisfy the Baire property. Thus, there are no analytic non-principal ultrafilters, and there are no $\mathbf{\Sigma}^1_2$ non-principal ultrafilters in models where all $\mathbf{\Sigma}^1_2$ sets are measurable or have the Baire property; and in the Solovay model, there are no non-principal ultrafilters at all. In a similar way, one can look at other objects (whose existence is usually established by the Axiom of Choice) from the point of view of descriptive set theory.

Before concluding this historical account, we should mention the role of *large cardinals* in the study of the real number continuum. Large cardinal axioms are additions to the standard set theoretic axioms postulating the existence of certain very large objects. Such postulates cannot be proved, but are generally considered natural enough to be taken up alongside the standard axioms. In the last few decades of the 20th century, much effort has been exerted into providing a connection between large cardinal axioms and the regularity of sets in the projective hierarchy. From Solovay's original result it already follows that if there exists a *measurable cardinal*, then all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable and satisfy the Baire property, and this proof can easily be adapted to show that, under this assumption, *all* regularity properties are satisfied on the $\mathbf{\Sigma}^1_2$ level. Further results followed, attempting to use ever stronger assumption in order to derive the regularity of sets higher up in the projective hierarchy. The culmination of this effort was the result of Donald A. Martin and John Steel [MS89] showing that if there are infinitely many *Woodin cardinals* then all projective sets are regular (via the so-called axiom of *Projective Determinacy*).

In spite of the beautiful structure provided by large cardinals, there is one substantial drawback: they blur the distinction between the different notions of regularity, by treating them all in the same way, putting them all in one basket, so to say. For instance, the fact that Lebesgue measure is, in various ways, stronger than the Baire property, becomes completely concealed if one considers models with large cardinals, and the same applies to other properties. Characterization theorems, which seem highly informative as such, become redundant if sufficiently large cardinals are assumed to exist. In this dissertation we will focus on the *individuality* of each regularity property, and on subtle ways to make the property hold without necessarily affecting other regularity properties. As a result, we shall not be assuming the existence of any large cardinals (with the exception of an inaccessible when proving something about the Solovay model), and most of the set-theoretic models making their appearance here are going to be relatively mild extensions of L obtained by an iteration of proper forcing, and will all lie within the realm of ZFC in terms of consistency strength.

1.2 Preliminaries

1.2.1 Set theory

Our basic axiomatic framework is ZFC, the *Zermelo-Frankel* axioms of set theory together with the Axiom of Choice.

We will not assume any additional axioms, e.g., large cardinal axioms, with the exception of a few times when proving theorems about the Solovay model; the Axiom of Choice will have to be dropped in a few instances when investigating consequences of determinacy.

We will assume complete familiarity with notions of naive set theory, such as the formalization of ordered pairs, relations, functions etc. as sets, as well as the formalization of natural numbers as von Neumann ordinals, and the definitions of the rational and real numbers as derived from the natural ones. We will also assume familiarity with ordinals, cardinals and concepts involving these, such as successor/limit ordinal, successor/limit cardinal, cofinality, regular and singular cardinals, and elementary properties of ordinal and cardinal arithmetic.

Moreover, we will assume some knowledge of elementary topology, in particular the concepts *open*, *closed*, *dense*, *nowhere dense* and *compact*, as well as notions of convergence, continuity and limits.

Basic logical tools and concepts will also be assumed, such as the syntax and semantics of first order logic and its application in the formalization of set theory. In particular, (class) models of set theory and truth of formulas in such models via relativization will be assumed as known.

Our notation mostly follows standard set theoretic convention, as found in textbooks such as [Jec03] and [Kan03]. As finite and infinite sequences play a prominent role, we briefly review the corresponding notation. If X is any set, X^ω denotes the set of all functions from ω to X and $X^{<\omega} = \bigcup_{n\in\omega} X^n$ denotes the set of finite sequences of elements from X. The length of a finite sequences $s \in X^{<\omega}$ is denoted by $|s|$. For $s \in X^{<\omega}$ and $n < |s|$, $s{\restriction}n$ refers to the initial segment of s consisting of n elements; likewise for $f{\restriction}n$ if $f \in X^\omega$. For two sequences $s, t \in X^{<\omega}$, $s^\frown t$ is the *concatenation* of s with t; likewise for $s^\frown f$ if $f \in X^\omega$. One convention follows: if $f \in X^\omega$ and we write "$s \subseteq f$", then this is assumed to imply that $s \in X^{<\omega}$, i.e., s is an initial segment of f, rather than some arbitrary subset. In most of our applications, X will be ω or $2 = \{0, 1\}$.

Other shorthand notation that we will often use is "$\forall^\infty$" and "$\exists^\infty$" to abbreviate "for all but finitely many" and "there are infinitely many", respectively.

1.2.2 Real numbers

In mathematics, the set of real numbers $\mathbb{R}$ is usually defined from the rational numbers using Dedekind cuts, equivalence classes of Cauchy sequences, or some such method. However, this object is somewhat cumbersome for foundational investigations, and in set theory it is usually preferable to work with simpler objects, which share all the essential logical, topological and structural properties of $\mathbb{R}$ but are more straightforward to study and easier to manipulate.

The most frequent incarnation of the real numbers in set theory is ω^ω, the set of functions from ω to ω. If, for every $s \in \omega^{<\omega}$, we define $[s] := \{x \in \omega^\omega \mid s \subseteq x\}$ to be the set of all functions extending s, then the collection $\{[s] \mid s \in \omega^{<\omega}\}$ forms a topology base for ω^ω, and the resulting topological space is called the *Baire space*. Clearly ω^ω has cardinality $2^{\aleph_0}$ and shares many other properties inherent to the real numbers. For example, it has a countable base of open sets, and each $x \in \omega^\omega$ can be approximated by a sequence of open neighbourhoods $\{[s] \mid s \subseteq x\}$. A metric, consistent with the topology, can be defined on ω^ω, by $d(x,y) := 1/2^n$ for the least n such that $x(n) \neq y(n)$ and $d(x,y) = 0$ if $x = y$. Convergence in the Baire space can be conveniently formulated as follows: $\lim_{n\to\infty} x_n = x$ iff $\forall s \subseteq x \, \forall^\infty n \, (s \subseteq x_n)$. Following common practice, we shall call elements of ω^ω "real numbers" or simply "reals". In this context, we can define the rationals as those $x \in \omega^\omega$ which are eventually 0. It is clear that every real is a limit of a countable sequence of rationals. The Baire space can be shown to be homeomorphic to the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ (in the standard sense).

In complete analogy, we define the *Cantor space* 2^ω to be the set of functions from ω to 2. The topology, metric, limit etc. are all defined similarly. This space is homeomorphic to Cantor's standard "$\frac{1}{3}$ set". Likewise, we can define n^ω for any $n \in \omega$ or even $\prod_i n_i$ for any sequence of natural numbers n_i.

Another incarnation of the real numbers is $[\omega]^\omega := \{x \subseteq \omega \mid |x| = \omega\}$, the set of all infinite subsets of ω. The space $[\omega]^\omega$ can either be identified with the Cantor space via characteristic functions, or with a subset of ω^ω via increasing enumerations. In either case, it gives rise to the same notion of a topology on $[\omega]^\omega$.

Many-dimensional real number spaces are defined in a standard way, with $(\omega^\omega)^n$ equipped with the product topology.

More information about basic topological and structural properties of the Baire and Cantor spaces can be found in classical textbooks such as [Kec95] and [Mos80].

1.2.3 Trees

The word "tree" in set theory can refer to many things. In descriptive set theory, however, a *tree on a set* X is a subset of $X^{<\omega}$ closed under initial segments. If T is a tree then $[T]$ denotes the set of all *branches through* T, defined as $[T] := \{f \in X^\omega \mid \forall n \, (x{\restriction}n \in T)\}$. If $X = \omega$ then $[T]$ is a subset of the Baire space. It is easy to see that every set $[T]$ is topologically closed (contains all its limit points) and, conversely, any closed set $C \subseteq \omega^\omega$ is of the form $[T]$ for some tree T; thus there is a one-to-one correspondence between trees on ω and closed subsets of ω^ω. More generally, if for an arbitrary set $A \subseteq \omega^\omega$ we define $T_A := \{x{\restriction}n \mid x \in A, n \in \omega\}$ then the operation $A \mapsto [T_A]$ is the topological closure of A. The same thing can be said of $X = 2$ and the Cantor space.

The following notation and terminology is used in the context of trees:

- For $t \in T$, the set of *immediate successors of* t is defined as

$$\mathrm{Succ}_T(t) := \{s \in T \mid \exists n \, (t^\frown \langle n \rangle = s)\}.$$

- A node $t \in T$ is called *splitting* if $|\mathrm{Succ}_T(t)| > 1$ and *non-splitting* otherwise.

- The *stem* of T is the longest $t \in T$ such that all $s \subseteq t$, $s \neq t$ are non-splitting.
- For $t \in T$, $T \uparrow t$ is the sub-tree $\{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$.

It is not hard to verify that $[T]$ is compact (in the topological sense) if and only if T is everywhere finitely branching, i.e., for every $t \in T$, $|\mathrm{Succ}_T(t)| < \omega$. In particular, the Cantor space, and any space of the form n^ω or $\prod_i n_i$ where the n_i are finite, is compact, whereas the Baire space is not.

Trees can also be defined in n dimensions. An n-dimensional tree on $X_0 \times \cdots \times X_{n-1}$ can interchangeably be viewed either as a subset of $(X_0 \times \cdots \times X_{n-1})^{<\omega}$, or as a subset of $\{(t_0, \dots, t_{n-1}) \mid t_i \in X_i^{<\omega} \text{ and } |t_0| = \cdots = |t_{n-1}|\}$, pointwisely closed under initial segments. For such an n-dimensional tree T, the set of branches through T is a subset of $X_0^\omega \times \cdots \times X_{n-1}^\omega$. As before, in the n-dimensional Baire space a set $C \subseteq (\omega^\omega)^n$ is closed if and only if there is an n-dimensional tree T such that $C = [T]$; the same applies for the Cantor space.

There are some specific trees that will be of importance.

Definition 1.2.1.

1. *A tree T on ω (or 2) is called a* perfect tree *if every node $t \in T$ has an extension $s \supseteq t$, $s \in T$, which is splitting.*
2. *A tree T on ω is called a* super-perfect tree*, or a* Miller tree*, if every node $t \in T$ has an extension $s \supseteq t$, $s \in T$ which is infinitely branching, i.e., such that $|\mathrm{Succ}_T(s)| = \omega$.*
3. *A tree T on ω is called a* Laver tree *if for every $s \in T$ longer than the stem of T, s is infinitely branching.*

In topology, a *perfect set* is a set C which is closed and contains no isolated points. It is easy to verify that in the Baire and Cantor spaces, $[T]$ is a perfect set if and only if T is a perfect tree. Perfect sets have cardinality $2^{\aleph_0}$ since the set of branches through the corresponding perfect tree can be put into a one-to-one correspondence with 2^ω.

1.2.4 Descriptive set theory

Classical descriptive set theory is the study of *definable* sets of reals, primarily the Borel and the projective hierarchy stemming from the work of Borel, Lebesgue, Luzin and Suslin in the early 20th century. We now give a systematic account of the main definitions. For convenience of the exposition, we will work with the Baire space ω^ω, but all definitions and results apply also to the Cantor space, as well as to the n-dimensional versions of the Baire and Cantor spaces.

The collection of the *Borel sets* $\mathcal{B}$ is defined to be the smallest collection of sets of reals satisfying the following properties:

1. every open set is in $\mathcal{B}$,
2. if $B \in \mathcal{B}$ then $(\omega^\omega \setminus B) \in \mathcal{B}$, and
3. if $B_i \in \mathcal{B}$ for every $i \in \omega$, then $\bigcup_i B_i \in \mathcal{B}$.

In other words, $\mathcal{B}$ is the σ-algebra generated by the open sets, and every Borel set is recursively built up from the open sets using the operations of complementation and countable union (or intersection). A more detailed look at the Borel sets allows us to define the *Borel hierarchy*, a stratification of the Borel algebra, by induction on $\alpha < \aleph_1$.

Definition 1.2.2. *For each $\alpha < \aleph_1$, the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ are defined by induction using the following rules:*

1. *a set $A \subseteq \omega^\omega$ is in $\mathbf{\Sigma}^0_1$ if and only if it is* open,
2. *a set A is in $\mathbf{\Pi}^0_\alpha$ if and only if its complement $\omega^\omega \setminus A$ is in $\mathbf{\Sigma}^0_\alpha$,*
3. *for $\alpha > 1$, a set A is in $\mathbf{\Sigma}^0_\alpha$ if and only if $A = \bigcup_{n\in\omega} A_n$, where each $A_n \in \mathbf{\Pi}^0_\beta$ for $\beta < \alpha$, and*
4. *a set A is in $\mathbf{\Delta}^0_\alpha$ if and only if it is in both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$.*

All these terms are typically used as adjectives, and we say things like "a set is $\mathbf{\Sigma}^0_\alpha$" or "A is a $\mathbf{\Sigma}^0_\alpha$ set".

It can be shown that this hierarchy is proper, in the sense that for every α there exists a set A which is $\mathbf{\Pi}^0_\alpha$ but not $\mathbf{\Sigma}^0_\alpha$. It is clear that a set B is Borel if and only if it is $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$ for some $\alpha < \aleph_1$, and the least α for which this is the case is called the *Borel rank* of B. Sets low in the hierarchy also have classical names: $\mathbf{\Sigma}^0_2$ sets are called F_σ (countable unions of closed sets) and $\mathbf{\Pi}^0_2$ sets are called G_δ (countable intersections of open sets).

As mentioned in the introduction, the Borel algebra is not closed under the natural operation of *projection.*

Definition 1.2.3.

1. *Let $A \subseteq (\omega^\omega)^2$ be a two-dimensional set of reals. The* projection of A *(onto the first coordinate) is*

$$p[A] := \{x \mid \exists y \in \omega^\omega\ ((x, y) \in A)\}.$$

2. *A set $A \subseteq \omega^\omega$ is called* analytic *if it is the projection of some Borel set $B \subseteq (\omega^\omega)^2$, and* co-analytic *if its complement is analytic.*

In [Sus17], Suslin showed that there are analytic sets which are not Borel, that A is analytic if and only if it is the projection of a *closed* set, and that the Borel sets are precisely those that are both analytic and co-analytic. Iterating the operation of projection and complementation, the *projective hierarchy* is obtained.

Definition 1.2.4. *For each $n \in \omega$, the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$ are defined by induction using the following rules:*

1. *A is $\mathbf{\Sigma}^1_1$ if and only if it is analytic,*
2. *A is $\mathbf{\Pi}^1_n$ if and only if $\omega^\omega \setminus A$ is $\mathbf{\Sigma}^1_n$,*
3. *A is $\mathbf{\Sigma}^1_{n+1}$ if and only if $A = p[A']$ for some $\mathbf{\Pi}^1_n$ set A', and*
4. *A is $\mathbf{\Delta}^1_n$ if and only if it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.*

Notice that the precise number of dimensions in which the sets and their projections are defined is not relevant, since any $(\omega^\omega)^n$ is homeomorphic to ω^ω. What matters is that if a set is in $(\omega^\omega)^n$ then its projection is defined in $(\omega^\omega)^{n-1}$. Just as the Borel hierarchy, the projective hierarchy is proper, i.e., for every n there is an A which is $\mathbf{\Pi}^1_n$ but not $\mathbf{\Sigma}^1_n$. A set is called *projective* if it is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ for some n.

So far, the Borel and projective hierarchies were presented from a purely topological point of view, but there is a straightforward connection to logic. Consider the language of *second order number theory* consisting of formulas with terms for natural numbers (first-order objects) as well as real numbers (second-order objects), and having first-order quantifiers $\exists^0, \forall^0$ and second-order quantifiers $\exists^1, \forall^1$. We use the notation $\mathbb{N}^2 \models \phi$ to say that the formula ϕ in the language of second-order number theory is true in the standard model. Formulas can use real numbers $r \in \omega^\omega$ as parameters, in which case we will write $\phi(r)$ to denote the fact that r appears in ϕ. A classification of formulas in this language can be defined according to the number of alternating natural number and real number quantifiers. Precisely:

Definition 1.2.5.

1. (a) *ϕ is Σ^0_0, or Π^0_0, if it is quantifier-free,*
   (b) *ϕ is Σ^0_{n+1} if it is of the form $\exists^0 k\, \psi$ where ψ is Π^0_n,*
   (c) *ϕ is Π^0_n if it is of the form $\neg\psi$ where ψ is Σ^0_n,*
   (d) *ϕ is Δ^0_n if it is both Σ^0_n and Π^0_n,*
   (e) *ϕ is* arithmetical *if it is Σ^0_n or Π^0_n for some n.*
2. (a) *ϕ is Σ^1_1 if it is of the form $\exists^1 x\, \psi$ where ψ is arithmetical (equivalently, quantifier-free),*

(b) ϕ *is* Π^1_n *if it is of the form* $\neg\psi$ *where* ψ *is* Σ^1_n,

(c) ϕ *is* Σ^1_{n+1} *if it is of the form* $\exists^1 x\, \psi$ *where* ψ *is* Π^1_n,

(d) ϕ *is* Δ^1_n *if it is both* Σ^1_n *and* Π^1_n,

(e) ϕ *is* projective *if it is* Σ^1_n *or* Π^1_n *for some* n.

So a formula is Σ^0_n iff it is of the form $\exists^0 k_0 \forall^0 k_1 \dots Q^0 k_n \phi$, and Π^0_n iff it is of the form $\forall^0 k_0 \exists^0 k_1 \dots Q^0 k_n \phi$, where ϕ is quantifier-free. Likewise, it is Σ^1_n iff it is of the form $\exists^1 x_0 \forall^1 x_1 \dots Q^0 x_n \phi$, and Π^1_n iff it is of the form $\forall^1 x_0 \exists^1 x_1 \dots Q^1 x_n \phi$, where ϕ is arithmetical (equivalently, quantifier-free). The classification of formulas allows us to classify sets definable in second-order number theory according to the complexity of the formula defining it.

Definition 1.2.6.

1. *A set* $A \subseteq \omega^\omega$ *is* Σ^i_n (Π^i_n) *if it can be written as*

$$A = \{x \in \omega^\omega \mid \mathbb{N}^2 \models \phi(x)\}$$

where ϕ *is* Σ^i_n (Π^i_n) *and free of parameters.*

2. *A set* $A \subseteq \omega^\omega$ *is* $\Sigma^i_n(r)$ (Π^i_n) *if it can be written as*

$$A = \{x \in \omega^\omega \mid \mathbb{N}^2 \models \phi(x, r)\}$$

where ϕ *is* Σ^i_n (Π^i_n) *and contains a real parameter* r.

As natural number quantifiers correspond to countable unions and intersections whereas real quantifiers correspond to the operation of projection, one can show that the "boldface" hierarchy defined by purely topological means, corresponds to the "lightface" hierarchy defined using logic, assuming that real parameters are allowed in the defining formula.

Fact 1.2.7. *A set* $A \subseteq \omega^\omega$ *is* $\boldsymbol{\Sigma}^i_n$ $(\boldsymbol{\Pi}^i_n)$ *iff* A *is* $\Sigma^i_n(r)$ $(\Pi^i_n(r))$ *for some* $r \in \omega^\omega$.

For a more detailed introduction to descriptive set theory, we refer the reader to classical textbooks such as [Kec95] and [Mos80].

1.2.5 Constructibility

In 1938, Gödel introduced the constructible universe L, an inner model of set theory defined similarly to the cumulative hierarchy V but using the definable power set operation rather than the full power set operation at successor steps of the construction.

Definition 1.2.8. *Let X be any set. A subset $Y \subseteq X$ is* (first order) definable over X *if there exists a first-order formula ϕ such that for every $x \in X$ we have*

$$x \in Y \text{ iff } X \models \phi(x, z_0, \dots, z_k),$$

where $z_0, \dots, z_k \in X$. Here, "$X \models \phi$" refers to the coded version of ϕ and the satisfaction relation $\models$ with respect to the set *model $(X, \in)$. Since the satisfaction relation for such set models is definable in* ZFC*, the predicate "being definable over X" is itself definable. Let $\mathrm{Def}(X) := \{Y \subseteq X \mid Y$ is definable over $X\}$.*

Definition 1.2.9. *For all ordinals α, define L_α by induction as follows:*

- $L_0 := \varnothing$,
- $L_{\alpha+1} := \mathrm{Def}(L_\alpha)$,
- $L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$ *for limit* λ.

Let $L := \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$ be the proper class of all sets in this hierarchy. This is called the constructible universe*, and sets in L are called* constructible.

By [Göd38], L is a model of set theory which, additionally, satisfies the Generalized Continuum Hypothesis (GCH) and many other mathematical statements. A definable version of the Axiom of Choice holds in L, i.e., there is a definable well-order of the entire universe of constructible sets, which we shall denote by $<_L$. It is defined recursively, by ordering each elements of L_α according to the formula defining it and the ordering of the parameters used in the definition, which come from some L_β for $\beta < \alpha$. We also denote by $<_{L_\alpha}$ the restriction of the well-order to an initial segment of the L, so $<_L = \bigcup_{\alpha \in \mathrm{Ord}} <_{L_\alpha}$.

The constructible universe L is the smallest inner model of set theory, in the sense that if M is any other proper class model, then $L \subseteq M$. The statement "all sets are constructible", typically abbreviated by "$V = L$", is absolute between L and every other model, hence L itself is a model of the statement "$V = L$". Here we will not present all known facts about L and refer to textbooks such as [Jec03, Kan03, Dev84] for further details. However, we are particularly interested in questions concerning definability.

Fact 1.2.10 (Gödel)**.**

1. *There is a sentence Θ (containing sufficiently much of* ZF$+V = L$ *to insure absoluteness of all the relevant definitions) such that for any set model M, if $M \models \Theta$ then M is isomorphic to L_δ for some limit ordinal δ.*
2. *There is a formula $\chi(x, y)$ such that if $x, y \in L_\delta$ then $L_\delta \models \chi(x, y)$ iff $x <_{L_\delta} y$.*

If δ is a countable limit ordinal then L_δ is countable and the structure $(L_\delta, \in)$ is isomorphic to (ω, E) for a well-founded relation E on ω. Conversely, if we are given a well-founded relation E on ω, and we know that $(\omega, E) \models \Theta$, then by Fact 1.2.10 (ω, E) must be isomorphic to some $(L_\delta, \in)$, and in that case we denote the transitive collapse of (ω, E) onto $(L_\delta, \in)$ by π_E.

Fact 1.2.11 (Gödel)**.** *In L, the canonical well-ordering of the reals (i.e., the set $\{(x, y) \mid x, y \in \omega^\omega$ and $x <_L y\}$) is a $\boldsymbol{\Delta}^1_2$ set.*

Proof. First note that, by virtue of the definition of the well-ordering, for any $x, y \in \omega^\omega \cap L_\delta$ we know that $x <_L y$ iff $x <_{L_\delta} y$ iff $L_\delta \models \chi(x, y)$. Therefore, for any $x, y \in \omega^\omega$, we may write $x <_L y$ iff $\exists \delta < \aleph_1\ (x, y \in L_\delta$ and $L_\delta \models \chi(x, y))$, which, in turn, may be written as follows: there exists $E \subseteq \omega \times \omega$ such that

1. E is well-founded,
2. $(\omega, E) \models \Theta$, and
3. $\exists n\, \exists m\ (n = \pi_E(x)$ and $m = \pi_E(y)$ and $(\omega, E) \models \chi(n, m))$.

Note that E can be considered a real number, so "there exists E" corresponds to a second order quantifier $\exists^1$. Moreover, the statement "E is well-founded" is $\boldsymbol{\Pi}^1_1$, whereas statements 2 and 3 are arithmetical (see e.g. [Kan03, Proposition 13.8]). It follows that $x <_L y$ is equivalent to a $\boldsymbol{\Sigma}^1_2$ statement.

To see that it is also $\boldsymbol{\Pi}^1_2$, apply the same trick to the statement $\forall \delta < \aleph_1\ (x, y \in L_\delta \to L_\delta \models \chi(x, y))$. □

This proof is paradigmatic for proving that in L, definitions by induction on a well-ordering of the reals can usually be modified to produce sets of low complexity. In particular, we will use this method many times for constructing $\boldsymbol{\Sigma}^1_2$, $\boldsymbol{\Delta}^1_2$ or $\boldsymbol{\Pi}^1_1$ counterexamples to regularity properties in L.

If a is any set, we may define $L[a]$ analogously to L but replacing definability by a first-order formula in the clause "$L_{\alpha+1} := \mathrm{Def}(L_\alpha)$" by definability with the parameter a. The hierarchy generated is $L_\alpha[a]$, and both the individual levels and the entire class $L[a]$ share most properties with L. In our setting, a will most often be a real number.

1.2.6 Absoluteness

To say that a formulas ϕ is *absolute* between V and some model M is to say that $M \models \phi$ if and only if $V \models \phi$. We are specifically interested in formulas ϕ in second-order number theory, as these are used to classify sets of reals.

Fact 1.2.12 (Analytic absoluteness)**.** *Let M be any model (countable or otherwise) of set theory. Every Σ^1_1 (hence Π^1_1) formula is absolute between M and V.*

As a result, Σ^1_2 formulas are upwards absolute and Π^1_2 formulas are downwards absolute between any M and V, but not the other way around.

Fact 1.2.13 (Shoenfield absoluteness)**.** *Let M be a model such that $\omega_1 \subseteq M$ (in particular, M cannot be countable). Every Σ^1_2 (hence Π^1_2) formula is absolute between M and V.*

As a result, Σ^1_3 formulas are upwards absolute and Π^1_3 formulas are downwards absolute between any M with $\omega_1 \subseteq M$ and V, but not the other way around.

Analytic absoluteness will most often be used in a context where M is some countable *elementary* submodel of a sufficiently large structure. Shoenfield absoluteness will typically be used between V and L, or some $L[r]$ for $r \in \omega^\omega$.

Absoluteness of sentences involving Borel sets is particularly interesting. Note that every Borel set comes together with a description of its own construction using the basic operations. Therefore, Borel sets can be coded by reals in an effective manner (see [Jec03, p 504–507] for details). Such reals are called *Borel codes*, and if $c \in \omega^\omega$ is such a Borel code, let B_c denote the Borel set encoded by c. If a model M contains the code c of a Borel set, then it can interpret the set B_c^M. This is not the same set as B_c as M has less reals than V, but, for all practical purposes, it is *the same Borel set*, i.e., it is how a different model interprets the same definition. Most simple operations involving Borel sets, if considered as operations on the codes rather than the sets themselves, are absolute.

Fact 1.2.14. *The statements "$x \in B_c$", "$B_c = \varnothing$", "$B_c \subseteq B_d$", "$B_c = \omega^\omega \setminus B_d$", "$B_c = B_d \cup B_e$", "$B_c = B_d \cap B_e$" etc. are all analytic or co-analytic, and therefore absolute between V and any model M containing c, d, e and x.*

Shoenfield absoluteness is intimately connected with tree representation of $\boldsymbol{\Sigma}^1_2$ sets.

Fact 1.2.15 (Shoenfield)**.**

1. *If A is $\Sigma^1_2(r)$ then there exists a tree T on $\omega \times \omega_1$ (i.e., $T \subseteq \omega^{<\omega} \times \omega_1^{<\omega}$), such that $T \in L[r]$, and such that for all x, $x \in A$ iff $\exists h \in \omega_1^\omega$ s.t. $(x, h) \in [T]$ iff $\exists h \in \omega_1^\omega\, \forall n\, ((x{\restriction}n, h{\restriction}n) \in [T])$.*
2. *If A is $\Sigma^1_2(r)$ and $\aleph_1^{L[r]} = \aleph_1$, then $A = \bigcup_{\alpha<\aleph_1} B_\alpha$, where B_α are Borel sets, and whose Borel codes are contained in $L[r]$.*

Both facts, especially the second, will be used numerous times in the analysis of $\boldsymbol{\Sigma}^1_2$ sets.

1.2.7 Forcing

In 1964, Paul Cohen [Coh63, Coh64] discovered the method of *forcing* by which models of set theory could be extended to larger models in a controlled manner, by adding so-called *generic objects*. This could be viewed as the counterpart to Gödel's method of inner models.

As we cannot reproduce the entire method of forcing here, we refer the readers to textbooks such as [Kun80] and [Bel85]. The main principle of forcing is the use of a partial order $(\mathbb{P}, \leq)$ contained in some ground model M of ZFC, and a "generic" object $G \subseteq \mathbb{P}$ outside M which can be adjoined to form a larger model of ZFC, $M[G]$. The combinatorial properties of the partial order $\mathbb{P}$ determine which additional statements are true in $M[G]$. Elements $p \in \mathbb{P}$ are called *forcing conditions* or simply *conditions*, and $q \leq p$ is interpreted as "q is stronger than p", "q contains more information than p", or "q extends p". Although M does not contain the objects in $M[G]$, it contains *names* τ for such objects, which are interpreted by an object $\tau_G \in M[G]$. In a syntactic way, the *forcing relation* $\Vdash$ is then defined which, in M, decides the truth-value of statements in $M[G]$. Specifically, the *Forcing theorem* states the following two things:

1. if $M \models$"$p \Vdash \phi(\tau)$" and $p \in G$ then $M[G] \models \phi(\tau_G)$, and
2. if $M[G] \models \phi(x)$ then there is a $p \in G$ and a name τ such that $\tau_G = x$ and $M \models$"$p \Vdash \phi(\tau)$".

We will assume familiarity with the technical aspects of forcing, in particular the concepts *compatible, dense, predense, dense/predense below p, antichain* and $\mathbb{P}$-*generic filter*, as well as the technical definition of names and (some) formalization of the forcing relation $\Vdash$; all these can be found in the textbooks mentioned above and in other literature. A forcing $\mathbb{P}$ is said to have the *countable chain condition*, or *c.c.c.*, if every maximal antichain is at most countable.

Although, formally, adjoining a $\mathbb{P}$-generic filter G is only possible if M is a set-sized transitive model, it is common for set theorists to talk of generic extensions $V[G]$ of the universe V. This is understood as follows: prior to the adjoining of G, one thinks of V as the universe of all sets (so a generic G cannot exist). However, when we adjoin G we take a "step out" of V, and look at it from the point of view of some larger (unspecified) universe, in which V is a set-sized model and a $\mathbb{P}$-generic object G over V exists. Classical textbooks on forcing show how such an argument can be formalized without being nonsensical (in fact, there are several possible approaches to formalization—see [Kun80, Chapter VII §9]). We shall not be concerned with these issues and will take the liberty to extend the universe V to a larger one $V[G]$ whenever convenient.

Iterations of forcing will be used throughout this dissertation. Intuitively, after extending a model M to $M[G_0]$ by adding a $\mathbb{P}$-generic G_0, the process can be repeated and a $\mathbb{P}$-generic G_1 can be added to $M[G_0]$ producing the larger model

$M[G_0][G_1]$; that, in turn, can be extended to $M[G_0][G_1][G_2]$, etc. An iteration of $\mathbb{P}$ of length α is the result of repeating this process for α steps. However, in order to specify what happens at limit stages of such an iteration, the formal approach is somewhat different: a forcing partial order $\mathbb{P}_\alpha$ is defined directly in the ground model M in such a way that adding a $\mathbb{P}_\alpha$-generic filter G to M *once* amounts to adding α many $\mathbb{P}$-generic filters G_β for $\beta < \alpha$, in sequence. Iterations can have *finite support* or *countable support*, depending on how the construction at limit stages is defined. In most of our application an intuitive understanding of iterations will suffice, although in Chapter 5 some of the more technical aspects of iterations will be relevant. For a detailed introduction on forcing iterations, see [Kun80, Chapter VIII §5], and for applications of it in the study of the continuum, see [BJ95, Chapters 5, 6 and 7].

If a forcing partial order $\mathbb{P}$ has the countable chain condition (c.c.c.) then it preserves $\aleph_1$, i.e., $\aleph_1^{M[G]} = \aleph_1^M$. An iteration of $\mathbb{P}$ with finite support does so, too. For partial orders without the c.c.c., preservation of $\aleph_1$ is established by alternative methods. The main modern device for this is the notion of *proper forcing*, which we now introduce. It is a crucial concept which we will be used a lot in our work.

In discussions of proper forcing, it is customary to consider generic extensions $V[G]$ of the universe V. On the other hand, M typically denotes a *countable elementary submodel* of some $\mathcal{H}_\theta$, where $\mathcal{H}_\theta$ is the collection of all sets hereditarily of cardinality $< \theta$, and θ is a *sufficiently large* cardinal, meaning that $\mathcal{H}_\theta$ contains all information necessary for the argument we are currently interested in. The precise value of θ is left unspecified, but usually it is sufficient for θ to be larger than $2^{|\mathbb{P}|}$. The model M can be seen as a miniature version of V, containing all the essential logical information relevant for the current argument, while being itself only countable.

If G is a $\mathbb{P}$-generic filter over V, we denote by $M[G]$ the set of all G-interpretations of names τ which lie in M. This set $M[G]$ might, or might not, be a generic extension of M. This leads to the following sequence of definitions:

Definition 1.2.16. *Let G be $\mathbb{P}$-generic over V.*

1. *We call G $(M, \mathbb{P})$-*generic *if $M[G]$ is a generic extension of M. Formally, this means that $G \cap D \cap M \neq \varnothing$ for every dense set $D \in M$.*

2. *A condition $p \in \mathbb{P}$ is called an $(M, \mathbb{P})$-master condition if $p \Vdash$ "$\dot{G}$ is $(M, \mathbb{P})$-generic", where $\dot{G}$ is the canonical name for the generic filter (over V).*

3. *A forcing $\mathbb{P}$ is called* proper *if for every countable $M \prec \mathcal{H}_\theta$ and every $p \in \mathbb{P} \cap M$, there exists a $q \leq p$ which is a $(M, \mathbb{P})$-master condition.*

The reference to M and $\mathbb{P}$ will often be dropped when clear from the context. Note that in order for $p \Vdash \dot{G} \cap D \cap M \neq \varnothing$ to be true, it is sufficient that $(D \cap M)$

be predense below p. Therefore, an equivalent definition of properness is the following: for every countable $M \prec \mathcal{H}_\theta$ and every $p \in \mathbb{P} \cap M$, there exists a $q \leq p$ such that for every dense $D \in M$, $(D \cap M)$ is predense below q.

Every c.c.c. forcing is proper, and every proper forcing preserves $\aleph_1$. Moreover, this is preserved in iterations of proper forcing notions with countable support.

We refer the reader to [Abr10] for a detailed introduction on proper forcing and its applications.

Below we give the definitions of some standard forcing partial orders. We will need this list for reference in subsequent chapters.

Definition 1.2.17.

1. Cohen forcing*, denoted by* $\mathbb{C}$*, consists of conditions* $s \in \omega^{<\omega}$ *ordered by* $t \leq s$ *iff* $s \subseteq t$*.*

2. Random forcing*, denoted by* $\mathbb{B}$*, consist of Borel (or closed) sets of* positive Lebesgue measure *(see Definition 1.3.1), ordered by* $B \leq C$ *iff* $B \subseteq C$*.*

3. Hechler forcing*, denoted by* $\mathbb{D}$*, consists of conditions* $(s, f) \in \omega^{<\omega} \times \omega^\omega$ *such that* $s \subseteq f$*, ordered by* $(s', f') \leq (s, f)$ *iff* $s \subseteq s'$ *and* $\forall n\, (f(n) \leq f'(n))$*.*

4. Sacks forcing*, denoted by* $\mathbb{S}$*, consists of perfect trees* $T \subseteq 2^{<\omega}$*, ordered by inclusion (i.e.* $S \leq T$ *iff* $S \subseteq T$*).*

5. Miller forcing*, denoted by* $\mathbb{M}$*, consists of super-perfect (Miller) trees* $T \subseteq \omega^{<\omega}$ *ordered by inclusion (see Definition 1.2.1).*

6. Laver forcing*, denoted by* $\mathbb{L}$*, consists of Laver trees* $T \subseteq \omega^{<\omega}$ *ordered by inclusion (see Definition 1.2.1).*

7. Mathias forcing*, denoted by* $\mathbb{R}$*, consists of conditions* $(s, A) \subseteq [\omega]^{<\omega} \times [\omega]^\omega$ *such that* $\max(s) < \min(A)$*, ordered by* $(s', A') \leq (s, A)$ *iff* $s \subseteq s'$ *and* $A' \subseteq A$*, and* $s' \setminus s \subseteq A$*.*

All the forcing partial orders mentioned above add a *generic real* x_G, canonically derived from the generic filter G. For Cohen forcing, we can define $x_G := \bigcup\{s \mid s \in G\}$, and for $\mathbb{S}, \mathbb{M}$ and $\mathbb{L}$: $x_G := \bigcup\{\text{stem}(T) \mid T \in G\}$. For $\mathbb{D}$, $x_G := \bigcup\{s \mid (s, f) \in G \text{ for some } f\}$, and similarly for $\mathbb{R}$. For random forcing $\mathbb{B}$, the generic real is the unique real such that $\{x_G\} = \bigcap\{B \mid B \in G\}$. In all cases, the generic filter G can be reconstructed from x_G; thus $V[G] = V[x_G]$. We will often talk about $\mathbb{P}$-generic reals rather than $\mathbb{P}$-generic filters in our applications of forcing.

All forcings in Definition 1.2.17 are proper, and $\mathbb{C}, \mathbb{B}$ and $\mathbb{D}$ are c.c.c. whereas the others are not.

Lastly, we introduce a very different kind of forcing, the *Lévy collapse* used to build the Solovay model.

Definition 1.2.18. *Let κ be an inaccessible cardinal, and let* $\mathrm{Col}(\omega, {<}\kappa)$ *be the partial order of finite functions p such that*

1. $\mathrm{dom}(p) \subseteq \kappa \times \omega$,
2. *if* $(\alpha, n) \in \mathrm{dom}(p)$ *then* $p(\alpha, n) < \alpha$.

From a $\mathrm{Col}(\omega, {<}\kappa)$-generic filter G one can obtain a function $f_G : \kappa \times \omega \to \kappa$ defined by $f_G := \bigcup G$, and for every $\alpha < \kappa$, a function $f_{G,\alpha} : \omega \to \alpha$ defined by $f_{G,\alpha}(n) := f_G(\alpha, n)$. Standard genericity arguments show that each $f_{G,\alpha}$ is a surjection; thus, in the generic extension by $\mathrm{Col}(\omega, {<}\kappa)$, κ is collapsed onto $\aleph_1$. The *Solovay model* is defined as an inner model of the $\mathrm{Col}(\omega, {<}\kappa)$-generic extension $V[G]$.

Definition 1.2.19.

1. *A set A is* definable from a sequence of ordinals *if there is an $s \in \mathrm{Ord}^\omega$, i.e., a countable sequence of ordinals, and a formula φ, such that*
$$x \in A \iff \varphi(s, x).$$
2. HOD^ω *is the class of all sets* hereditarily definable from a sequence of ordinals*, i.e., the class of all A such that every set in the transitive closure of A is definable from a sequence of ordinals.*
3. *By the* Solovay model *we refer to* HOD^ω *defined within $V[G]$, where V is a model with an inaccessible cardinal κ and $V[G]$ the* $\mathrm{Col}(\omega, {<}\kappa)$*-generic extension.*

HOD^ω is an inner model satisfying $\mathsf{ZF} + \mathsf{DC}$ (Axiom of Dependent Choices), though, in general, not the full Axiom of Choice. Also, it is easy to see that every projective set is definable from a sequence of ordinals.

The following fundamental property of the Lévy collapse is instrumental for proving that sets of reals in the Solovay model satisfy many nice properties.

Lemma 1.2.20 (Solovay)**.** *Let κ be an inaccessible cardinal in V and $V[G]$ be the* $\mathrm{Col}(\omega, {<}\kappa)$*-generic extension. For every formula φ, there is a formula $\tilde{\varphi}$ such that for $s \in \mathrm{Ord}^\omega$ and $x \in \omega^\omega$:*
$$V[G] \models \varphi(s, x) \iff V[s][x] \models \tilde{\varphi}(s, x).$$

Proof. See e.g. [Jec03, Lemma 26.17] or [Kan03, Lemma 11.12]. □

As soon as one proves that, in $V[G]$, all sets of reals definable from a sequence of ordinals are "nice" in some certain way, the following two results are immediately obtained: Con(ZFC+"all projective sets are 'nice' ") and Con(ZF+DC+"all sets are 'nice' ").

1.2.8 Determinacy

Suppose two players, I and II, are playing a game picking integers x_i, y_i in turns, and continue doing this ω-many times:

$$\begin{array}{c|cccccccc} \text{I}: & x_0 & & x_1 & & x_2 & & \dots & \\ \hline \text{II}: & & y_0 & & y_1 & & y_2 & & \dots \end{array}$$

After infinitely many moves, a real $z := \langle x_0, y_0, x_1, y_1, \dots \rangle$ is produced. Given a pre-determined "pay-off" set $A \subseteq \omega^\omega$, player I wins this game if $z \in A$, otherwise II does. Despite this unrealistic scenario, such so-called *infinite, two-person games of perfect information* are mathematically well-defined and play a crucial role in descriptive set theory. We let $G(A)$ stand for the game as described above with the winning condition for player I determined by the set A. The following sequence of definitions explains the importance of infinite games in the study of the continuum.

Definition 1.2.21.

1. *A* strategy for player I *is a function $\sigma : \omega^{<\omega} \to \omega$, the intended interpretation of which is that $\sigma(p)$ determines the integer x_i for player I to move, in the game where the moves played so far have been $p = \langle x_0, y_0, \dots, x_{i-1}, y_{i-1} \rangle$. A* strategy for player II *is a function $\tau : \omega^{<\omega} \to \omega$ with the analogous interpretation.*

2. *If $y = \langle y_0, y_1, \dots \rangle$ is a real, then $\sigma * y$ denotes the result of the game in which I follows strategy σ and II plays the sequence of integers given by y, and similarly for $x * \tau$ where x is the sequence of integers played by I.*

3. *In a fixed game $G(A)$, σ is a* winning strategy for player I *if for all $y \in \omega^\omega$, $\sigma * y \in A$, and τ is a* winning strategy for player II *if for all $x \in \omega^\omega$, $x * \tau \notin A$.*

4. *The game $G(A)$ is* determined *if player I or player II has a winning strategy. A set $A \subseteq \omega^\omega$ is* determined *if the game $G(A)$ is determined.*

It is easy to show that open and closed sets A are determined—this is known as the Gale-Stewart theorem and is due to [GS53]. A much more difficult result is the theorem of Donald A. Martin [Mar75] showing that all Borel sets A are determined. The following axiom has been proposed in [MS62].

Definition 1.2.22. *The* Axiom of Determinacy*, AD, is the statement that all sets of reals $A \subseteq \omega^\omega$ are determined.*

AD contradicts the Axiom of Choice, since one can use the well-ordering of the reals to construct a non-determined set. However, AD is still often considered as an *alternative* to AC, and the system ZF + AD is consistent assuming the existence of infinitely many *Woodin cardinals* with a *measurable cardinal* above them, by results of Hugh Woodin. Unlike full AD, the following weaker axiom is not contradictory with ZFC.

Definition 1.2.23. *The* Axiom of Projective Determinacy*, PD, is the statement that all projective sets of reals $A \subseteq \omega^\omega$ are determined.*

By the result of Martin and Steel [MS89], PD is true assuming that there are infinitely many Woodin cardinals.

Another variation of the Axiom of Determinacy involves the possibility for players I and II to chose real numbers rather then integers. Let $A \subseteq (\mathbb{R})^\omega$ be the pay-off set, and let $G_\mathbb{R}(A)$ be the corresponding game with real moves. Here we write $\mathbb{R}$ following custom, but this typically refers to the Baire or Cantor space. The concepts of a strategy, winning strategy and determinacy can be defined analogously.

Definition 1.2.24. *The* Axiom of Real Determinacy*, $\mathsf{AD}_\mathbb{R}$, is the statement that every set $A \subseteq (\mathbb{R})^\omega$ is determined (i.e., every game $G_\mathbb{R}(A)$ is determined).*

$\mathsf{AD}_\mathbb{R}$ is stronger than AD, and its consistency can be deduced from an assumption slightly stronger than the existence of infinitely many Woodin cardinals (see [Kan03, Theorem 23.19]).

1.2.9 Cardinal invariants

Cardinal invariants (sometimes called *cardinal coefficients*) are cardinal numbers that have a combinatorial definition but may have different values in different models of set theory. We will only be interested in cardinal invariants of the continuum. The most famous cardinal invariant (if it may be called such) is the cardinality of the continuum itself, $2^{\aleph_0}$. The other invariants $\mathfrak{k}$ are typically defined as the least cardinality of a set of reals with a certain property, and usually have value $\aleph_0 < \mathfrak{k} \leq 2^{\aleph_0}$. Although this dissertation is primarily about questions of definability, certain cardinal invariants will play a crucial role too, so we will give the most important definitions. A detailed introduction can be found e.g. in [Bla10].

Definition 1.2.25.

1. *Let $x, y \in \omega^\omega$. We say that y* dominates *x, notation $x \leq^* y$, if $\forall^\infty n\,(x(n) < y(n))$.*
2. *Let $x, y \in [\omega]^\omega$. We say that y* splits *x if both $x \cap y$ and $x \setminus y$ are infinite.*

Definition 1.2.26.

1. $\mathfrak{b}$, *the* bounding number, *is the least cardinality of a set* $A \subseteq \omega^\omega$ *such that there is no real* y *which dominates every real in* A.

2. $\mathfrak{d}$, *the* dominating number, *is the least cardinality of a set* $A \subseteq \omega^\omega$ *such that every real* x *is dominated by some real in* A.

3. $\mathfrak{r}$, *the* reaping number, *is the least cardinality of a set* $A \subseteq \omega^\omega$ *such that there is no real* y *which splits every real in* A.

4. $\mathfrak{s}$, *the* splitting number, *is the least cardinality of a set* $A \subseteq \omega^\omega$ *such that every real* x *is split by some real in* A.

The next set of invariants concerns σ-ideals on the reals.

Definition 1.2.27. *Let* I *be a* σ*-ideal on* ω^ω*, i.e., an* $I \subseteq \mathscr{P}(\omega^\omega)$ *such that*

1. *if* $A \in I$ *and* $B \subseteq A$ *then* $B \in I$,

2. *if* $A_n \in I$ *for* $n \in \omega$*, then* $\bigcup_n A_n \in I$.

By convention, singletons $\{x\}$ *are assumed to be in the ideal* I *but the entire space* ω^ω *is not. For each such ideal* I *we define the following cardinal invariants:*

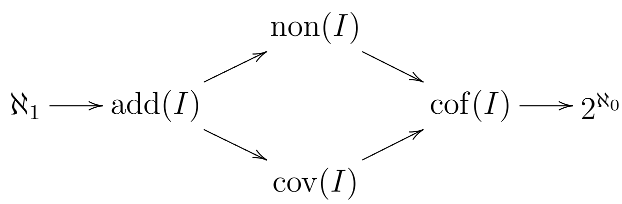
1. $\mathrm{cov}(I)$, *the* covering number of I, *is the least size* κ *of a family* $\{A_\alpha \mid \alpha < \kappa\}$ *of sets in* I *such that* $\bigcup_{\alpha<\kappa} A_\alpha = \omega^\omega$.

2. $\mathrm{add}(I)$, *the* additivity number of I, *is the least size* κ *of a family* $\{A_\alpha \mid \alpha < \kappa\}$ *of sets in* I *such that* $\bigcup_{\alpha<\kappa} A_\alpha \notin I$.

3. $\mathrm{non}(I)$, *the* uniformity number of I, *is the least cardinality of a set* $A \subseteq \omega^\omega$ *such that* $A \notin I$.

4. $\mathrm{cof}(I)$, *the* cofinality number of I, *is the least size* κ *of a family* $\{A_\alpha \mid \alpha < \kappa\}$ *of sets in* I *such that* $\forall A \in I\, \exists \alpha < \kappa\, (A \subseteq A_\alpha)$.

It is easy to see that the inequalities $\aleph_1 \leq \mathrm{add}(I)$, $\mathrm{add}(I) \leq \mathrm{non}(I)$, $\mathrm{add}(I) \leq \mathrm{cov}(I)$, $\mathrm{non}(I) \leq \mathrm{cof}(I)$, $\mathrm{cov}(I) \leq \mathrm{cof}(I)$ and $\mathrm{cof}(I) \leq 2^{\aleph_0}$ are provable in ZFC, as represented in the following diagram (where "$\rightarrow$" represents "$\leq$").



The next cardinal invariants involve the space $[\omega]^\omega$ of infinite subsets of ω. For $a, b \in [\omega]^\omega$, $a \subseteq^* b$ denotes the statement "a is a subset of b modulo a finite set", i.e., $|a \setminus b| < \omega$.

Definition 1.2.28.

1. *A set $A \subseteq [\omega]^\omega$ has the* finite intersection property *(f.i.p.) if for every $a_0, \dots, a_k \in A$, $\bigcap_{i=0}^k a_i$ is infinite. A real $b \in [\omega]^\omega$ is a* pseudo-intersection *of A if $b \subseteq^* a$ for all $a \in A$. Clearly, if A has a pseudo-intersection then it also has the f.i.p., but the converse need not be true. The* pseudo-intersection number $\mathfrak{p}$ *is the smallest size of a set $A \subseteq [\omega]^\omega$ with the f.i.p. but without a pseudo-intersection.*

2. *A* tower *is a collection $A \subseteq [\omega]^\omega$ ordered by reverse almost-inclusion $\supseteq^*$ (i.e., $A = \{a_\alpha \mid \alpha < \kappa\}$ such that if $\alpha < \beta$ then $a_\beta \subseteq^* a_\alpha$) but which does not have a pseudo-intersection. The* tower number $\mathfrak{t}$ *is the smallest size of a tower.*

3. *A collection $D \subseteq [\omega]^\omega$ is* dense *(in $[\omega]^\omega$) if $\forall a \in [\omega]^\omega\ \exists b \subseteq a$ such that $b \in D$; it is* open *if whenever $b \in D$ and $b' \subseteq^* b$, then also $b' \in D$. The* distributivity number $\mathfrak{h}$ *is the least cardinality κ of a set $\{D_\alpha \mid \alpha < \kappa\}$ of open dense sets D_α such that $\bigcap_{\alpha<\kappa} D_\alpha = \varnothing$.*

4. *A collection $A \subseteq [\omega]^\omega$ is* almost disjoint (a.d.) *if $a \cap b$ is finite for every $a, b \in A$. It is* maximal almost disjoint (mad) *if it is infinite, almost disjoint and maximal with regard to that property. The* almost disjointness number $\mathfrak{a}$ *is the least size of a mad family.*

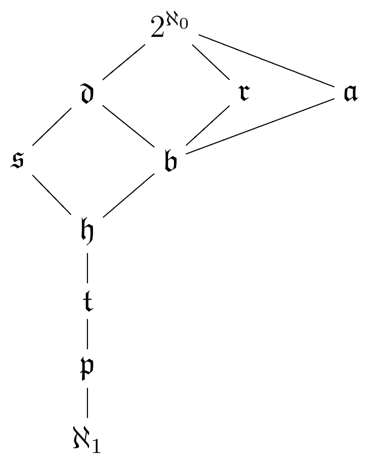


Figure 1.1: Van Douwen's diagram

It is not hard to see that each of the cardinal invariants we have so far defined must at least be uncountable. Also, if the Continuum Hypothesis holds then all cardinal invariants have value $2^{\aleph_0}$, so they are really only interesting in models of $\neg\mathsf{CH}$. It is consistent for each invariant to be larger than $\aleph_1$, and in the absence of CH, a statement involving a cardinal invariant, such as "$\mathfrak{b} > \aleph_1$" or "$\mathfrak{b} < \mathfrak{d}$", tells us something about the structure of the continuum and can be a useful axiom in applications of set theory.

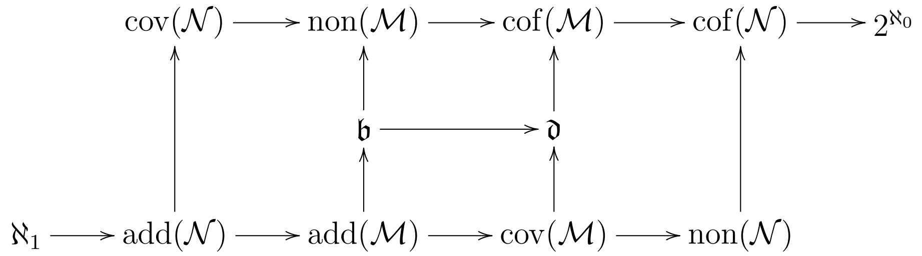


Figure 1.2: Cichoń's diagram

Figures 1.1 and 1.2, known as the *van Douwen diagram* and *Cichoń's diagram*, respectively, show all known ZFC-provable inequalities between the cardinal invariants we have defined, as well as those that involve the *meager* and *Lebesgue null* ideals $\mathcal{M}$ and $\mathcal{N}$ (see Definition 1.3.1).

In these diagrams, all inequalities except $\mathfrak{p} \leq \mathfrak{t}$ have been proven to be consistently strict, i.e., for every such pair $\mathfrak{k}, \mathfrak{l}$ there is a model in which $\mathfrak{k} < \mathfrak{l}$.

1.3 Regularity properties

1.3.1 Definitions

We begin by giving a precise definition of the three classical regularity properties (adapted to the Baire and Cantor spaces rather than the original $\mathbb{R}$).

Definition 1.3.1. *Lebesgue measurability is defined together with a measure function μ mapping subsets of ω^ω or 2^ω to the interval* $[0, 1]$*.*

- *If $s \in \omega^{<\omega}$ is a finite sequence with $|s| = n$, then $\mu([s]) := \prod_{i=0}^{n} \frac{1}{2^{s(i)+1}}$. If we are dealing with $s \in 2^{<\omega}$ then the definition is simply $\mu([s]) := \frac{1}{2^n}$. Note that this is set up so that the size of the whole space is* 1*.*

- *Following a standard measure-theoretic construction, μ can be extended to all Borel sets, by induction on the operations of negation and countable union.*

- *A Borel set B is called* Lebesgue null *if $\mu(B) = 0$, and an arbitrary set A is called* Lebesgue null *if $A \subseteq B$ for some Borel set B with $\mu(B) = 0$, in which case we define $\mu(A) := 0$.*
- *Finally, a set A is called* Lebesgue measurable *if there exists a Borel set B such that $(A \setminus B) \cup (B \setminus A)$ is Lebesgue null, in which case we define $\mu(A) := \mu(B)$.*

The last two lines provide an extension of the notion of a Lebesgue null set and a Lebesgue measurable set beyond the Borel sets on which it was originally defined. We let $\mathcal{N}$ denote the σ-ideal of Lebesgue null sets.

Definition 1.3.2. *The property of Baire is topological in nature.*

- *Recall that in a topological space, a set A is* nowhere dense *if for every basic open set O there is a basic open subset $U \subseteq O$ such that $U \cap A = \varnothing$. A set A is* meager *(also called* of first category*) if it is the countable union of nowhere dense sets.*
- *A set $A \subseteq \omega^\omega$ or 2^ω satisfies the* property of Baire *if there is an open set O such that $(A \setminus O) \cup (O \setminus A)$ is meager.*

The σ-ideal of meager sets is denoted by $\mathcal{M}$.

Definition 1.3.3. *A set $A \subseteq \omega^\omega$ or 2^ω satisfies the* perfect set property *if it is either countable or there exists a perfect set P such that $P \subseteq A$.*

There is a multitude of other notions that could adequately be described "regularity properties", appearing in the most diverse fields of mathematics. It would be impossible to give a complete list, so we will only introduce the more well-known ones, with a special emphasis on those that will play a role in this dissertation.

Our first definition is loosely related to the property of Baire and was first introduced by Edward Marczewski[1] in 1935 ([Szp35]).

Definition 1.3.4. *A set $A \subseteq \omega^\omega$ or 2^ω is* Marczewski measurable *(sometimes called a* Marczewski set*, or* having property (s)*) if for every perfect set P there is a perfect subset $Q \subseteq P$ such that $Q \subseteq A$ or $Q \cap A = \varnothing$.*

The above can clearly be expressed using perfect trees T and the set of their branches $[T]$, and in this setting Marczewski measurability is related to Sacks forcing $\mathbb{S}$ (see Definition 1.2.17). We also obtain the properties $\mathbb{M}$-*Marczewski measurable* and $\mathbb{L}$-*Marczewski measurable* if we replace perfect trees by superperfect (Miller) and Laver trees, respectively ($\mathbb{M}$ and $\mathbb{L}$ standing for the Miller and Laver forcing partial orders).

[1] Before 1940, Marczewski's surname was Szpilrajn; thus, in his pre-1940 publications he is cited with that name.

The above definition can be generalized to an arbitrary collection $\mathbb{X}$ of sets of reals. Thus we can call a set $A \subseteq \omega^\omega$ $\mathbb{X}$-*Marczewski measurable* if $\forall P \in \mathbb{X} \exists Q \in \mathbb{X}$ s.t. $Q \subseteq P$ and $Q \subseteq A$ or $Q \cap A = \varnothing$, although some additional requirements on $\mathbb{X}$ may be necessary for this to be a good notion of regularity (otherwise it may fail for very simple, e.g. closed, sets). This notion was previously investigated by the author in [Kho09], where it was called the *Marczewski-Burstin algebra*, and a similar notion was used by Daisuke Ikegami in [Ike10a]. We will return to this in Chapter 2.

The next property is motivated by infinitary combinatorics. A classical theorem of Frank P. Ramsey [Ram29] says the following: for any partition of $[\omega]^2$ (the set of two-element subsets of ω) into disjoint sets A and B, there is an infinite set $H \subseteq \omega$ such that $[H]^2$ is either completely contained in A or completely contained in B (such an H is called a *homogeneous* set). This easily extends to any natural number n in place of 2, and the interesting question concerns $[\omega]^\omega$. This gives rise to the following definition.

Definition 1.3.5. *A set $A \subseteq [\omega]^\omega$ satisfies the* Ramsey property *if and only if there exists an $x \in [\omega]^\omega$ such that $[x]^\omega \subseteq A$ or $[x]^\omega \cap A = \varnothing$.*

Another, less well-known, though close relative of the Ramsey property is the following:

Definition 1.3.6. *A set $A \subseteq [\omega]^\omega$ satisfies the* doughnut property *if and only if there exist $x, y \in [\omega]^\omega$ such that $y \setminus x$ is infinite and such that $\{z \mid x \subseteq z \subseteq y\} \subseteq A$ or $\{z \mid x \subseteq z \subseteq y\} \cap A = \varnothing$.*

The perfect set property has many relatives too, specifically if we replace perfect trees by different kind of trees.

Definition 1.3.7.

1. *A set $A \subseteq \omega^\omega$ is called K_σ-*regular *if either there is a real x that dominates all $a \in A$ (see Definition 1.2.25), or there is a super-perfect tree T such that $[T] \subseteq A$.*

2. *A tree T on ω is called a* Spinas tree *(due to [Spi94]) if it is super-perfect with the additional requirement that for every ω-splitting node $t \in T$, if s_1 and s_2 extend t and are both ω-splitting, then $|s_1| = |s_2|$; in other words, the next splitting nodes are all a fixed distance away from t.*

   A set $A \subseteq \omega^\omega$ is called a dominating set *if for every $x \in \omega^\omega$ there exists $a \in A$ such that $x \leq^* a$. A is* u-regular *if it is either not a dominating set or there is a Spinas tree T with $[T] \subseteq A$.*

3. *If x and f are two reals, we say that x* strongly dominates f *if* $\forall^\infty n\,(x(n+1) > f(x(n)))$. *A set $A \subseteq \omega^\omega$ is called* strongly dominating *if* $\forall f \in \omega^\omega\, \exists x \in A$ *s.t. x strongly dominates f. $A \subseteq \omega^\omega$ is called* Laver-regular*, or* ℓ-regular*, if it is either not strongly dominating or there is a Laver tree T such that $[T] \subseteq A$.*

These can be understood as certain dichotomies, saying of a set $A \subseteq \omega^\omega$ that it is either "small" in some particular sense, or else contains a certain "large" kind of object. Their isolation is due to [Kec77], [Spi94] and [GRSS95], respectively.

Finally, rather than thinking about regularity we can think about special irregular objects, for example, those defined as the maximal possible sets satisfying a certain property. Consider non-principal ultrafilters U on ω. By an identification of $[\omega]^\omega$ with the Baire or Cantor space, one can easily show that an ultrafilter is not Lebesgue-measurable and does not have the property of Baire. Likewise, one can easily show that an object explicitly derived from U does not have the Ramsey or the doughnut property. So, an ultrafilter can be seen as a special kind of *irregular object*. In other words, the property of *not* being an ultrafilter is considered a notion of regularity. Another object with a similar attitude is a mad family (see Definition 1.2.28).

1.3.2 Regularity of projective sets

All the properties defined above can be violated assuming the Axiom of Choice. On the other hand, all are true for analytic sets. This is due to Suslin [Sus17] for the three classical properties; Marczewski [Szp35] for Marczewski measurability; Silver [Sil70] for the Ramsey property; and Kechris [Kec77, Theorem 4 (i)], Spinas [Spi94, Theorem 1.4] and Goldstern, Repický, Shelah and Spinas [GRSS95, Lemma 2.3] for the three dichotomy-style properties, respectively. Furthermore, there are no analytic ultrafilters (folklore), and no analytic mad families by a result of Adrian Mathias [Mat77, Corollary 4.7]. Other properties we mentioned (doughnut, $\mathbb{M}$- and $\mathbb{L}$-Marczewski measurability), as well as many we have not, also hold on the analytic level. The Baire property, Lebesgue measure, Ramsey, doughnut and all Marczewski-style properties are satisfied by co-analytic sets too, by virtue of the symmetry between the regularity of sets and their complements.

If we wish to continue up the projective hierarchy, we immediately face undecidability issues. Recall that $A \subseteq \omega^\omega$ is a *Bernstein set* if neither A nor its complement contains a perfect set. The Bernstein set is a counterexample to virtually every regularity property, most certainly the ones we have defined above (although this does not apply to irregular objects, i.e., a Bernstein set is not necessarily an ultrafilter or a mad family).

Fact 1.3.8 (Gödel)**.** *In L, there is a $\mathbf{\Delta}^1_2$ Bernstein set.*

Proof. Let $\{P_\alpha \mid \alpha < \aleph_1\}$ enumerate all perfect sets in L, and by induction on $\alpha < \aleph_1$ produce two sets $A = \{a_\alpha \mid \alpha < \aleph_1\}$ and $B = \{b_\alpha \mid \alpha < \aleph_1\}$ as follows: if $\{a_\beta \mid \beta < \alpha\}$ and $\{b_\beta \mid \beta < \alpha\}$ have already been defined, let a_α be the $<_L$-least real in $P_\alpha \setminus (\{a_\beta \mid \beta < \alpha\} \cup \{b_\beta \mid \beta < \alpha\})$, and then let b_α be the $<_L$-least real in $P_\alpha \setminus (\{a_\beta \mid \beta \leq \alpha\} \cup \{b_\beta \mid \beta < \alpha\})$. Both operations are possible because $|P_\alpha| = \aleph_1$ but $\alpha < \aleph_1$.

Clearly $A \cap B = \varnothing$ and both sets contain at least one point from every P_α, so both A and B are Bernstein sets. To see that A (and B) is $\mathbf{\Sigma}^1_2$, we use the same trick as in the proof of Fact 1.2.11. Notice that if $a_\alpha \in L_\delta$ for some countable limit ordinal δ, then the initial segments $\{a_\beta \mid \beta < \alpha\}$ and $\{b_\beta \mid \beta < \alpha\}$ are also in L_δ. Moreover, picking the $<_L$-least element can be expressed within L_δ using an absolute formula defining the initial segment of the well-ordering $<_L$ (see Fact 1.2.10). Therefore the definition of A is absolute between L and some L_δ for a sufficiently large limit ordinal. Then, for every $x \in \omega^\omega$, we may write $x \in A$ iff $\exists \delta < \aleph_1$ ($x \in L_\delta$ and $L_\delta \models x \in A$), which, in turn, may be written as follows: there exists $E \subseteq \omega \times \omega$ such that

1. E is well-founded,

2. $(\omega, E) \models \Theta$, and

3. $\exists n$ ($x = \pi_E(n)$ and $(\omega, E) \models n \in \pi_E^{-1}[A]$).

As in the proof of Fact 1.2.11, the above statement is $\mathbf{\Sigma}^1_2$.

To see that A (and B) is also $\mathbf{\Pi}^1_2$, apply the same trick to the statement $\forall \delta < \aleph_1$ ($x \in L_\delta \rightarrow L_\delta \models x \in A$). □

So, in L, the statement "all $\mathbf{\Delta}^1_2$ sets are regular" fails for all notions of regularity (to prove that there are no $\mathbf{\Delta}^1_2$ ultrafilters requires a separate, though not more difficult, proof; for mad families, see Section 5.1).

We mentioned Solovay's characterization theorem for $\mathbf{\Sigma}^1_2$ sets in the introduction, a significant result linking regularity of $\mathbf{\Sigma}^1_2$ sets with a statement regarding forcing over L. We are now in a position to state it precisely (see Section 1.2.7 for relevant definitions).

Theorem 1.3.9 (Solovay)**.**

1. *All $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable if and only if for every r, $\{x \in \omega^\omega \mid x$ is not random-generic over $L[r]\}$ has measure zero.*

2. *All $\mathbf{\Sigma}^1_2$ sets satisfy the Baire property if and only if for every r, $\{x \in \omega^\omega \mid x$ is not Cohen-generic over $L[r]\}$ is meager.*

We will give a proof of this theorem in a more general setting, see Corollary 2.3.8. A similar characterization holds for $\mathbf{\Delta}^1_2$ sets, due to [IS89, Theorem 3.1].

Theorem 1.3.10 (Judah-Shelah)**.**

1. *All $\mathbf{\Delta}^1_2$ sets are Lebesgue measurable if and only if for every r, there exists a random-generic real over $L[r]$.*

2. *All $\mathbf{\Delta}^1_2$ sets satisfy the Baire property if and only if for every r, there exists a Cohen-generic real over $L[r]$.*

Other properties have their own characterization theorems. We list a few of the more important ones.

Theorem 1.3.11 (Brendle-Löwe)**.**

1. *The following are equivalent:*
   - *(a) all $\mathbf{\Sigma}^1_2$ sets are Marczewski measurable,*
   - *(b) all $\mathbf{\Delta}^1_2$ sets are Marczewski measurable,*
   - *(c) for every r, there exists a real not in $L[r]$.*

2. *The following are equivalent:*
   - *(a) all $\mathbf{\Sigma}^1_2$ sets are $\mathbb{M}$-Marczewski measurable,*
   - *(b) all $\mathbf{\Delta}^1_2$ sets are $\mathbb{M}$-Marczewski measurable,*
   - *(c) for every r, there exists a real y which is unbounded over $L[r]$, i.e., such that no real $x \in L[r]$ dominates y.*

3. *The following are equivalent:*
   - *(a) all $\mathbf{\Sigma}^1_2$ sets are $\mathbb{L}$-Marczewski measurable,*
   - *(b) all $\mathbf{\Delta}^1_2$ sets are $\mathbb{L}$-Marczewski measurable,*
   - *(c) for every r, there exists a real y which is dominating over $L[r]$.*

Proof. See Theorems 7.1, 6.1 and 4.1 from [BL99], respectively. □

Theorem 1.3.12 (Kechris; Spinas; Brendle-Löwe)**.** *The following are equivalent:*

1. *all $\mathbf{\Sigma}^1_2$ are K_σ-regular,*

2. *all $\mathbf{\Pi}^1_1$ are K_σ-regular,*

3. *all $\mathbf{\Sigma}^1_2$ are u-regular,*

4. *all $\mathbf{\Pi}^1_1$ are u-regular,*

5. *all $\mathbf{\Sigma}^1_2$ are Laver-regular,*

6. *for every r, there exists a real y which is dominating over $L[r]$.*

Proof. For the equivalence between 1, 2 and 6, see [Kec77, Section 4], [Iho88, Theorem 1.1] and [IS89, Theorem 3.2]; for 3, 4 and 6, see [Spi94, Theorem 4.2] and [BHS95, Theorem 2.1]; and for 5 and 6, see [BL99, Proposition 4.2]. □

We have deliberately left the perfect set property to the end, because, from the properties we have mentioned so far, it is the only one that has large cardinal strength already on the $\mathbf{\Pi}^1_1$ level.

Theorem 1.3.13 (Mansfield; Solovay; Specker)**.** *The following are equivalent:*

1. *all $\mathbf{\Sigma}^1_2$ sets satisfy the perfect set property,*
2. *all $\mathbf{\Pi}^1_1$ sets satisfy the perfect set property,*
3. $\forall r \in \omega^\omega\ (\aleph_1^{L[r]} < \aleph_1)$.

In this theorem, the direction from 3 to 1 follows from a more general result, usually called the *Mansfield-Solovay theorem*, interesting in its own right.

Theorem 1.3.14 (Mansfield; Solovay)**.** *If A is a $\mathbf{\Sigma}^1_2$ set then either $A \subseteq L$ or A contains a perfect set.*

Proof. This is due to [Man70] as well as Solovay's main work [Sol70]. For an easy proof, see [Jec03, Theorem 25.23]. □

The Mansfield-Solovay theorem can also be relativized to a real r, i.e., every $\Sigma^1_2(r)$ set is either in $L[r]$ or contains a perfect set, and in this form it is clear how the implication $3 \Rightarrow 1$ from Theorem 1.3.13 is obtained. The direction $2 \Rightarrow 3$, originally due to Specker [Spe57], is remarkable because of the following fact:

Fact 1.3.15. *If $\forall r \in \omega^\omega\ (\aleph_1^{L[r]} < \aleph_1)$ then $\aleph_1^V$ is an inaccessible cardinal in L.*

Proof. It is clear that the cardinal $\aleph_1^V$ remains regular in L, so assume, towards contradiction, that it is not a limit cardinal there (which is sufficient since L satisfies GCH). Let α be an ordinal such that, in L, it is a cardinal and $L \models$"$\aleph_1^V = \alpha^+$". Since $\alpha < \aleph_1$ in V, there is a real r which codes α. But then $L[r] \models$"α is countable" and so $L[r] \models$"$\aleph_1^V = \alpha^+ = \aleph_1^{L[r]}$", contradicting the assumption. □

Finally, we should mention that for irregular objects, a characterization theorem is often missing. The following questions are open:

Question 1.3.16.

1. *Is there a statement involving "transcendence over L" equivalent to the statement "there are no $\mathbf{\Sigma}^1_2$ ultrafilters"?*
2. *Is there a statement involving "transcendence over L" equivalent to the statement "there are no $\mathbf{\Sigma}^1_2$ mad families"?*

Another object, closely related to mad families, seems even more mysterious. Call two functions $x, y \in \omega^\omega$ *eventually different* if $\forall^\infty n\ (f(n) \neq g(n))$. Then $A \subseteq \omega^\omega$ is called a *maximal eventually different* (m.e.d.) family if all $x, y \in A$ are eventually different and A is maximal with respect to that property. Unlike mad families, there is no analogue of Mathias' theorem saying that there are no analytic m.e.d. families. More surprisingly, even the following basic question remains open:

Question 1.3.17. *Does there exist a closed m.e.d. family $A \subseteq \omega^\omega$?*

1.3.3 The strength of projective regularity hypotheses

A large part of this dissertation is concerned with the strength of hypotheses involving the regularity of sets (low) in the projective hieararchy. Let Reg be a placeholder for some particular regularity property, let $\boldsymbol{\Gamma}$ denote a projective pointclass (such as analytic, $\boldsymbol{\Sigma}^1_2$, etc), and let $\boldsymbol{\Gamma}(\mathsf{Reg})$ abbreviate the statement "all sets in $\boldsymbol{\Gamma}$ satisfy property Reg". As is now clear, the statements $\boldsymbol{\Sigma}^1_2(\mathsf{Reg})$, $\boldsymbol{\Delta}^1_2(\mathsf{Reg})$, and in some cases $\boldsymbol{\Pi}^1_1(\mathsf{Reg})$ are independent of ZFC. Therefore, it makes sense to consider such a statement as an additional hypothesis and see how strong it is, in the sense of implying other such hypotheses or being itself a consequence of a similar hypothesis.

For example, by [RS85] and [Bar84], $\boldsymbol{\Sigma}^1_2$(Lebesgue) implies $\boldsymbol{\Sigma}^1_2$(Baire), while the converse is false. Likewise, "$\boldsymbol{\Delta}^1_2$(Lebesgue) $\Rightarrow$ $\boldsymbol{\Delta}^1_2$(Baire)" and "$\boldsymbol{\Delta}^1_2$(Baire) $\Rightarrow$ $\boldsymbol{\Delta}^1_2$(Lebesgue)" are both false. In [IS89] it is shown that $\boldsymbol{\Sigma}^1_2$(Baire) implies $\boldsymbol{\Sigma}^1_2/\boldsymbol{\Pi}^1_1$($K_\sigma$-regularity) and $\boldsymbol{\Sigma}^1_2$(Ramsey) implies $\boldsymbol{\Sigma}^1_2/\boldsymbol{\Pi}^1_1$($K_\sigma$-regularity), whereas both converses are false. Other work dealing with similar questions includes [Iho88, BL99, Hal03, BHL05, BL11, Ike10a].

If we have a characterization theorem, we gain some control over the truth of the statements $\boldsymbol{\Sigma}^1_2(\mathsf{Reg})$ and $\boldsymbol{\Delta}^1_2(\mathsf{Reg})$ in various models, using the method of *iterated forcing* briefly described in Section 1.2.7. All the characterization theorems we mentioned have the following form: $\boldsymbol{\Delta}^1_2(\mathsf{Reg})$ holds if and only if for every r, there exist a certain type of *transcendent real* over the ground model $L[r]$; and $\boldsymbol{\Sigma}^1_2(\mathsf{Reg})$ holds if and only if for every r, there exist *many* transcendent reals of this type over the ground model $L[r]$ (where "many" is defined using some ideal closely related to the transcendence property). Therefore, if we find a forcing $\mathbb{P}$ which adds this specific type of reals to the ground model and extend L using an $\aleph_1$-iteration of $\mathbb{P}$ (with finite support if $\mathbb{P}$ is c.c.c. or countable support when $\mathbb{P}$ is proper but non-c.c.c.), we obtain a model in which $\boldsymbol{\Delta}^1_2(\mathsf{Reg})$ holds. Likewise, if we find another forcing $\mathbb{P}'$ which adds many reals of the required type to the ground model, then a forcing iteration of length $\aleph_1$ will yield a model where $\boldsymbol{\Sigma}^1_2(\mathsf{Reg})$ is true. Conversely, if we know of a forcing $\mathbb{Q}$ that it does *not* add such reals (neither in a one-step extension nor in the iteration), then an iteration (or in some cases a product) of $\mathbb{Q}$ will yield a model in which $\boldsymbol{\Delta}^1_2(\mathsf{Reg})$ is false;

similarly for $\mathbf{\Sigma}^1_2(\textsf{Reg})$ and "not adding many such reals".

Now suppose we are dealing with two properties, $\textsf{Reg}_1$ and $\textsf{Reg}_2$, each of them related to reals of type 1 and type 2, respectively, via characterization theorems. How can we compare the strength of hypotheses involving these two properties? If we can prove that the existence of type 1 reals leads to the existence of type 2 reals, then we have proved $\mathbf{\Delta}^1_2(\textsf{Reg}_1) \Rightarrow \mathbf{\Delta}^1_2(\textsf{Reg}_2)$. On the other hand, if we can find a forcing $\mathbb{P}$ which adds reals of type 1 but not of type 2, then we can produce a model of $\mathbf{\Delta}^1_2(\textsf{Reg}_1) + \neg\mathbf{\Delta}^1_2(\textsf{Reg}_2)$. The same can be done on the $\mathbf{\Sigma}^1_2$ level and the existence of "many" reals. Thus, in the presence of characterization theorems, comparing the strength of regularity hypothesis for $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ sets boils down to proving something about adding or not adding specific types of reals.

Example 1.3.18.

1. It is well-known that random forcing does not add Cohen reals, and Cohen forcing does not add random reals. Therefore, using the equivalence in Theorem 1.3.10, in the *random model* (i.e., product of random forcing over L) $\mathbf{\Delta}^1_2$(Lebesgue) holds but $\mathbf{\Delta}^1_2$(Baire) fails, whereas in the *Cohen model* (iteration/product of Cohen forcing over L), $\mathbf{\Delta}^1_2$(Baire) holds and $\mathbf{\Delta}^1_2$(Lebesgue) fails. So (consistently) there are no implications between these two statements.

2. By parts 2 and 3 of Theorem 1.3.11, $\mathbb{L}$-measurability is connected to dominating reals and $\mathbb{M}$-measurability to unbounded reals (i.e., reals that are not dominated by a real from the ground model). As a dominating real is, by definition, unbounded, we have (in $\textsf{ZFC}$) the implication $\mathbf{\Sigma}^1_2(\mathbb{L}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{M})$. Conversely, it is known that Miller forcing adds unbounded reals but not dominating reals. Therefore, the *Miller model* (iteration of Miller forcing) is a witness of $\mathbf{\Sigma}^1_2(\mathbb{M}) \not\Rightarrow \mathbf{\Sigma}^1_2(\mathbb{L})$.

Note that there is a "strongest possible" and a "weakest possible" hypothesis in this context. The strongest hypothesis is the statement "$\forall r\ (\aleph_1^{L[r]} < \aleph_1)$". As this implies that $\omega^\omega \cap L[r]$ is countable for every r, any other "transcendence" over $L[r]$ can be deduced by a diagonal argument. Indeed, this statement implies $\mathbf{\Sigma}^1_2(\textsf{Reg})$ for nearly all known notions of regularity, and it is the only hypothesis of this kind that can *not* be obtained by any forcing iteration starting from L (as it has the consistency strength of an inaccessible cardinal). We have seen this to be equivalent to $\mathbf{\Pi}^1_1$(perfect set property), and there are other natural statements equivalent to it, for example $\mathbf{\Sigma}^1_2$(Baire property in the dominating topology), see [BL99, Theorem 5.11].

On the other hand, $\mathbf{\Sigma}^1_2(\mathbb{S})$ (where $\mathbb{S}$ abbreviates Marczewski-measurability, due to its relation to Sacks forcing) is the weakest possible hypothesis, since by 1 of Theorem 1.3.11 it is equivalent to the statement $\forall r\ (\omega^\omega \cap L[r] \neq \omega^\omega)$ which will certainly hold in any (non-trivial) forcing extension.

All other hypotheses of the form $\mathbf{\Sigma}^1_2(\textsf{Reg})$, $\mathbf{\Delta}^1_2(\textsf{Reg})$ and $\mathbf{\Pi}^1_1(\textsf{Reg})$ are located somewhere in between these two extreme cases, and can be seen as asserting that the set-theoretic universe is larger than L in some specific sense.

There is a close relationship between these types of questions and cardinal invariants of the continuum. Suppose we have the two properties $\textsf{Reg}_1$ and $\textsf{Reg}_2$, connected to type 1 and type 2 "transcending" reals via a characterization theorem. Define the cardinal invariants

- $\mathfrak{k}_1 :=$ least size of $A \subseteq \omega^\omega$ such that there are no type 1 reals over A, and
- $\mathfrak{k}_2 :=$ least size of $A \subseteq \omega^\omega$ such that there are no type 2 reals over A.

Then the following can be said:

1. $\mathfrak{k}_1 > \aleph_1 \;\Rightarrow\; \mathbf{\Delta}^1_2(\textsf{Reg}_1)$ and $\mathfrak{k}_2 > \aleph_1 \;\Rightarrow\; \mathbf{\Delta}^1_2(\textsf{Reg}_2)$ (provably in $\textsf{ZFC}$).
2. If $\mathfrak{k}_1 \leq \mathfrak{k}_2$ is provable in $\textsf{ZFC}$, then, most likely, the same proof will show that $\mathbf{\Delta}^1_2(\textsf{Reg}_1) \Rightarrow \mathbf{\Delta}^1_2(\textsf{Reg}_2)$.
3. If it is consistent that $\mathfrak{k}_1 < \mathfrak{k}_2$, then, most likely, the same model will show that $\mathbf{\Delta}^1_2(\textsf{Reg}_2) \not\Rightarrow \mathbf{\Delta}^1_2(\textsf{Reg}_1)$.

Here, point 1 is a straightforward $\textsf{ZFC}$-theorem, but 2 and 3 are not. In 2, the idea is the following: if $\mathfrak{k}_1 \leq \mathfrak{k}_2$ is provable in $\textsf{ZFC}$, then *most likely*, the proof involves showing that, for any $A \subseteq \omega^\omega$, if there is a type 1 transcendent real over A or over a closely related set A', then there is a type 2 transcendent real over A. This is also a proof of $\mathbf{\Delta}^1_2(\textsf{Reg}_1) \Rightarrow \mathbf{\Delta}^1_2(\textsf{Reg}_2)$, substituting $L[r]$ in place of the sets A and A'. Point 3 is even more vague: the idea there is that if $\mathfrak{k}_1 < \mathfrak{k}_2$ is consistent *and if the proof uses an iterated forcing argument*, then, *most likely*, it involves a forcing $\mathbb{P}$ which adds reals of type 2 (in order to make $\mathfrak{k}_2$ large) but does not add reals of type 1 (in order to keep $\mathfrak{k}_1$ small). Then an iteration of this forcing will show that $\mathbf{\Delta}^1_2(\textsf{Reg}_2) \not\Rightarrow \mathbf{\Delta}^1_2(\textsf{Reg}_1)$. However, there may be other methods of proving that $\mathfrak{k}_1 < \mathfrak{k}_2$ is consistent, and those will not necessarily lead to the same conclusion. Points 2 and 3 cannot be turned into precise $\textsf{ZFC}$-theorems since the converse of point 1 does not hold: for example, in any $\aleph_1$-iteration there will still be just $\aleph_1$ many reals so all cardinal invariants will have value $\aleph_1$, whereas $\mathbf{\Delta}^1_2(\textsf{Reg})$ may hold due to the characterization theorem. The same will also be true in any model of $\textsf{CH}$ with a measurable cardinal. So the concepts of *cardinality* and *regularity* are inherently different.

Nevertheless, the above insight is very useful because it allows us to apply results from the (much more well-researched) field of cardinal invariants to questions of regularity. For example, part of Cichoń's diagram can be translated to a diagram involving regularity hypotheses, as represented in Figure 1.3. Here, note that $\mathfrak{b}$ and $\mathfrak{d}$ correspond to $\mathbb{L}$- and $\mathbb{M}$-Marczewski measurability, respectively, the

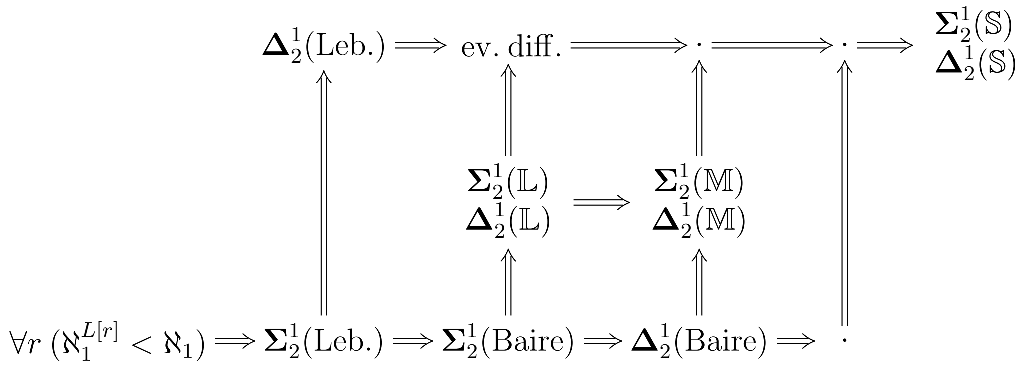


Figure 1.3: Cichoń's diagram for regularity hypotheses.

covering numbers of the null and the meager ideals correspond to Lebesgue measurability and the Baire property on the $\mathbf{\Delta}^1_2$ level, and the additivity numbers of these ideals to the same properties on the $\mathbf{\Sigma}^1_2$ level. The bottom left and top right corners show the strongest and the weakest hypotheses, respectively. The statement "ev. diff." abbreviates $\forall r\, \exists x$ (x is eventually different from all reals in $L[r]$) and has been included for completeness, with the relationship to $\mathrm{non}(\mathcal{M})$ due to [BJ95, Theorem 2.4.7]. It is currently unknown whether this statement is equivalent to some natural regularity hypothesis, see Example 2.5.6 (4) for more about this. Likewise, it is not clear whether a regularity hypothesis can be put in the places left empty in the diagram.

Regularity hypothesis	Transcendence over $L[r]$	Cardinal invariant
$\forall r\ (\aleph_1^{L[r]} < \aleph_1)$	"co-countable" set of new reals	$\aleph_1$
$\mathbf{\Sigma}^1_2$(Lebesgue)	measure-one set of random reals	$\mathrm{add}(\mathcal{N})$
$\mathbf{\Delta}^1_2$(Lebesgue)	random reals	$\mathrm{cov}(\mathcal{N})$
$\mathbf{\Sigma}^1_2$(Baire)	co-meager set of Cohen reals	$\mathrm{add}(\mathcal{M})$
$\mathbf{\Delta}^1_2$(Baire)	Cohen reals	$\mathrm{cov}(\mathcal{M})$
?	eventually different reals	$\mathrm{non}(\mathcal{M})$
$\mathbf{\Delta}^1_2(\mathbb{L})$ / $\mathbf{\Sigma}^1_2(\mathbb{L})$	dominating reals	$\mathfrak{b}$
$\mathbf{\Delta}^1_2(\mathbb{M})$ / $\mathbf{\Sigma}^1_2(\mathbb{M})$	unbounded reals	$\mathfrak{d}$
$\mathbf{\Delta}^1_2(\mathbb{S})$ / $\mathbf{\Sigma}^1_2(\mathbb{S})$	new reals	$2^{\aleph_0}$

Table 1.1: Correspondence between regularity, transcendence and cardinal invariants.

Each implication in Figure 1.3 follows from the same argument that proves the corresponding cardinal inequality. Moreover, there are (consistently) no other implications between the properties in this diagram, and this fact, too, is witnessed by the same model that witnesses the strict cardinal inequality. The connection between cardinal invariants and regularity statements in general will become more apparent in Sections 2.3 when we introduce *quasi-generic reals.*

1.4 Summary of results

All results in this dissertation are related to the questions described above. In Chapter 2, we present a systematic treatment of regularity properties, in a framework that is heavily influenced by the methods of forcing. This allows us to extract common features out of many proofs involving regularity and forcing, and to generalize known results. A similar enterprise was already undertaken by Daisuke Ikegami in [Ike10a, Ike10b], and part of this chapter can be seen as an extension of his approach to a more general framework. Although many theorems are stated in a new way, the methods and proofs used there were, for the most part, known prior to our work. The main purpose here is the unification of ideas and methods from different areas, and the making precise of various intuitions, heuristics and unpublished results in this field. Many interesting questions are isolated in this process which seem worthy of further study. Furthermore, many results of this chapter will be used for reference in subsequent chapters.

In Chapter 3, we investigate a recently isolated partition property on the second level of the projective hierarchy. We prove a number of results about implications and non-implications between this and other regularity properties. The most interesting result is:

Theorem 3.5.3. *It is consistent that* $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ *holds whereas both* $\mathbf{\Delta}^1_2(\mathbb{M})$ *and* $\mathbf{\Delta}^1_2(\text{doughnut})$ *fail.*

The proof involves a combinatorially involved forcing notion, one among the many so-called *creature forcings* of Saharon Shelah. As this type of forcing does not fit the framework of Chapter 2, special proofs are required for many results in this chapter.

In Chapter 4, we turn our attention to *Hausdorff gaps*, a classical object first constructed by Felix Hausdorff in 1936 in [Hau36]. In [Tod96, Theorem 1], Stevo Todorčević proved that such objects do not exist on the analytic level, suggesting that Hausdorff gaps are irregular objects, i.e., that the lack of such gaps can be considered a regularity hypothesis. Our main results are the following:

Corollary 4.3.10. *The following are equivalent:*

1. *there is no* $(\mathbf{\Sigma}^1_2, \cdot)$*-Hausdorff gap,*

2. *there is no* $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$*-Hausdorff gap,*

3. *there is no* $(\mathbf{\Pi}^1_1, \cdot)$*-Hausdorff gap,*

4. *there is no* $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$*-Hausdorff gap,*

5. $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$.

Corollary 4.4.3. Con(ZFC+ *"there are no projective Hausdorff gaps"*) *and* Con(ZF + DC+ *"there are no Hausdorff gaps").*

Corollary 4.5.4. $\mathsf{AD}_{\mathbb{R}}$ *implies that there are no Hausdorff gaps.*

In Chapter 5, we consider mad families from the descriptive-theoretic point of view. We define a new cardinal invariant, the *Borel-almost-disjointness number* $\mathfrak{a}_B$, which is related to the existence of $\mathbf{\Sigma}^1_2$ mad families in a similar way as the additivity/covering numbers of $\mathcal{M}$ and $\mathcal{N}$ are related to $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2$(Baire) and $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2$(Lebesgue). The main result in this chapter is that $\mathfrak{a}_B < \mathfrak{b}$ is consistent. More precisely, we prove the following:

Theorem 5.3.3 *In the* $\aleph_2$*-iteration of Hechler forcing (with finite support) starting from a model of* CH*,* $\mathfrak{b} = \aleph_2$ *while* $\mathfrak{a}_B = \aleph_1$.

With a minimal modification in the proof of this theorem, we obtain the consistency of $\mathfrak{b} > \aleph_1+$ "there is a $\mathbf{\Sigma}^1_2$ mad family", answering a question posed by Friedman and Zdomskyy in [FZ10, Question 16]. Likewise, we obtain the consistency of $\mathbf{\Sigma}^1_2(\mathbb{L})+$"there is a $\mathbf{\Sigma}^1_2$ mad family".

Chapter 2

Idealized regularity

Since the developments of forcing in the 1960s and Solovay's celebrated result [Sol70] establishing the consistency of "$\mathsf{ZF} + \mathsf{DC}+$ all sets of reals are Lebesgue measurable, have the property of Baire and the perfect set property", it gradually became commonplace to associate regularity properties with a notion of forcing. Random forcing was specifically designed by Solovay to prove the measurability result, Cohen forcing is naturally related to the Baire property, and we have already seen in Section 1.3.3 that Marczewski-style properties can be viewed in a forcing context. Frequently, a regularity property is isolated because of its significance for the combinatorics of certain forcings, and conversely, understanding a regularity property usually greatly benefits from finding a forcing that corresponds to it.

At first, one might have the hope that *all* regularity properties can be formulated in terms of forcing. Unfortunately, this seems over-ambitious and in subsequent chapters we will consider properties that do not seem to fall into this category. Nevertheless, a large number of regularity properties *can* be directly formulated as properties of a certain forcing, and it turns out that the framework of *idealized forcing* introduced by Jindřich Zapletal is very well suited for this purpose. The goal of this chapter is to develop a systematic theory of regularity properties in this framework.

An important inspiration for this chapter is the work of Daisuke Ikegami in [Ike10a, Ike10b], who considered a wide class of forcing notions called *strongly arboreal forcings* and showed that many regularity properties can be stated directly in terms of a forcing from this class. In Section 2.3 we pay special credit to these results and generalize the main theorem of [Ike10a].

It should be noted that despite the novel framework, most proofs in this chapter are not really new, but variations on, or generalizations of, arguments found in various other sources, such as the original result of Solovay, the work of Zapletal and Ikegami, and folklore results. In Sections 2.5 we present a slightly different point of view, raising interesting questions suitable for further research.

2.1 Idealized forcing

An *ideal* on ω^ω is a set $I \subseteq \mathscr{P}(\omega^\omega)$ which is closed under subsets (if $B \in I$ and $A \subseteq B$ then $A \in I$) and unions (if $A, B \in I$ then $A \cup B \in I$). By standard convention, we also assume that all singletons $\{x\}$ are in the ideal and that the whole space ω^ω is not. A *σ-ideal* is an ideal that is additionally closed under countable unions. For convenience we will usually talk of the Baire space when giving definitions, proving theorems etc., but in most cases this can easily be adapted to the Cantor space, the space $[\omega]^\omega$, or any other incarnation of the real numbers. Sets that lie in the ideal I will be called *I-small* and those that do not will be called *I-positive*, following standard practice.

In [Zap04] and [Zap08], Jindřich Zapletal developed an extensive theory of *idealized forcing*, i.e., using the partial order of I-positive Borel sets of reals, ordered by inclusion, as a forcing notion. We start by reviewing a few of the basic concepts and results.

Definition 2.1.1. *Let I be a σ-ideal on ω^ω. Let $\mathbb{P}_I := \mathcal{B}(\omega^\omega) \setminus I$ denote the partial order of all Borel I-positive subsets of ω^ω ordered by inclusion.*

Fact 2.1.2 (Zapletal)**.**

1. *If G is a $\mathbb{P}_I$-generic filter, then there is a unique real $x_G \in V[G]$ such that for all Borel sets coded in V, $x_G \in B^{V[G]}$ iff $B^V \in G$. This is called the* generic real*, and the generic filter can be recovered from the generic real using the previous characterization, so $V[G] = V[x_G]$.*

2. *If $\dot{x}_G$ is the name for the generic real, then every I-positive B forces $\dot{x}_G \in B$, and for every $B \in I$, $\Vdash \dot{x}_G \notin B$.*

Proof. See [Zap08, Proposition 2.1.2]. □

In particular, a real x is $\mathbb{P}_I$-generic over a transitive model M if $x \in \bigcup D$ for every dense $D \in M$. When M is a non-transitive elementary submodel then, in accordance to common usage, we will say that a real x is $(M, \mathbb{P}_I)$-generic if $x \in \bigcup(D \cap M)$ for every dense $D \in M$. This is equivalent to saying that x is the real derived from an $(M, \mathbb{P}_I)$-generic filter (see Definition 1.2.16). We will often drop the reference to $\mathbb{P}_I$ if it is clear from context.

Recall from Definition 1.2.16 that a forcing notion $\mathbb{P}$ is *proper* if for every countable elementary submodel $M \prec \mathcal{H}_\theta$ of a sufficiently large structure and every $p \in \mathbb{P} \cap M$, there is a *master condition* $q \leq p$, that is, a condition q such that $q \Vdash$ "$\dot{G}$ is an M-generic filter"—or alternatively, $q \Vdash$ "$\dot{x}_G$ is an M-generic real".

Fact 2.1.3 (Zapletal).

1. *Let I be a σ-ideal and $M \prec \mathcal{H}_\theta$ a countable elementary submodel of a sufficiently large structure. Then for every $B \in \mathbb{P}_I \cap M$, the set*

$$C := \{x \in B \mid x \text{ is } M\text{-generic}\}$$

is Borel.

2. *The forcing $\mathbb{P}_I$ is proper iff for every countable $M \prec \mathcal{H}_\theta$ and every $B \in \mathbb{P}_I \cap M$, the set C from above is I-positive.*

Proof. See [Zap08, Proposition 2.2.2]. □

This set C *is* the master condition, having the additional property that *every* real $x \in C$ is M-generic. This highly useful aspect of properness was used numerous times in [Zap04, Zap08] and we shall use it to good effect in many of our arguments, too.

Another essential feature of properness is the generation of new reals from the generic real, by Borel functions encoded in the ground model.

Theorem 2.1.4 (Zapletal). *Let $\mathbb{P}_I$ be a proper forcing and $\dot{x}$ a name for a real. Then there is a Borel function f and a condition B such that $B \Vdash \dot{x} = f(\dot{x}_G)$.*

Proof. See [Zap08, Proposition 2.3.1]. □

All the ideals we consider will have an absolute definition. To be precise, if B is a Borel set, then the statement "$B \in I$" will be $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2$ (formally this means that the sentence $\phi(x)$, saying that "the Borel set encoded by the real number x is in I", has complexity $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2$). In particular, the membership of Borel sets in I will be absolute between transitive models containing ω_1, by Shoenfield absoluteness.

Idealized forcings are related to more standard forcings (using simple combinatorial objects) via dense embeddings. Suppose that I is a σ-ideal and Q a partial order consisting of simple sets (e.g., closed), also ordered by inclusion, and such that every $q \in Q$ is I-positive and every Borel I-positive set contains some $q \in Q$ as a subset. Then, clearly, there is a dense embedding from Q to $\mathbb{P}_I$ (denoted by $Q \hookrightarrow_{\mathrm{d}} \mathbb{P}_I$) so the two are forcing equivalent. At the heart of this embedding lies a *dichotomy theorem*: every Borel set is either in I or contains a set $q \in Q$. Such theorems are typically hard to prove and require some familiarity with the specific combinatorics of the objects and the ideal. There are at least three different methods for this: the classical method, using direct combinatorial properties; the "forcing" method, using forcing with Q and some absoluteness results; and the game-theoretic method, using the Borel determinacy of a corresponding game. Notice that this is exactly the same as saying that all Borel sets satisfy a certain *regularity property*, namely the property of either being I-small or containing a (large) object $q \in Q$. We have already mentioned the perfect set property, K_σ-regularity and Laver-regularity (see Definition 1.3.7), which are all satisfied by analytic, and therefore Borel, sets. The following list shows some typical examples of this phenomenon (see Definition 1.2.17 for the forcing partial orders).

Example 2.1.5.

1. Let ctbl be the σ-ideal of countable sets, and recall the Sacks forcing partial order $\mathbb{S}$ consisting of perfect sets. Since Borel sets satisfy the perfect set property, it follows that there is a dense embedding $\mathbb{S} \hookrightarrow_{\mathrm{d}} \mathcal{B}(\omega^\omega) \setminus \mathsf{ctbl}$.

2. Let K_σ be the ideal of σ-compact sets, i.e., sets A such that some x dominates all $a \in A$, and recall the Miller forcing partial order $\mathbb{M}$ consisting of super-perfect trees. Since Borel sets satisfy K_σ-regularity, there is a dense embedding $\mathbb{M} \hookrightarrow_{\mathrm{d}} \mathcal{B}(\omega^\omega) \setminus K_\sigma$.

3. Let $I_{\mathbb{L}}$ be the *Laver ideal*, defined as the ideal of all sets A which are *not* strongly dominating (see Definition 1.3.7 (3)). Recall the Laver partial order $\mathbb{L}$. Since Borel sets satisfy the Laver-regularity, there is a dense embedding $\mathbb{L} \hookrightarrow_{\mathrm{d}} \mathcal{B}(\omega^\omega) \setminus I_{\mathbb{L}}$.

4. Consider the Lebesgue null ideal $\mathcal{N}$. Random forcing is the algebra $\mathcal{B}(\omega^\omega) \setminus \mathcal{N}$, but by classical results it is known that every Borel set of positive measure contains a closed set of positive measure. Therefore, the collection of all closed subsets of ω^ω with positive Lebesgue measure is forcing equivalent to $\mathcal{B}(\omega^\omega) \setminus \mathcal{N}$.

5. Recall the Mathias forcing partial order $\mathbb{R}$. Each condition $(s, S) \in \mathbb{R}$ gives rise to the closed set $[s, S] := \{s \cup a \mid a \in [S]^\omega\}$. Such sets generate the *Ellentuck topology*, due to Ellentuck [Ell74], and we can consider the σ-ideal I_{RN} of sets meager in this topology, equal to the σ-ideal of nowhere dense sets in this topology, also called the *Ramsey-null ideal.* By Ellentuck's original proof [Ell74], every Borel set is either in I_{RN} or contains a set of the form $[s, S]$. Therefore there is a dense embedding $\mathbb{R} \hookrightarrow_{\mathrm{d}} \mathcal{B}([\omega]^\omega) \setminus I_{\mathrm{RN}}$.

6. The last two examples we exhibit are somewhat more involved. Let E_0 be the equivalence relation on 2^ω given by xE_0y iff $\forall^\infty n\ (x(n) = y(n))$. A *partial E_0-transversal* is a set A which contains at most one element from each E_0-equivalence class, in other words, $\forall x, y \in A$: if $x \neq y$ then $\exists^\infty n\ (x(n) \neq y(n))$. Let I_{E_0} be the σ-ideal generated by Borel partial E_0-transversals.

The Borel equivalence relation E_0 is well-known among descriptive set theorists and it played a key role in the study of the Glimm-Effros dichotomy in [HKL90]. The ideal I_{E_0} was investigated by Zapletal who, among other things, isolated the notion of an E_0-*tree*.

Definition 2.1.6. *(Zapletal) An* E_0-tree *is a perfect tree* $T \subseteq 2^{<\omega}$ *such that*

(a) there is a stem s_0 *with* $|s_0| = k_0$*, and*

(b) there are numbers $k_0 < k_1 < k_2 < \dots$ *and for each* i *exactly two sequences* $s_0^i, s_1^i \in {}^{[k_i,k_{i+1})}2$*, such that*

$$[T] = \{s_0{}^\frown s^0_{z(0)}{}^\frown s^1_{z(1)}{}^\frown s^2_{x(2)}{}^\frown \cdots \mid z \in 2^\omega\}.$$

Let $\mathbb{E}_0$ denote the forcing partial order of E_0-trees ordered by inclusion. A standard fusion argument can be used to show the properness of this forcing. Zapletal [Zap04, Lemma 2.3.29] proved the corresponding dichotomy: every Borel (even analytic) set is either in I_{E_0} or contains $[T]$ for some $T \in \mathbb{E}_0$. It follows that $\mathbb{E}_0 \hookrightarrow_{\mathrm{d}} \mathcal{B}(2^\omega) \setminus I_{E_0}$.

7. Similarly to the above, let G be the relation on 2^ω given by xGy iff there is exactly one n such that $(x(n) \neq y(n))$. A Borel set B is G-*independent* if any two distinct elements $x, y \in B$ are not G-related, i.e., if $x \neq y$ then x and y differ in at least two digits. Let I_G be the σ-ideal generated by Borel G-independent sets. This definition is due to [Zap04, Section 2.3.11].

Definition 2.1.7. *A perfect tree* $T \subseteq 2^{<\omega}$ *is called a* Silver tree *if for every* $s, t \in T$*, if* $|s| = |t|$ *then* $\{i \in 2 \mid s^\frown\langle i\rangle \in T\} = \{i \in 2 \mid t^\frown\langle i\rangle \in T\}$*, i.e., the branching at each node depends only on the length of that node. Another way to put this is:* $[T] = \prod_{n\in\omega} Z_n$ *for some sequence* $\{Z_n \mid n \in \omega\}$ *such that each* Z_n *is either* $\{0\}$*,* $\{1\}$*, or* $\{0,1\}$*, and such that* $\exists^\infty n\,(Z_n = \{0,1\})$*. The partial order of Silver trees ordered by inclusion is called* Silver forcing *and typically abbreviated by* $\mathbb{V}$*.*

By [Zap04, Lemma 2.3.37], every Borel (even analytic) set is either in the ideal I_G or contains $[T]$ for some Silver tree T. Therefore $\mathbb{V} \hookrightarrow_{\mathrm{d}} \mathcal{B}(2^\omega) \setminus I_G$.

There are other situations (often when the forcing $\mathbb{P}_I$ satisfies the c.c.c.) when there is no strict dichotomy in the above sense, but rather one in the sense of "modulo I". Suppose that for every I-positive Borel set B there is a $q \in Q$ such that $(q \setminus B) \in I$. In this case there is no dense embedding from Q to $\mathbb{P}_I$ but only to the algebra $\mathcal{B}(\omega^\omega)/I$ of Borel sets modulo I. Since there is always a dense embedding from $\mathbb{P}_I$ to $\mathcal{B}(\omega^\omega)/I$, the two notions Q and $\mathbb{P}_I$ are still forcing equivalent.

Example 2.1.8.

1. Consider the meager ideal $\mathcal{M}$ and Cohen forcing $\mathbb{C}$. From the fact that Borel sets satisfy the Baire property, it follows that for any Borel non-meager set there is a basic open set $[s]$ contained in B modulo meager. Therefore $\mathbb{C} \hookrightarrow_{\mathrm{d}} \mathcal{B}(\omega^\omega)/\mathcal{M}$ so $\mathbb{C}$ and $\mathcal{B}(\omega^\omega) \setminus \mathcal{M}$ are forcing equivalent.

2. Recall the Hechler partial order $\mathbb{D}$. Each condition $(s, f) \in \mathbb{D}$ gives rise to the closed set $[s, f] := \{x \in \omega^\omega \mid s \subseteq x \text{ and } \forall n\ (f(n) < x(n))\}$, and these sets generate the *dominating topology*, a non-second-countable topology refining the standard topology on ω^ω. Let $\mathcal{M}_\mathbb{D}$ be the ideal of sets meager in the dominating topology. For the same reason as with Cohen forcing, $\mathbb{D}$ densely embeds into $\mathcal{B}(\omega^\omega)/\mathcal{M}_\mathbb{D}$, and hence is forcing equivalent to $\mathcal{B}(\omega^\omega) \setminus \mathcal{M}_\mathbb{D}$.

Many other interesting examples can be found in [Zap04, Zap08] to which we refer the reader for further study.

We will mostly be interested in ideals that are *Borel generated*, in the sense that every $A \in I$ is contained in some Borel set $B \in I$. Even when an ideal I does not have this property by nature, for our purposes it will be sufficient (and often, more interesting—cf. Section 2.4) to consider its Borelized version $I^\mathcal{B} := \{A \in \mathscr{P}(\omega^\omega) \mid A \subseteq B \text{ for some Borel set } B \in I\}$.

If I is a Borel generated σ-ideal, we can use it to derive a "Marczewski-null-style" ideal.

Definition 2.1.9. *Let I be a Borel generated σ-ideal. Define $\mathcal{N}_I$ by stipulating*

$$A \in \mathcal{N}_I \ :\Longleftrightarrow\ \forall B \in \mathbb{P}_I\ \exists C \leq B\ (C \cap A = \varnothing).$$

Lemma 2.1.10. *Let I be a Borel generated σ-ideal such that $\mathbb{P}_I$ is proper.*

1. *$\mathcal{N}_I$ is a σ-ideal extending I and coinciding with I on Borel sets.*

2. *If $\mathbb{P}_I$ satisfies the c.c.c., then $\mathcal{N}_I = I$.*

Proof.

1. It is clear that if $A \in I$ then there is a $B \supseteq A$ is such that $B \in I$, and any $C \in \mathbb{P}_I$ can be extended to $C \setminus B \in \mathbb{P}_I$ which is disjoint from A, so $A \in \mathcal{N}_I$. Also it is clear that if B is Borel and I-positive, then B itself is a witness to the fact that $B \notin \mathcal{N}_I$.

   To prove that $\mathcal{N}_I$ is a σ-ideal we use properness: let A_n be sets in $\mathcal{N}_I$ and let $A = \bigcup_n A_n$. For each n, let $D_n := \{B \in \mathbb{P}_I \mid B \cap A_n = \varnothing\}$. By definition all D_n are dense. Fix some B and let M be a countable model containing all the D_n and B. Let $C := \{x \in B \mid x \text{ is } M\text{-generic}\}$. By properness and Fact 2.1.3 C is I-positive, and for every $x \in C$ and every n we have $x \in \bigcup(D_n \cap M)$, implying that $x \notin A_n$. Therefore $C \cap A = \varnothing$ as had to be shown.

2. Let $A \in \mathcal{N}_I$ and define $D := \{B \in \mathbb{P}_I \mid B \cap A = \varnothing\}$. Since D is dense, let $E \subseteq D$ be a maximal antichain. By the c.c.c. it is countable, so $C := \omega^\omega \setminus \bigcup E$ is a Borel set and $A \subseteq C$. Since C is disjoint from all $B \in E$ it must be I-small, otherwise it would contradict E's maximality. □

The σ-ideal $\mathcal{N}_I$ is usually not Borel generated when $\mathbb{P}_I$ is not c.c.c. In the presence of a dense embedding $Q \hookrightarrow_{\mathrm{d}} \mathbb{P}_I$, it is equivalent to the standard *Marczewski null ideal* for partial orders. For example, $\mathcal{N}_{\mathsf{ctbl}}$ is the classical Marczewski-null ideal, i.e., the ideal of sets A such that for every perfect set p there is a perfect subset $q \subseteq p$ with $q \cap A = \varnothing$. The same holds for $\mathcal{N}_{K_\sigma}$ and $\mathcal{N}_{I_\mathbb{L}}$ with perfect sets replaced by super-perfect resp. Laver trees.

One of the reasons for introducing the derived ideal $\mathcal{N}_I$ is that it is closely related to density in the forcing-theoretic sense. It will be useful in several places in the subsequent sections.

2.2 From ideals to regularity

Recall the notion of *Marczewski measurability*, and the generalized $\mathbb{X}$-*Marczewski measurability*, defined in Section 1.3.1. Adapting it to the idealized forcing context, we get the following regularity property, which we will call *I-regularity.*

Definition 2.2.1. *Let I be a σ-ideal, assume that $\mathbb{P}_I$ is proper, and let A be an arbitrary subset of ω^ω (or a similar space). We say that A is* I-regular*, denoted by* Reg(I)*, if*

$$\forall B \in \mathbb{P}_I \; \exists C \in \mathbb{P}_I \textit{ s.t. } C \leq B \textit{ and } (C \subseteq A \;\; \textit{or} \;\; C \cap A = \varnothing).$$

Lemma 2.2.2. *The collection of I-regular sets forms a σ-algebra.*

Proof. By definition, this collection is closed under complements. Let A_n be I-regular sets, and let B be an I-positive Borel set. It is clear that $B \cap A_n$ is still I-regular. Now, if $(A_n \cap B) \in \mathcal{N}_I$ for all n, then by Lemma 2.1.10 we have $\bigcup_n (A_n \cap B) \in \mathcal{N}_I$ so there exists $C \leq B$ disjoint from $\bigcup_n A_n$. On the other hand, if some $(A_n \cap B)$ is not in $\mathcal{N}_I$, then by I-regularity there must be a $C \leq B$ such that $C \subseteq A_n$. □

So the property of being I-regular is equivalent to being $\mathbb{P}_I$-Marczewski measurable (according to the defined in Section 1.3.1), and in the presence of a dense embedding $Q \hookrightarrow_{\mathrm{d}} \mathbb{P}_I$ it is equivalent to being Q-Marczewski measurabile. In the case of a dense embedding modulo I, it is only equivalent to an analogous statement with "inclusion" replaced by "inclusion modulo I".

Furthermore, in many cases when Q generates a topology and I is the σ-ideal of sets meager in that topology, I-regularity is equivalent to the Baire property (in the respective topology). This applies to Cohen, Hechler and Mathias forcing, the latter fact being the essence of Ellentuck's proof that all analytic sets are Ramsey [Ell74]. For the σ-ideal of Lebesgue null sets $\mathcal{N}$, our notion of regularity coincides with Lebesgue measurability.

Note that when dealing with the regularity of all sets within a given projective pointclass $\mathbf{\Gamma}$, the first quantifier from Definition 2.2.1 can usually be dropped. In most cases the σ-ideal is *homogeneous*, meaning that for every Borel I-positive set B there is a Borel function f which is a bijection between B and ω^ω and preserves membership in I. Such a function can easily be used to transform the set $A \cap B$ to another set A' without increasing its complexity. Thus the first clause "$\forall B$" in the definition can be eliminated. This is important in such situations as Mathias forcing: formally, I_{RN}-regularity is the property usually called *being completely Ramsey*, but on the level of projective pointclasses $\mathbf{\Gamma}$, it is equivalent to the Ramsey property (as given in Definition 1.3.5). The same may be said about Silver forcing: here, the homogeneity of the corresponding ideal I_G together with the dense embedding implies that I_G-regularity is equivalent to the property of containing or being disjoint from a set $[T]$ where T is a Silver tree. By identifying infinite subsets of ω with their characteristic functions, it is easy to see that this property is equivalent to the *doughnut property* (see Definition 1.3.6).

So, we see that a fairly wide class of regularity properties can be captured by a single definition in the idealized forcing context. Table 2.1 sums up a few of the standard forcing notions with their corresponding ideals and regularity properties.

	Forcing	σ-ideal I	I-regularity
c.c.c.	Cohen ($\mathbb{C}$)	$\mathcal{M}$	Baire property
	random ($\mathbb{B}$)	$\mathcal{N}$	Lebesgue measurable
	Hechler ($\mathbb{D}$)	$\mathcal{M}_{\mathbb{D}}$	$\mathbb{D}$-Baire property
non-c.c.c.	Sacks ($\mathbb{S}$)	ctbl	Marczewski measurable; not a Bernstein set
	Miller ($\mathbb{M}$)	K_σ	$\mathbb{M}$-Marczewski measurable
	Laver ($\mathbb{L}$)	$I_{\mathbb{L}}$	$\mathbb{L}$-Marczewski measurable
	Mathias ($\mathbb{R}$)	I_{RN}	Ramsey; Baire property in Ellentuck topology
	E_0-tree ($\mathbb{E}_0$)	I_{E_0}	$\mathbb{E}_0$-Marczewski measurable
	Silver ($\mathbb{V}$)	I_G	doughnut property

Table 2.1: Some standard regularity properties

We will now prove a number of general results about I-regularity and definability, following the pattern described in Section 1.3. Since our definition was based on forcing, it is not surprising that all the proofs proceed via forcing-theoretic methods (even when the result is a ZFC-theorem). From now on, we will always assume that I is a σ-ideal such that $\mathbb{P}_I$ is a *proper* forcing partial order. As usual, $\boldsymbol{\Gamma}(\mathrm{Reg}(I))$ is shorthand for the statement "all sets in $\boldsymbol{\Gamma}$ are I-regular."

Proposition 2.2.3. *All analytic sets are I-regular.*

Proof. Let A be an analytic set, defined by a Σ^1_1 formula ϕ with parameter r (we will suppress r for convenience of notation). Let B be any I-positive Borel set. Let M be a countable elementary submodel containing B and r. In M, there is a stronger condition $B' \leq B$ such that $B' \Vdash \phi(\dot{x}_G)$ or $B' \Vdash \neg\phi(\dot{x}_G)$. Assume the former, and let $C := \{x \in B' \mid x \text{ is } M\text{-generic}\}$ be the Master-condition. By properness it is I-positive, and since $C \Vdash \dot{x}_G \in \dot{B}'$, for every $x \in C$ we have $M[x] \models \phi(x)$. Since ϕ is Σ^1_1 it is absolute between $M[x]$ and V and hence $\phi(x)$ holds in V. Therefore $C \subseteq A$. The case when $B' \Vdash \neg\phi(\dot{x}_G)$ proceeds analogously noting that Π^1_1-absoluteness between $M[x]$ and V also holds. In that case we get $C \cap A = \varnothing$. $\square$

Proposition 2.2.4. *If $V = L$ then there is a $\boldsymbol{\Delta}^1_2$ non-I-regular set.*

Proof. By Fact 1.3.8 we know that if $V = L$ then there is a $\boldsymbol{\Delta}^1_2$ *Bernstein set*, i.e., a set A such that there is no perfect set completely contained in A or completely disjoint from A. By our assumption that all singletons, and hence all countable sets, are I-small, it follows that every I-positive Borel set must be uncountable, so by the perfect set property it must contain a perfect set. Therefore the Bernstein set cannot be I-regular. $\square$

Proposition 2.2.5. *If for every $r \in \omega^\omega$ there is a $\mathbb{P}_I$-generic real over $L[r]$ then all $\boldsymbol{\Delta}^1_2$ sets are I-regular.*

Proof. Let A be a $\boldsymbol{\Delta}^1_2$ set, defined by Σ^1_2 formulas ϕ and ψ with parameter r. Let B be any I-positive Borel set, and let c be its code. Let x be $\mathbb{P}_I$-generic over $L[r,c]$. We know that either $\phi(x)$ or $\psi(x)$ is true, and by Shoenfield absoluteness, the same holds in $L[r,c,x]$. Hence, by the forcing theorem, we can find a $B' \leq B$ in $L[r,c]$ such that $B' \Vdash \phi(\dot{x}_G)$ or $B' \Vdash \psi(\dot{x}_G)$. Since both ϕ and ψ are Σ^1_2, the situation is symmetrical, so without loss of generality we may assume the former. Let M be a countable elementary submodel containing B', r and c. Let $C := \{x \in B' \mid x \text{ is } M\text{-generic}\}$ be the master condition. Now in V, for every $x \in C$, $M[x] \models \phi(x)$, and by upwards Σ^1_2-absoluteness we get $\phi(x)$ in V. Therefore $C \subseteq A$. In the situation where $B' \Vdash \psi(\dot{x}_G)$ we would get $C \cap A = \varnothing$ by an identical argument. $\square$

Proposition 2.2.6. *If for every $r \in \omega^\omega$ the set $\{x \mid x$ is not $\mathbb{P}_I$-generic over $L[r]\}$ is in $\mathcal{N}_I$, then all $\boldsymbol{\Sigma}^1_2$ sets are I-regular.*

Proof. Let A be a $\mathbf{\Sigma}^1_2$ set, defined by a formula ϕ with parameter r. Let B be an I-positive Borel set, and let c be its code. In $L[r, c]$, let $B' \leq B$ be such that $B' \Vdash \phi(\dot{x}_G)$ or $B' \Vdash \neg\phi(\dot{x}_G)$, without loss of generality the former. Now in V, we know by assumption that there exists an I-positive $C \leq B'$ such that every $x \in C$ is $\mathbb{P}_I$-generic over $L[r, c]$. Hence, for every $x \in C$ we have $L[r, c, x] \models \phi(x)$ and by Shoenfield absoluteness $\phi(x)$ holds in V, so $C \subseteq A$. The case when $B' \Vdash \neg\phi(\dot{x}_G)$ is identical and we get $C \cap A = \varnothing$. □

Corollary 2.2.7. *If* $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$*, then all* $\mathbf{\Sigma}^1_2$ *sets are* I*-regular.*

Proof. Fix any r. Since $\aleph_1^V$ is inaccessible in $L[r]$ the collection of dense sets in $(\mathbb{P}_I)^{L[r]}$ is countable in V. Let $\{D_n \mid n < \omega\}$ enumerate it and for each n let $A_n := \omega^\omega \setminus \bigcup D_n$. Clearly each $A_n \in \mathcal{N}_I$. Now note that if x is not $\mathbb{P}_I$-generic over $L[r]$, then $x \notin \bigcup D_n$ for some D_n, i.e., $x \in \bigcup_n A_n$. Since $\mathcal{N}_I$ is a σ-ideal by Lemma 2.1.10, it follows that $\{x \mid x \text{ is not } \mathbb{P}_I\text{-generic over } L[r]\} \subseteq \bigcup_n A_n \in \mathcal{N}_I$. □

Proposition 2.2.5 gives an easy way to construct models where the regularity hypothesis $\mathbf{\Delta}^1_2(\mathrm{Reg}(I))$ holds. An $\aleph_1$-iteration of the (proper) forcing $\mathbb{P}_I$, with countable support, starting from L, will yield such a model. Obtaining $\mathbf{\Sigma}^1_2(\mathrm{Reg}(I))$ is more difficult. By Theorem 1.3.11, we know that for I being the K_σ or the Laver ideal, $\mathbf{\Sigma}^1_2(\mathrm{Reg}(I))$ follows from $\mathbf{\Delta}^1_2(\mathrm{Reg}(I))$, so such cases are trivial. In other well-known cases (e.g., Cohen forcing, random forcing) we need to use a different forcing notion—often referred to as an *Amoeba forcing*—to add many $\mathbb{P}_I$-generic reals simultaneously. However, we also know from [BL99, Theorem 5.11] and [BL11, Theorem 7] that in some cases, notably Hechler forcing, $\mathbf{\Sigma}^1_2(\mathrm{Reg}(I))$ is already the strongest possible hypothesis, i.e., it is equivalent to $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$, and therefore cannot be obtained by a forcing iteration for reasons of consistency strength. On the other hand, a hypothesis of the form $\mathbf{\Delta}^1_2(\mathrm{Reg}(I))$ can never be strongest possible (for the same reason).

Our last result concerns the Solovay model, which was discussed in Section 1.2.7.

Proposition 2.2.8. *In the Solovay model, all sets are* I*-regular.*

Proof. Let V be a model with an inaccessible cardinal κ and $V[G]$ the corresponding Lévy collapse of κ. In $V[G]$, let A be a set of reals definable from a countable sequence of ordinals. Let $s \in \mathrm{Ord}^\omega$ be such that A is definable by $\varphi(s, x)$. By standard properties of the Lévy collapse (Lemma 1.2.20), there is a formula $\tilde{\varphi}$ such that for all x, $V[G] \models \varphi(s, x)$ iff $V[s][x] \models \tilde{\varphi}(s, x)$.

Let B be an I-positive Borel set in $V[G]$. Without loss of generality we may assume that the code of B is contained in s. Now consider the forcing $\mathbb{P}_I$ in $V[s]$. There is $B' \leq B$ in $V[s]$ such that $B' \Vdash \tilde{\varphi}(\dot{x}_G)$ or $\mathcal{B}' \Vdash \neg\tilde{\varphi}(\dot{x}_G)$. Assume the former. Since $\kappa = \aleph_1^{V[G]}$ is inaccessible in $V[s]$, the collection of all dense sets in

$\mathbb{P}_I^{V[s]}$ is countable in $V[G]$. Just as in the proof of Corollary 2.2.7, the collection $\{x \mid x \text{ is not } \mathbb{P}_I\text{-generic over } V[s]\}$ is in $\mathcal{N}_I$. Therefore, in $V[G]$, there is a Borel I-positive set $C \leq B'$ such that every $x \in C$ is $\mathbb{P}_I$-generic over $V[s]$. Hence, for every such x we have $V[s][x] \models \tilde{\varphi}(s, x)$, which implies that $V[G] \models \varphi(s, x)$, i.e., $x \in A$. The case that $B' \Vdash \neg\tilde{\varphi}(\dot{x}_G)$ is analogous. □

Corollary 2.2.9. Con(ZFC+ *"all projective sets are I-regular"*) *and* Con(ZF + DC+ *"all sets are I-regular"*).

2.3 Quasi-generic reals and characterization results

Propositions 2.2.5 and 2.2.6 tell us that sufficient transcendence over L implies I-regularity for $\boldsymbol{\Delta}^1_2$ or $\boldsymbol{\Sigma}^1_2$ sets, but they are not yet characterization theorems, i.e., their converse does not necessarily hold. In Section 1.3.2 we saw that the converse is true for Cohen and random forcing. However, we also saw that for Sacks, Miller and Laver forcing the characterization involved new reals, unbounded reals and dominating reals rather than Sacks-, Miller- and Laver-generic reals, respectively. The difference can be explained by introducing the following definition:

Definition 2.3.1. *Let I be a σ-ideal on ω^ω and M a transitive model of set theory. A real x is called* I-quasi-generic *over M if for every Borel set $B \in I$ whose Borel code lies in M, $x \notin B$. We will also use the term* Q-quasi-generic *if we know that $Q \hookrightarrow_{\mathrm{d}} \mathcal{B}(\omega^\omega) \setminus I$.*

An obvious generalization of Cohen and random reals, the concept of *quasi-generic reals* was first explicitly introduced in [BHL05, Section 1.5] in the context of Silver forcing. In [Ike10a] the notion was fully exploited in order to prove a characterization theorem for *arboreal forcings* in an abstract setting. Note that by Fact 2.1.2 every real x which is $\mathbb{P}_I$-generic over M is also I-quasi-generic over M. The converse is true for all c.c.c. forcings.

Lemma 2.3.2 (Ikegami)**.** *Let I be a σ-ideal such that $\mathbb{P}_I$ is c.c.c., and let M a transitive model. Then every real x is $\mathbb{P}_I$-generic over M iff it is I-quasi-generic over M.*

Proof. Let x be I-quasi-generic over M. For every maximal antichain E in M, let $B_E := \omega^\omega \setminus \bigcup E$. By the c.c.c., this is a Borel set, it is coded in M, and is in $\mathcal{N}_I$, hence in I. By I-quasi-genericity $x \notin B_E$, hence $x \in \bigcup E$. Therefore x is $\mathbb{P}_I$-generic. □

For non-c.c.c. forcings, the notion of I-quasi-genericity is usually different from $\mathbb{P}_I$-genericity. For example, the following is easy to show (see Lemma 2.5.3 for a proof):

- a real is $\mathbb{S}$-quasi-generic (i.e., ctbl-quasi-generic) over M iff it is not in M,
- a real is $\mathbb{M}$-quasi-generic (i.e., K_σ-quasi-generic) over M iff it is unbounded over M,
- a real is $\mathbb{L}$-quasi-generic (i.e., $I_{\mathbb{L}}$-quasi-generic) over M iff it is strongly dominating over M (see Definition 1.3.7 (3)).

Concerning the last point: if x is strongly dominating over M then it is also dominating over M. The converse is false; however, it is not hard to see that *if* there is a dominating real over M *then* there is also a strongly dominating real over M. Therefore, as far as statements about transcendence over a model go, dominating and strongly dominating reals amount to the same thing.

We can extend Ikegami's characterization theorems to the idealized forcing context. First, we show that quasi-generics are sufficient to ensure I-regularity on the second projective level, giving stronger versions of Propositions 2.2.5 and 2.2.6.

Proposition 2.3.3. *If for every $r \in \omega^\omega$, for every I-positive set B, there is an $x \in B$ which is I-quasi-generic over $L[r]$, then all $\boldsymbol{\Delta}^1_2$ sets are I-regular.*

If we assume sufficient homogeneity of the ideal then the additional clause "for every I-positive set B" may be omitted.

Proof. Let A be a $\Delta^1_2(r)$ set and let B be any I-positive set. We may assume that $\exists s \in \omega^\omega$ $(\aleph_1^{L[s]} = \aleph_1)$ since otherwise the result trivially follows from Corollary 2.2.7. Also we may assume that $\aleph_1^{L[r]} = \aleph_1$ without loss of generality (otherwise consider $L[r, s]$). By Shoenfield's classical analysis of $\boldsymbol{\Sigma}^1_2$ sets (Fact 1.2.15) we may write $(A \cap B) = \bigcup_{\alpha<\aleph_1} C_\alpha$ and $(B \setminus A) = \bigcup_{\alpha<\aleph_1} D_\alpha$, with C_α and D_α Borel sets coded in $L[r]$. Let $x \in B$ be I-quasi-generic. Then $x \in C_\alpha$ or $x \in D_\alpha$ for some α, so either C_α or D_α is the required I-positive subset of B.

If I is homogeneous we can transform $A \cap B$ to some A' without increasing the complexity, and run the same argument with A'. □

Proposition 2.3.4. *If for every $r \in \omega^\omega$, $\{x \mid x$ is not I-quasi-generic over $L[r]\} \in \mathcal{N}_I$, then all $\boldsymbol{\Sigma}^1_2$ sets are I-regular.*

Proof. Let A be $\Sigma^1_2(r)$ and again assume that $\aleph_1^{L[r]} = \aleph_1$. Let B be an I-positive set. If there are no I-quasi-generics in $A \cap B$, then by assumption there is $C \leq B$ disjoint from A so we are done. So assume $x \in A \cap B$ is I-quasi-generic, and, as before, write $A \cap B = \bigcup_{\alpha<\aleph_1} C_\alpha$ with C_α Borel and coded in $L[r]$. Since x is in some C_α, this C_α will be the I-positive subset of B contained in A. □

With the stronger notion of quasi-genericity instead of genericity, a suitable converse of Propositions 2.3.3 and 2.3.4 can indeed be proved, but *only* if we assume that the ideal I (i.e., the membership of Borel sets in I) is $\mathbf{\Sigma}^1_2$, as shown by the results of Ikegami [Ike10a]. For its statement we require two additional definitions.

Definition 2.3.5. *A forcing $\mathbb{P}$ is* Σ^1_3-absolute *if every Σ^1_3 formula is absolute between V and $V[G]$, for any $\mathbb{P}$-generic G.*

Since upwards Σ^1_3-absoluteness holds between every model and an extension of it by a forcing preserving ω_1, it is only *downwards* Σ^1_3-absoluteness which matters. The next definition is due to Zapletal [Zap08, Proposition 2.3.4].

Definition 2.3.6. *For a projective pointclass $\mathbf{\Gamma}$, we say that* $\mathbf{\Gamma}$-I-uniformization *holds if for every I-positive B and every $A \subseteq B \times \omega^\omega$ such that $A \in \mathbf{\Gamma}$ and $\forall x \in B\ \exists y\ ((x, y) \in A)$, there is an I-positive Borel set $C \subseteq B$ and a Borel function $g : C \to \omega^\omega$ uniformizing A, i.e., such that $\forall x \in C\ ((x, g(x)) \in A)$.*

Zapletal already showed that analytic I-uniformization holds [Zap08, Proposition 2.3.4], and that $\mathbf{\Pi}^1_1$-I-uniformization fails in L [Zap08, Example 2.3.5]. The following theorem shows that it is another transcendence property over L. Despite our more general framework, the proof of this theorem is essentially due to Ikegami [Ike10a].

Theorem 2.3.7 (Ikegami)**.** *Let I be a σ-ideal such that $\mathbb{P}_I$ is proper. The following are equivalent:*

1. *All $\mathbf{\Delta}^1_2$ sets are I-regular,*
2. *$\mathbf{\Pi}^1_1$-I-uniformization holds, and*
3. *$\mathbb{P}_I$ is Σ^1_3-absolute.*

If I is $\mathbf{\Sigma}^1_2$ then it is also equivalent to

4. *$\forall r \in \omega^\omega$, $\forall B \in \mathbb{P}_I$, there is an $x \in B$ which is I-quasi-generic over $L[r]$.*

Proof.

- $(1 \Rightarrow 2)$. Let B be I-positive and let $A \subseteq B \times \omega^\omega$ be $\mathbf{\Pi}^1_1$. By Kondô's uniformization theorem, let f be a $\mathbf{\Pi}^1_1$ function with $\mathrm{dom}(f) = B$ uniformizing A. Let $D_{n,m} := \{B' \leq B \mid \forall x \in B'\ (f(x)(n) = m)\}$, and let $D_n := \bigcup_m D_{n,m}$.

  Claim. D_n is dense below B.

  Proof. Fix n, let $B' \leq B$, and consider $A_m := \{x \in B' \mid f(x)(n) = m\}$. Since (the graph of) f is $\mathbf{\Pi}^1_1$, each A_m is a $\mathbf{\Delta}^1_2$ set, therefore I-regular. Now, if $A_m \in \mathcal{N}_I$ for every m, then $B' = \bigcup_m A_m \in \mathcal{N}_I$ by Lemma 2.1.10, hence $B' \in I$, contradicting the assumption. Therefore at least one A_m must be $\mathcal{N}_I$-positive, and since it is I-regular, there must be a $B'' \leq B'$ such that $B'' \subseteq A_m$. Then $B'' \in D_{n,m}$. □(Claim.)

Now let M be a countable elementary submodel, containing B and all the D_n. Let $C := \{x \in B \mid x \text{ is } M\text{-generic}\}$ be the I-positive M-master condition. Define $g : C \to \omega^\omega$ by $g(x)(n) = m$ iff $x \in \bigcup(D_{n,m} \cap M)$. This is a Borel function since $(D_{n,m} \cap M)$ is countable. As x is M-generic and D_n is dense, $x \in \bigcup(D_n \cap M)$ so there is an m such that $x \in \bigcup(D_{n,m} \cap M)$. Also, it is clear that there can be at most one m such that $x \in \bigcup(D_{n,m} \cap M)$ since $\bigcup D_{n,m}$ and $\bigcup D_{n,m'}$ are disjoint for all $m \neq m'$. By definition of $D_{n,m}$ it follows that $g(x)(n) = m$ iff $f(x)(n) = m$, so $g{\restriction}C = f{\restriction}C$ and the result follows.

- $(2 \Rightarrow 3)$. Let ϕ be a Σ^1_1 formula (with parameter r, which we suppress for ease of notation) and let "$\exists x\, \forall y\, \phi(x, y)$" be the Σ^1_3 formula in question. We must show that it is downwards absolute; so assume $V[G] \models \exists x\, \forall y\, \phi(x, y)$. Let B be a condition and $\dot{x}$ a name for a real, such that $B \Vdash \forall y\, \phi(\dot{x}, y)$. By Fact 2.1.4, there is a Borel function f such that $B \Vdash \forall y\, \phi(f(\dot{x}_G), y)$. We now claim that the statement "$\exists x\, \forall y\, \phi(x, y)$" must hold in V as well. Towards contradiction, suppose it does not. Then for every $x \in B$ there is a y such that $\neg\phi(f(x), y)$. Using $\mathbf{\Pi}^1_1$-I-uniformization let $C \leq B$ be I-positive, and let g be a Borel function uniformizing the $\mathbf{\Pi}^1_1$ set $\{(x, y) \in B \times \omega^\omega \mid \neg\phi(f(x), y)\}$. Then for every $x \in C$, $\neg\phi(f(x), g(x))$ holds. Since both f and g are Borel functions, the statement "$\forall x \in C\ \neg\phi(f(x), g(x))$" is $\mathbf{\Pi}^1_1$, hence absolute. Therefore $C \Vdash \neg\phi(f(\dot{x}_G), g(\dot{x}_G))$. But this contradicts $B \Vdash \forall y\, \phi(f(\dot{x}_G), y)$, so we are forced to conclude that "$\exists x\, \forall y\, \phi(x, y)$" is downwards absolute.
- $(3 \Rightarrow 1)$. Let A be a $\mathbf{\Delta}^1_2$ set, defined by Σ^1_2 formulas ϕ and ψ. The statement "$\forall x\ (\phi(x) \leftrightarrow \neg\psi(x))$" is Π^1_3, so by assumption it is true in the extension. Now proceed as in the proof of Proposition 2.2.5. For every I-positive B there is an I-positive $B' \leq B$ such that $B' \Vdash \phi(\dot{x}_G)$ or $B' \Vdash \psi(\dot{x}_G)$. Assume the former, let M be a countable elementary submodel containing B' and let $C := \{x \in B' \mid x \text{ is } M\text{-generic}\}$. For every $x \in C$, $M[x] \models \phi(x)$, and by upwards Σ^1_2-absoluteness $\phi(x)$ holds. The situation with $B' \Vdash \psi(\dot{x}_G)$ is analogous because ψ is also Σ^1_2.

For the equivalence with (4), suppose I is $\mathbf{\Sigma}^1_2$. Let B be I-positive. Then the statement "$\exists x\ (x \in B$ and x is I-quasi-generic over $L[r])$" is $\Sigma^1_3(r, B)$, and is clearly forced by B to be true in the extension $V[G]$ (it is true of the generic real x_G, which is I-quasi-generic over V and hence also over $L[r]$). So by Σ^1_3-absoluteness it is true in V. The direction $(4) \Rightarrow (1)$ is Proposition 2.3.3. □

Corollary 2.3.8. *Assume that I is $\mathbf{\Sigma}^1_2$. Then the following are equivalent:*

1. *For every $r \in \omega^\omega$, $\{x \mid x$ is not I-quasi-generic over $L[r]\} \in \mathcal{N}_I$, and*
2. *All $\mathbf{\Sigma}^1_2$ sets are I-regular.*

Proof. Assume that all $\mathbf{\Sigma}^1_2$ sets are I-regular. If I is $\mathbf{\Sigma}^1_2$ then $\{x \mid x$ is not I-quasi-generic over $L[r]\}$ is also $\mathbf{\Sigma}^1_2$, hence I-regular. If it is not in $\mathcal{N}_I$ then there must be an I-positive Borel set B contained in it, i.e., such that for every $x \in B$, x is not I-quasi-generic over $L[r]$. But then by Theorem 2.3.7 $\mathbf{\Delta}^1_2(\mathrm{Reg}(I))$ would be false, which certainly contradiction the assumption. □

We can now also be somewhat more precise about the relationship between regularity hypotheses and cardinal invariants which we mentioned in Section 1.3.3. Recall that the *covering number* $\mathrm{cov}(I)$ is the least number of I-small sets needed to cover the whole space ω^ω, and the *additivity number* $\mathrm{add}(I)$ is the least number of I-small sets whose union is not I-small. A variation of the covering number is $\mathrm{cov}^*(I)$ defined as the least number of I-small sets needed to cover some I-positive Borel set. In the case of homogeneity of I, $\mathrm{cov}^*(I) = \mathrm{cov}(I)$.

Lemma 2.3.9.

1. $\mathrm{cov}^*(I) > \aleph_1 \implies \mathbf{\Delta}^1_2(\mathrm{Reg}(I))$,
2. $\mathrm{add}(\mathcal{N}_I) > \aleph_1 \implies \mathbf{\Sigma}^1_2(\mathrm{Reg}(I))$.

Proof. This follows immediately from Propositions 2.3.3 and 2.3.4, noting that $\{x \mid x$ is not I-quasi-generic over $L[r]\}$ can be written as $\bigcup\{B \in \mathcal{B} \mid B \in I$ and B is coded by a real in $L[r]\}$ and that there are only $\aleph_1$ reals in $L[r]$. □

As before, the converse is clearly false, for instance in $\aleph_1$-iteration of $\mathbb{P}_I$ starting from L, or in models of CH with a measurable cardinal (the latter due to Corollary 2.2.7). Nevertheless, we can say a little more if we look at the details of cardinal inequality proofs. Assuming that I and J are $\mathbf{\Sigma}^1_2$ ideals, we can say the following:

1. if $\mathrm{cov}^*(I) \leq \mathrm{cov}^*(J)$ is provable in ZFC, then, *most likely*, $\mathbf{\Delta}^1_2(\mathrm{Reg}(I)) \Rightarrow \mathbf{\Delta}^1_2(\mathrm{Reg}(J))$ is also provable in ZFC, and
2. if it is consistent that $\mathrm{cov}^*(I) < \mathrm{cov}^*(J)$, then, *most likely*, it is consistent that $\mathbf{\Delta}^1_2(\mathrm{Reg}(J)) \not\Rightarrow \mathbf{\Delta}^1_2(\mathrm{Reg}(I))$.
3. The same holds regarding $\mathrm{add}(\mathcal{N}_I)$ and $\mathbf{\Sigma}^1_2(\mathrm{Reg}(I))$.

To say a bit more about point 1, if $\mathrm{cov}^*(I) \leq \mathrm{cov}^*(J)$ is a theorem, then *most likely* the proof is as follows: "Given a collection $\{B_\alpha \in J \mid \alpha < \kappa\}$ for some $\kappa < \mathrm{cov}^*(I)$, transform it to another collection $\{B'_\alpha \in I \mid \alpha < \kappa\}$, find a real

x outside $\bigcup_{\alpha<\kappa} B'_\alpha$, and then transform it again to get another real y outside $\bigcup_{\alpha<\kappa} B_\alpha$, proving that $\kappa < \text{cov}^*(J)$." If the transformation process in this proof is recursive, then B'_α will have a code in $L[r]$ whenever B_α has a code in $L[r]$. As a result, the same proof will show that $\mathbf{\Delta}^1_2(\text{Reg}(I)) \Rightarrow \mathbf{\Delta}^1_2(\text{Reg}(J))$, via Proposition 2.3.3 and Theorem 2.3.7. For $\text{add}(\mathcal{N}_I)$ and $\mathbf{\Sigma}^1_2(\text{Reg}(I))$, the same applies, using Proposition 2.3.4 and Corollary 2.3.8.

Concerning point 2, the idea is that if $\text{cov}^*(I) < \text{cov}^*(J)$ is consistent, then *most likely* the proof involves a forcing iteration with some $\mathbb{P}$ which adds J-quasi-generic reals but no I-quasi-generic reals (for example, $\mathbb{P}_J$ itself). In this case, it is clear that an iteration of this forcing will also yield a model in which $\mathbf{\Delta}^1_2(\text{Reg}(J))$ is true but $\mathbf{\Delta}^1_2(\text{Reg}(I))$ is false. For the additivity and the $\mathbf{\Sigma}^1_2$ level, we have to consider a $\mathbb{P}$ which adds a co-$\mathcal{N}_J$ set of J-quasi-generics but does not add a co-$\mathcal{N}_I$ set of I-quasi-generics.

However, the above is merely heuristics: for example, if the cardinal inequality is proved by some other means than described above, the conclusion may well fail to hold, and conversely, relationships between regularity hypotheses can be established without a direct link to cardinal invariants.

Finally, we would like to mention that Ikegami's equivalence theorems depend heavily on the complexity of the ideal I, and it is still open whether a suitable characterization theorem can be proved in case I is not $\mathbf{\Sigma}^1_2$. It so happens that many naturally occurring ideals (for example, most of the ideals in [Zap04]) are $\mathbf{\Sigma}^1_2$, and for some time it was considered unlikely that a natural counterexample would exist at all. However, in a recent development, Marcin Sabok [Sab10] proved that the Ramsey-null ideal I_{RN} is one such counterexample, i.e., it is $\mathbf{\Pi}^1_2$-complete. This naturally leads to the following question.

Question 2.3.10. *Does $\mathbf{\Delta}^1_2$(Ramsey) imply that $\forall r \in \omega^\omega$ there is a Ramsey-null-quasi-generic real over $L[r]$?*

2.4 Dichotomies

The notion of I-regularity was introduced as a generalization of many well-known regularity properties. There is another class of properties, however, which do not (and cannot) fall into this framework, and these are the dichotomy properties such as the perfect set property and its relatives K_σ-regularity, u-regularity and Laver-regularity (see Definition 1.3.7).

Proving this dichotomy for Borel sets is a necessary requirement for an embedding $Q \hookrightarrow_{\text{d}} \mathcal{B}(\omega^\omega) \setminus I$. Often the proof can be extended to cover analytic sets, and a natural question arises as to what happens at higher complexity levels. By Theorem 1.3.13 we know that "all $\mathbf{\Sigma}^1_2$ sets satisfy the perfect set property" is equivalent to "all $\mathbf{\Pi}^1_1$ sets satisfy the perfect set property", and equivalent to $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$: the strongest regularity hypothesis. Notice that this is very different from $\mathbf{\Sigma}^1_2(\mathbb{S})$, which is equivalent to $\forall r\ (L[r] \cap \omega^\omega \neq \omega^\omega)$: the weakest regularity hypothesis. Though both properties involve the ideal ctbl and the partial order $\mathbb{S}$ of perfect sets, the resulting hypotheses could hardly be more different.

To be able to study such dichotomies in our idealized framework, we introduce a new definition.

Definition 2.4.1. *Let I be a Borel generated σ-ideal. A set A satisfies the I-*dichotomy*, denoted by* Dich(I)*, if it is either in I or there is an I-positive Borel set B such that $B \subseteq A$.*

Notice that in the presence of a dense embedding $Q \hookrightarrow_{\mathrm{d}} \mathbb{P}_I$, the I-dichotomy, as we defined it above, is *exactly* the original dichotomy, involved in the proof of the embedding.

Notice also that this definition is really only interesting when the ideal I is Borel generated. For example, if we would replace I by $\mathcal{N}_I$ and define "$\mathcal{N}_I$-dichotomy" analogously, it would be equivalent to I-regularity and would not give us anything interesting. It is also clear that $\mathbf{\Gamma}(\mathrm{Dich}(I))$ implies $\mathbf{\Gamma}(\mathrm{Reg}(I))$ for projective pointclasses $\mathbf{\Gamma}$, and that the two notions coincide if $I = \mathcal{N}_I$, which is the case for all c.c.c.-ideals. In particular, Dich$(\mathcal{N})$ and Dich$(\mathcal{M})$ are Lebesgue measurability and the Baire property, respectively.

Originally, it was our intention to prove abstract results about the I-dichotomy similar to the ones we proved about I-regularity. It turns out, however, that I-dichotomy is a much more mysterious property. For example, we were not able to prove that all analytic sets satisfy the I-dichotomy, although we don't know of any counterexamples. Strangely enough, on the $\mathbf{\Sigma}^1_2$ level results similar to Propositions 2.2.6 and 2.3.4 can be proved.

Proposition 2.4.2. *If for every $r \in \omega^\omega$, $\{x \mid x$ is not $\mathbb{P}_I$-generic over $L[r]\} \in I$, then $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ holds.*

Proof. Let A be a $\mathbf{\Sigma}^1_2$ set, defined by a formula ϕ with parameter r. If there are no reals in A which are $\mathbb{P}_I$-generic over $L[r]$ then we are done since $A \in I$. So suppose there is an $x \in A$ which is $\mathbb{P}_I$-generic over $L[r]$. Since by Shoenfield absoluteness $L[r,x] \models \phi(x)$, by the forcing theorem there is a $B \in L[r]$ such that $B \Vdash \phi(\dot{x}_G)$. Now let M be a countable elementary submodel containing r and B, and let $C := \{x \in B \mid x$ is M-generic$\}$ be the master condition. For every $x \in C$ we have $M[x] \models \phi(x)$ and by upwards Σ^1_2-absoluteness $\phi(x)$ holds in V, hence $C \subseteq A$. □

Currently we do not know of any applications of this theorem, except where $\mathbb{P}_I$ has the c.c.c., in which case it says the same as Proposition 2.2.6. Also, note that, unlike Proposition 2.2.6, here we cannot conclude that $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ follows from the strongest hypothesis $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$ (the reason will soon become clear). Also, if we wish to prove an analogous propositions with *quasi-generics* instead of *generics*, we must assume that $\aleph_1$ is not inaccessible in L.

Proposition 2.4.3. *If* $\forall r\ \{x \mid x$ *is not* I*-quasi-generic over* $L[r]\} \in I$ *and* $\exists r\,(\aleph_1^{L[r]} = \aleph_1)$*, then* $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ *holds. If additionally* I *is* $\mathbf{\Sigma}^1_2$*, then, conversely,* $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ *implies that* $\forall r\ \{x \mid x$ *is not* I*-quasi-generic over* $L[r]\} \in I$*.*

Proof. Let A be $\Sigma^1_2(r)$ and without loss of generality assume $\aleph_1^{L[r]} = \aleph_1$. Then A can be written as $\bigcup_{\alpha<\aleph_1} B_\alpha$ with B_α coded in $L[r]$. If there are no reals in A which are I-quasi-generic over $L[r]$ then $A \in I$, and if there is, then it is in some B_α which must then be I-positive.

For the converse direction, suppose I is $\mathbf{\Sigma}^1_2$. Since $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ implies $\mathbf{\Sigma}^1_2(\mathrm{Reg}(I)$ which implies $\mathbf{\Delta}^1_2(\mathrm{Reg}(I))$, by Theorem 2.3.7 there exists an I-quasi-generic real in every I-positive set B. Thus $\{x \mid x$ is not I-quasi-generic over $L[r]\}$ is a $\mathbf{\Sigma}^1_2$ set which does not contain an I-positive Borel set so by the dichotomy it is in I. □

Despite these results, the trouble with $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ is that it can be inconsistent! Zapletal [Zap08, Proposition 3.9.2] essentially proved the following strong result: if $\mathbb{P}_I$ is a forcing notion (satisfying additional requirements which are true in all natural cases), and $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ holds, then any intermediate extension N of the forcing extension $V^{\mathbb{P}_I}$ is either a c.c.c. extension of V or $N = V^{\mathbb{P}_I}$. This immediately implies that for several ideals I, among them the Borelized version of the Ramsey-null ideal $I^{\mathcal{B}}_{\mathrm{RN}}$ and the ideals I_{E_0} and I_G (see Example 2.1.5), the statement $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ is simply false. In the case of the E_0-ideal, there is even a direct diagonalization proof, also an (unpublished) result of Zapletal.

Proposition 2.4.4 (Zapletal)**.** *There is a* $\mathbf{\Sigma}^1_2$ *set not satisfying the* I_{E_0}*-dichotomy.*

Proof. First, construct a two-dimensional perfect tree T, inductively generated by the following clauses:

- $(\varnothing, \varnothing) \in T$,
- if $(s,t) \in T$ then the following pairs of extending sequences are in T:
  - $(s^\frown\langle 0\rangle, t^\frown s^\frown\langle 0\rangle^\frown\langle 0\rangle)$
  - $(s^\frown\langle 0\rangle, t^\frown s^\frown\langle 0\rangle^\frown\langle 1\rangle)$
  - $(s^\frown\langle 1\rangle, t^\frown s^\frown\langle 1\rangle^\frown\langle 0\rangle)$
  - $(s^\frown\langle 1\rangle, t^\frown s^\frown\langle 1\rangle^\frown\langle 1\rangle)$

It is clear that T generates a perfect tree, with $[T] \subseteq 2^\omega \times 2^\omega$ the set of branches through T. For $x \in 2^\omega$, let $T_x := \{t \mid \exists s \subseteq x\,((s,t) \in T)\}$. We claim that the following two conditions are satisfied:

(a) each T_x is an E_0-tree, and

(b) $\forall x \neq x'$, $\forall y \in [T_x], \forall y' \in [T_{x'}]$, $\neg(yE_0y')$.

Part (a) follows immediately from the construction. For part (b), note that the construction of T guarantees that if $s \subseteq x$ and $(x, y) \in [T]$, then the sequence s appears infinitely often in y. Therefore, if $x \neq x'$, there exists some n such that $s := x \restriction n$ differs from $s' := x' \restriction n$. Therefore, if $y \in [T_x]$ then s will appear infinitely often in y and if $y' \in [T_{x'}]$ then s' will appear infinitely often in y'; moreover, this will happen on the same digits of the respective reals y and y'. Hence, there will be infinitely many digits on which y and y' disagree, i.e., $\neg(yE_0y')$.

Now let Y be a universal analytic subset of $2^\omega \times 2^\omega$, i.e., a set which is itself analytic and such that every analytic subset of 2^ω is equal to some vertical section $(Y)_x := \{y \mid (x, y) \in Y\}$. Then $[T] \setminus Y$ is $\mathbf{\Pi}^1_1$ so by Kondô's uniformization theorem (see [Jec03, Theorem 25.36]) we can find a $\mathbf{\Pi}^1_1$ function g uniformizing it, i.e., such that for every x, if $([T] \setminus Y)_x \neq \varnothing$ then $g(x) \in ([T] \setminus Y)_x$. Let $A := \mathrm{ran}(g)$.

We claim that A is the required counterexample. It is clear that A is $\mathbf{\Sigma}^1_2$ because it is the range of a $\mathbf{\Pi}^1_1$ function. To show that $A \notin \mathrm{Dich}(I_{E_0})$, first suppose there is a Borel I_{E_0}-positive set $B \subseteq A$. Then there are at least two $y, y' \in A$ such that yE_0y'. Let x, x' be such that $y = g(x)$ and $y' = g(x')$. Then $y \in ([T])_x = [T_x]$ and $y' \in ([T])_{x'} = [T_{x'}]$ are related by E_0, contradicting assumption (b) above. Now, suppose $A \in I_{E_0}$. Then there exists a Borel, hence analytic, set $B \in I_{E_0}$ such that $A \subseteq B$. By the universal property of Y, there must be some x so that $B = (Y)_x$. Moreover, since by condition (a), $[T_x]$ is an E_0-tree, hence I_{E_0}-positive, the set $([T] \setminus Y)_x = [T_x] \setminus (Y)_x$ is non-empty, hence $x \in \mathrm{dom}(g)$. Therefore $g(x) \notin B$, but that contradicts $g(x) \in A$. □

A similar proof works for the ideal I_G. Combining this fact with Propositions 2.4.2 and 2.4.3, we see that for the ideals I_{E_0}, I_G and $I^{\mathcal{B}}_{\mathrm{RN}}$, the antecedents in these propositions must themselves be inconsistent.

We will not be able to say much more about I-dichotomy. As counterexamples exist in ZF we cannot prove $\mathrm{Dich}(I)$ for all sets of reals in the Solovay model, and this also seems to form an obstacle to proving $\mathbf{\Sigma}^1_1(\mathrm{Dich}(I))$. The following questions seem very interesting in this context:

1. Is there a general proof of $\mathbf{\Sigma}^1_1(\mathrm{Dich}(I))$, or are there natural counterexamples?

2. Under what conditions is $\mathbf{\Sigma}^1_2(\mathrm{Dich}(I))$ consistent?

3. Under what conditions does $\mathrm{Dich}(I)$ hold for all sets of reals in the Solovay model?

2.5 From transcendence to quasi-generics

In the previous sections we have taken a σ-ideal on the reals as a starting point and proved results connecting regularity to transcendence over L. We can take the reverse approach and consider a natural transcendence property as the starting point. We will assume this property to be simple enough, in any case given by a Borel relation between reals (in practice it is usually of low complexity in the arithmetic hierarchy).

Definition 2.5.1. *Let* R *be a Borel relation on* ω^ω *(or a similar space). We say:*

1. y *is* R-transcendent over A *if* $\forall x \in A\,(x \mathrel{\mathsf{R}} y)$, *and*
2. $\{y_n \mid n \in \omega\}$ *is* R-σ*-transcendent over* A *if* $\forall x \in A\, \exists n\,(x \mathrel{\mathsf{R}} y_n)$.

For a model M *we say that* y *is* R*-transcendent over* M *iff it is* R*-transcendent over* $\omega^\omega \cap M$*, and similarly for* R-σ*-transcendent.*

Intended examples of relations "$x \mathrel{\mathsf{R}} y$" are: y dominates x, y is not dominated by x, y is eventually different from x, y splits x, etc.

Definition 2.5.2. *Let* R *be a Borel relation. For* $x \in \omega^\omega$*, let*

$$K_x^{\mathsf{R}} := \{y \in \omega^\omega \mid \neg(x \mathrel{\mathsf{R}} y)\}.$$

Let I^{R} *be the* σ*-ideal generated by the sets* K_x^{R}*.*

The σ-ideal I^{R} is Borel generated, and it is easily seen to be $\mathbf{\Sigma}^1_2$. Given R, it is useful to think about the *dual relation* $\breve{\mathsf{R}}$, defined by $x \mathrel{\breve{\mathsf{R}}} y$ iff $\neg(y \mathrel{\mathsf{R}} x)$. Note that $A \in I^{\mathsf{R}}$ iff there exists $\{x_n \mid n < \omega\}$ which $\breve{\mathsf{R}}$-σ-transcends A.

Lemma 2.5.3. *Let* $\mathsf{R}, K_x^{\mathsf{R}}$ *and* I^{R} *be as above, and let* M *be a model such that* $\omega_1 \subseteq M$*. Then* y *is* R*-transcendent over* M *iff it is* I^{R}*-quasi-generic over* M*.*

Proof. First, suppose y is I^{R}-quasi-generic over M. For any $x \in \omega^\omega \cap M$, K_x^{R} is a Borel set in I, coded in M. Therefore $y \notin K_x^{\mathsf{R}}$, hence $x\mathsf{R}y$.

Conversely, suppose y is R-transcendent over M. Let $B \in I^{\mathsf{R}}$ be a Borel set coded in M. By Σ^1_2-absoluteness $M \models B \in I^{\mathsf{R}}$, so there are x_n in M such that $M \models B \subseteq \bigcup_n K_{x_n}^{\mathsf{R}}$, which by Π^1_1-absoluteness is also true in V. Now, if $y \in B$ then $y \in K_{x_n}^{\mathsf{R}}$ for some x_n, i.e., $\neg(x_n \mathrel{\mathsf{R}} y)$ for some x_n. But this contradicts the R-transcendence of y over M. Therefore $y \notin B$, which proves that y is I^{R}-quasi-generic. □

So forcing with $\mathbb{P}_{I^{\mathsf{R}}}$ is, in a sense, a very canonical way to add a R-transcendent real. The biggest problem is to verify whether $\mathbb{P}_I$ is proper, but if it is, then by Proposition 2.3.3, Corollary 2.3.8 and Proposition 2.4.3 it immediately follows that

- $\mathbf{\Delta}^1_2(\text{Reg}(I^{\mathsf{R}}))$ iff for all r there is a R-transcendent real over $L[r]$,
- $\mathbf{\Sigma}^1_2(\text{Reg}(I^{\mathsf{R}}))$ iff for all r, $\{x \mid x \text{ not } \mathsf{R}\text{-transcendent over } L[r]\} \in \mathcal{N}_{I^{\mathsf{R}}}$, and
- If $\exists r(\aleph_1^{L[r]} = \aleph_1)$, then $\mathbf{\Sigma}^1_2(\text{Dich}(I^{\mathsf{R}}))$ iff for all r, $\{x \mid x \text{ is not } \mathsf{R}\text{-transcendent over } L[r]\} \in I^{\mathsf{R}}$.

A little more can be proved assuming that R is a transitive relation.

Proposition 2.5.4. *Suppose that* R *is transitive. Then*

1. *all analytic sets satisfy the* I^{R}*-dichotomy, and*
2. *the following are equivalent:*
   (a) $\mathbf{\Delta}^1_2(\text{Reg}(I^{\mathsf{R}}))$,
   (b) $\mathbf{\Sigma}^1_2(\text{Reg}(I^{\mathsf{R}}))$,
   (c) $\mathbf{\Sigma}^1_2(\text{Dich}(I^{\mathsf{R}}))$.

Proof.

1. Let A be analytic, and let $A = \bigcup_{\alpha<\aleph_1} B_\alpha$ be its Borel decomposition. If one of the B_α's is I^{R}-positive we are done, so suppose they are all in I^{R}. Then for every α there are $\{x^\alpha_n \mid n < \omega\}$ s.t. for all $y \in B_\alpha$, $\exists n$ s.t. $\neg(x^\alpha_n \mathrel{\mathsf{R}} y)$. Consider $V[G]$, the forcing extension with $\mathbb{P}_{I^{\mathsf{R}}}$. There the generic real x_G is R-transcendent over V, hence $x^\alpha_n \mathsf{R} x_G$ holds for all α, n. By Shoenfield absoluteness $A = \bigcup_{\alpha<\aleph_1} B_\alpha$ still holds in the extension. Therefore, in $V[G]$, for all $y \in A$ there are α, n such that $\neg(x^\alpha_n \mathrel{\mathsf{R}} y)$, and by transitivity of R, $\neg(x_G \mathrel{\mathsf{R}} y)$ (otherwise $x^\alpha_n \mathrel{\mathsf{R}} x_G \mathrel{\mathsf{R}} y$). Therefore $V[G] \models A \in I^{\mathsf{R}}$, and since this statement is $\mathbf{\Sigma}^1_2$ it is true in V, too.
2. The implications $(c) \Rightarrow (b) \Rightarrow (a)$ are trivial, so we have to show $(a) \Rightarrow (c)$. If $\mathbf{\Delta}^1_2(\text{Reg}(I^{\mathsf{R}}))$ holds then by Theorem 2.3.7 Σ^1_3-absoluteness holds between V and the forcing extension $V[G]$ by $\mathbb{P}_{I^{\mathsf{R}}}$. Now apply exactly the same argument as in part 1, using Σ^1_3-absoluteness instead of Shoenfield absoluteness. $\square$

Below are some examples of familiar transcendence properties.

Example 2.5.5.

1. Let R be defined by $x \mathrel{\mathsf{R}} y$ iff $x \neq y$. Then $I^{\mathsf{R}} = \mathsf{ctbl}$, the corresponding forcing is Sacks forcing, and a real is I^{R}-quasi-generic over M iff it is not in M. Moreover, I^{R}-dichotomy is the perfect set property.
2. Let R be defined by $x \mathrel{\mathsf{R}} y$ iff $x \not\geq^* y$. Then $I^{\mathsf{R}} = K_\sigma$, the corresponding forcing is Miller forcing, and a real is K_σ-quasi-generic over M iff it is unbounded over M. Moreover, I^{R}-dichotomy is K_σ-regularity.

3. Let R be defined by $x \mathsf{R} y$ iff y dominates x. The corresponding ideal consist of all sets A which are *not* dominating. There are two ways to look at this: considering x, y as members of ω^ω, or of $\omega^{\uparrow\omega}$ (strictly increasing sequences). The latter case is easier because by Borel u-regularity (see Definition 1.3.7 (2)), there is a dense embedding from the set of Spinas trees into $\mathcal{B}(\omega^{\uparrow\omega}) \setminus I^{\mathsf{R}}$. This is a proper forcing notion which, by [BHS95, Theorem 5.1], is equivalent to Laver forcing. If we consider the ω^ω situation, then there is a dense embedding consisting of so-called *nice sets* (see [BHS95, p. 294]), which are combinatorially more complicated.

4. Let R be defined by $x \mathsf{R} y$ iff y strongly dominates x (Definition 1.3.7 (3)). Now the corresponding ideal is the Laver ideal $I_{\mathbb{L}}$, and a real x is strongly dominating iff it is Laver-quasi-generic.

Next we consider several examples of transcendence properties which are well-known, but have not yet been studied from the point of view of regularity. As always, the main problem is determining whether the corresponding forcing is proper.

Example 2.5.6.

1. For $x, y \in [\omega]^\omega$, let R be defined by $x \mathsf{R} y$ iff y *splits* x, i.e., $x \cap y$ and $x \setminus y$ are infinite. The corresponding ideal I^{R} has been studied by Spinas in [Spi04, Spi08]. In his terminology, a set A was called *countably splitting* iff $A \notin I^{\mathsf{R}}$. Spinas isolated the following notion of a *splitting tree:*

   Definition 2.5.7. *A tree T on $2^{<\omega}$ is called a* splitting tree *if for every s in T there exists $N \in \omega$ such that $\forall n \geq N$ there exist two extensions t_0 and t_1 in T such that $t_0(n) = 0$ and $t_1(n) = 1$. Let $\mathbb{SPL}$ denote the partial order of splitting trees ordered by inclusion.*

   By [Spi04, Theorem 1.2], there is a dense embedding $\mathbb{SPL} \hookrightarrow_{\mathrm{d}} \mathbb{P}_{I^{\mathsf{R}}}$. Since splitting treas easily allow a standard fusion construction, it follows that $\mathbb{P}_{I^{\mathsf{R}}}$ is proper, so all the preceding theorems apply. In particular, all $\boldsymbol{\Delta}^1_2$ sets are I^{R}-regular iff for every r there exists a splitting real over $L[r]$. However, R is not transitive and it is not clear whether $\boldsymbol{\Sigma}^1_2(\mathrm{Dich}(I^{\mathsf{R}})$ is consistent and whether $\boldsymbol{\Delta}^1_2(\mathrm{Reg}(I^{\mathsf{R}}))$ and $\boldsymbol{\Sigma}^1_2(\mathrm{Reg}(I^{\mathsf{R}}))$ are equivalent.

2. Consider the dual relation, i.e., $x \mathsf{R} y$ iff x does not split y. In [Spi08], Spinas studied the corresponding ideal and attempted to find a dense combinatorial object, but without success. Until this has been achieved, it is not clear whether the corresponding forcing $\mathbb{P}_{I^{\mathsf{R}}}$ is proper, and whether any of the preceding theorems hold.

3. Let R be defined by: $x \mathsf{R} y$ iff x and y are *infinitely often equal*, i.e., $\exists^\infty n (x(n) = y(n))$. In [Spi08] Spinas has isolated the notion of an i.o.e.-tree:

Definition 2.5.8. *A tree T on ω is an* i.o.e.-tree *if every $s \in T$ has an extension $t \in T$ such that $|\mathrm{Succ}_T(t)| = \omega$.*

Spinas [Spi08, Theorem 3.3] showed that the partial order of i.o.e.-trees densely embeds into $\mathbb{P}_{I^{\mathsf{R}}}$, which, again, implies that $\mathbb{P}_{I^{\mathsf{R}}}$ is proper, hence the equivalence theorems hold. In particular, all $\boldsymbol{\Delta}^1_2$ sets are I^{R}-regular iff for every r there exists an infinitely often equal real over $L[r]$. It is open whether $\boldsymbol{\Sigma}^1_2(\mathrm{Dich}(I^{\mathsf{R}}))$ is consistent and whether $\boldsymbol{\Delta}^1_2(\mathrm{Reg}(I^{\mathsf{R}}))$ and $\boldsymbol{\Sigma}^1_2(\mathrm{Reg}(I^{\mathsf{R}}))$ are equivalent.

4. Now consider the dual notion again, i.e., $x \mathrel{\mathsf{R}} y$ iff x and y are *eventually different*. The corresponding ideal has not received much attention so far, and it is currently unknown whether there is a dense partial order consisting of trees or other simple objects, and whether the forcing is proper.

2.6 Questions and further research

We would like to conclude this chapter by pointing out a number of open questions, or ideas for further research, that have come up in the development of this general theory.

Question 2.6.1. *Concerning I-regularity:*

1. *Does the direction "(3) $\Rightarrow$ (4)" in Ikegami's theorem (Theorem 2.3.7) work without the assumption on the complexity of the ideal I?*

2. *A test-case for the above: does $\boldsymbol{\Sigma}^1_2$(Ramsey) imply the existence of Ramsey-null-quasi-generics over $L[r]$?*

3. *Under which conditions is $\boldsymbol{\Delta}^1_2(\mathrm{Reg}(I))$ and $\boldsymbol{\Sigma}^1_2(\mathrm{Reg}(I))$ equivalent?*

Question 2.6.2. *Concerning I-dichotomy:*

1. *Is $\boldsymbol{\Sigma}^1_1(\mathrm{Dich}(I))$ true?*

2. *Under which assumptions is $\boldsymbol{\Sigma}^1_2(\mathrm{Dich}(I))$ consistent?*

3. *Under which assumptions is $\mathrm{Dich}(I)$ true for all sets in the Solovay model?*

4. *What can we say about $\boldsymbol{\Pi}^1_1(\mathrm{Dich}(I))$? When is it equivalent to $\boldsymbol{\Sigma}^1_2(\mathrm{Dich}(I))$?*

Question 2.6.3. *Concerning R-transcendence:*

1. *Are the forcings related to the unsplit reals and the eventually different reals (Example 2.5.6 (2) and (4)) proper?*

In this chapter, we have not really talked about infinite games or the axiom of determinacy. The main reason is that the framework of idealized forcing seems too general to allow us to prove any results. Of course, we know that AD (the axiom of determinacy) implies the Baire property and Lebesgue measurability for all sets of reals, as well as many of the I-dichotomy properties. One might ask whether the following holds:

Question 2.6.4. *Does* AD *imply that all sets are* I*-regular?*

Although no counterexample to this implication is currently known, it is also known to be a very difficult open problem for many regularity properties. Most notably, it is still open whether AD implies that all sets are Ramsey. On the other hand, the Ramsey property does follow from the *axiom of real determinacy* $\mathsf{AD}_{\mathbb{R}}$, by [Pri76, Kas83]. The same holds for many other Marczewski-style properties, as shown in [Löw98]. In a sense, it is more natural to use games with real moves when talking about more complicated regularity properties. So, one might at least wonder whether the following is true:

Question 2.6.5. *Does* $\mathsf{AD}_{\mathbb{R}}$ *imply that all sets are* I*-regular?*

Unfortunately we do not have a proof of this in general.

Chapter 3

Polarized partitions

In this chapter we turn our attention to a regularity property motivated by the study of partition combinatorics on the real line. It is a close relative of the classical Ramsey property (see Definition 1.3.5), but is combinatorially more involved, and has only become the object of systematic study quite recently, in the works of DiPrisco, Llopis, Todorčević, Shelah and Zapletal [DPLT01, DPT03, Tod10, ZS10].

The term *polarized partition* has been in use in the literature, and we shall use the term *polarized partition property* to refer to the regularity property studied in this chapter. It comes in two guises: an *unbounded* and a *bounded* version, with the former being a consequence of the Ramsey property (on the level of projective pointclasses $\boldsymbol{\Gamma}$) but the latter not. We will look at the strength of hypotheses stating that all $\boldsymbol{\Delta}^1_2$ or $\boldsymbol{\Sigma}^1_2$ sets satisfy the bounded or unbounded polarized partition property, comparing them with other regularity and transcendence statements.

One interesting aspect of this property is that it does not seem to fall into the framework described in Chapter 2. The best candidates of forcing notions related to it (the forcing used in [ZS10] and our own $\mathbb{P}$ defined in Section 3.5) belong to the class of "creature forcings" introduced by Shelah, which, in general, are not equivalent to idealized forcings. Perhaps for this reason, we were not able to prove a characterization theorem in the style of the results in Section 1.3.2. Nevertheless, we were able to prove many other results comparing this property to other well-known regularity properties, the most interesting being the content of Section 3.5 stating that the bounded version of the property can be forced to hold on the $\boldsymbol{\Sigma}^1_2$ level without adding unbounded or splitting reals.

The results of this chapter are joint work with Jörg Brendle.

3.1 Motivation.

The property studied in this chapter is motivated by the following combinatorial question: suppose we are given a partition of the Baire space ω^ω into two pieces, say, A and $\omega^\omega \setminus A$, and an infinite sequence $\langle m_i \mid i < \omega \rangle$ of integers ≥ 2. Can we find an infinite sequence $\langle H_i \mid i < \omega \rangle$ of subsets of ω, with $|H_i| = m_i$, which is homogeneous for the partition, i.e., such that the product $\prod_i H_i$ is completely contained in A or completely disjoint from A?

Without placing any definability conditions on A, it is easy to construct a counterexample using AC. For instance, if $\preceq$ is a well-ordering of ω^ω and if for every x we denote by y_x the $\preceq$-least real eventually equal to x, then the following set is a counterexample:

$$A := \{x \in \omega^\omega \mid |\{n \mid x(n) \neq y_x(n)\}| \text{ is even}\}.$$

This is because if there were a sequence $\langle H_i \mid i \leq \omega \rangle$ with $|H_i| \geq 2$ such that, say, $\prod_i H_i \subseteq A$, then any $x \in \prod_i H_i$ could be changed to $x' \in \prod_i H_i$ by altering just one digit, so that $y_x = y_{x'}$ but $|\{n \mid x'(n) \neq y_{x'}(n)\}|$ is odd, yielding a contradiction. This also immediately shows that if $V = L$, then the property fails for $\mathbf{\Delta}^1_2$ partitions, as we can use the $\mathbf{\Delta}^1_2$ well-ordering $<_L$ of the reals instead of $\preceq$.

For definable partitions A, the situation is quite different. Recall that a set $A' \subseteq [\omega]^\omega$ satisfies the *Ramsey property* if there exists an $x \in [\omega]^\omega$ such that $[x]^\omega \subseteq A'$ or $[x]^\omega \cap A' = \varnothing$.

Lemma 3.1.1 (folklore). *Let $\mathbf{\Gamma}$ be any projective pointclass and assume that all sets in $\mathbf{\Gamma}$ are Ramsey. Then the partition problem has a positive solution for all partitions in $\mathbf{\Gamma}$.*

Proof. Suppose $A \subseteq \omega^\omega$ is a given set of complexity $\mathbf{\Gamma}$, and $m_0, m_1, \dots$ are integers ≥ 2. Let $A' := \{x \in A \mid x \text{ is strictly increasing}\}$, i.e., $A' = A \cap \omega^{\uparrow\omega}$, and let $A'' := \{\text{ran}(x) \mid x \in A'\}$ (i.e., identify infinite sequences with their increasing enumeration). Since A'' is still in $\mathbf{\Gamma}$, by assumption there is an $x \in [\omega]^\omega$ which is homogeneous for A'', i.e., such that $[x]^\omega \subseteq A''$ or $[x]^\omega \cap A'' = \varnothing$. Now, simply take as H_0 the first m_0 values of x, as H_1 the next m_1 values of x, and so on. Clearly, for every $y \in \prod_i H_i$ we have $\text{ran}(y) \subseteq x$ and hence either $\prod_i H_i \subseteq A' \subseteq A$ or $\prod_i H_i \cap A' = \varnothing$. Since $\prod_i H_i$ only contains increasing sequences, the latter case implies $\prod_i H_i \cap A = \varnothing$. □

As an immediate corollary, we see that the answer to our combinatorial question is positive for

- analytic and co-analytic partitions,
- $\mathbf{\Sigma}^1_2$ partitions in the iterated Mathias model,
- all partitions in the Solovay model.

The homogeneous x obtained from the proof of Lemma 3.1.1 can grow quite rapidly, and in general there is no upper bound on its rate of growth. Hence the homogeneous sequence $\langle H_i \mid i \leq \omega \rangle$ obtained from x is also potentially unbounded. We could ask what happens if we tighten the conditions of the original question so as to rule out these "unbounded" solutions. Suppose that, this time, we are given a partition A and two sequences of integers ≥ 2: $m_0, m_1, \ldots$ and $n_0, n_1, \ldots$. Can we find $\langle H_i \mid i < \omega \rangle$ such that $|H_i| = m_i$ and $H_i \subseteq n_i$ which is homogeneous for A? Here, we want the n_i to increase at a much quicker rate then the m_i, since otherwise this property will fail even for very simple partitions (e.g., closed).

This time, a positive solution clearly cannot follow from the Ramsey property. In [DPT03], DiPrisco and Todorčević first computed explicit upper bounds $\vec{n}$ as a function of $\vec{m}$ and proved that with these bounds the problem does have a positive solution for analytic partitions, as well as for all partitions in the Solovay model. The computation of $\vec{n}$ in terms of $\vec{m}$ used a recursive but non-primitive-recursive function (an Ackermann-style function) which was improved by Shelah and Zapletal [ZS10] to a direct, primitive-recursive computation using the methods of creature forcing.

It is clear that both the bounded and the unbounded partition problems discussed above lead to the definition of a new *regularity property.*

Definition 3.1.2.

1. *Whenever $H = \langle H_i \mid i \in \omega \rangle$ is an infinite sequence of finite subsets, we shall use the shorthand notation $[H]$ instead of $\prod_i H_i$. This corresponds to identifying the sequence H with a finitely branching uniform perfect tree, so that $[H]$ is the set of branches through this tree.*

2. *Let $m_0, m_1, \ldots$ be fixed integers. A set $A \subseteq \omega^\omega$ satisfies the (*unbounded*)* polarized partition property

$$\begin{pmatrix} \omega \\ \omega \\ \cdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \cdots \end{pmatrix}$$

if there is an $H = \langle H_i \mid i \in \omega \rangle$ with $|H_i| = m_i$, such that $[H] \subseteq A$ or $[H] \cap A = \varnothing$.

3. *Let $m_0, m_1, \ldots$ and $n_0, n_1, \ldots$ be fixed integers ≥ 2 such that the n_i's are recursive in the m_i's. A set $A \subseteq \omega^\omega$ (or $\subseteq \prod_i n_i$) satisfies the* bounded polarized partition property

$$\begin{pmatrix} n_0 \\ n_1 \\ \cdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \cdots \end{pmatrix}$$

if there is an $H = \langle H_i \mid i \in \omega \rangle$ with $|H_i| = m_i$ and $H_i \subseteq n_i$, such that $[H] \subseteq A$ or $[H] \cap A = \varnothing$.

We will mostly use inline notation and denote by $\mathbf{\Gamma}(\vec{\omega} \to \vec{m})$ and $\mathbf{\Gamma}(\vec{n} \to \vec{m})$ the hypotheses that all sets in $\mathbf{\Gamma}$ satisfy the properties "$(\vec{\omega} \to \vec{m})$" and "$(\vec{n} \to \vec{m})$", respectively.

Our first observation is that as long as we are only interested in projective pointclasses $\mathbf{\Gamma}$, the precise value of the right-hand-side integers $m_0, m_1, \dots$ is irrelevant.

Lemma 3.1.3. *Let $\mathbf{\Gamma}$ be a pointclass and $m_0, m_1, \dots$ and $m'_0, m'_1, \dots$ two sequences of integers ≥ 2. Then*

1. *$\mathbf{\Gamma}(\vec{\omega} \to \vec{m})$ holds if and only if $\mathbf{\Gamma}(\vec{\omega} \to \vec{m}')$ holds.*
2. *If $\mathbf{\Gamma}(\vec{n} \to \vec{m})$ holds for some (sufficiently large) $\vec{n}$, then there are $\vec{n'}$ such that $\mathbf{\Gamma}(\vec{n}' \to \vec{m}')$ holds.*

Proof.

1. It is clear that decreasing any of the m_i's only makes the partition property easier to satisfy. Suppose we know $\mathbf{\Gamma}(\vec{\omega} \to \vec{m})$ and we are given $\vec{m}'$. Find $0 = k_{-1} < k_0 < k_1 < \dots$ such that for all i we have $m_{k_{i-1}} \cdot m_{k_{i-1}+1} \cdot \dots \cdot m_{k_i - 1} \geq m'_i$:

$$(\overbrace{m_0, m_1, \dots, m_{k_0-1}}^{\text{product is } \geq m'_0} \overbrace{m_{k_0}, m_{k_0+1}, \dots, m_{k_1-1}}^{\text{product is } \geq m'_1} \overbrace{m_{k_1}, m_{k_1+1}, \dots, m_{k_2-1}}^{\text{product is } \geq m'_2} \dots).$$

   Now let $\varphi : \omega^\omega \longrightarrow \omega^\omega$ be the continuous function given by

$$\varphi(x) := (\langle x(0), \dots, x(k_0 - 1)\rangle, \langle x(k_0), \dots, x(k_1 - 1)\rangle, \dots)$$

   where $\langle \dots \rangle$ is the canonical (recursive) bijection between ω and $\omega^{k_i - k_{i-1}}$, for the respective i. Let $A \subseteq \omega^\omega$ be a set in $\mathbf{\Gamma}$. Then $A' := \varphi^{-1}[A]$ is in $\mathbf{\Gamma}$ so by assumption there is an H' such that $\forall i\ (|H'_i| = m_i)$ and $[H'] \subseteq A'$ or $[H'] \cap A' = \varnothing$. Define H by $H_i := \{\langle r_0, \dots, r_{(k_i - k_{i-1})-1}\rangle \mid r_j \in H'_{k_{i-1}+j}\}$. Then clearly $|H_i| = m_{k_{i-1}} \cdot \dots \cdot m_{k_i - 1} \geq m'_i$ and it only remains to show that $[H] = \varphi``[H']$. But that follows immediately from the definition of φ.

2. Here, use the same function φ but now note that we may choose H' to be bounded by $\vec{n}$, so that each $H'_{k_{i-1}+j}$ is bounded by $n_{k_{i-1}+j}$. Therefore the possible elements of H_i are bounded by $\langle n_{k_{i-1}}, n_{k_{i-1}+1}, \dots, n_{k_i - 1}\rangle$ (assuming that the coding is monotonous). □

Because of this, we will use the generic notations $(\vec{\omega} \to \vec{m})$ and $(\vec{n} \to \vec{m})$ to refer to the unbounded resp. bounded partition properties, leaving $\vec{n}$ and $\vec{m}$ unspecified if it is irrelevant.

By Lemma 3.1.1 and [DPT03, Corollary 3.8], $\mathbf{\Sigma}^1_1(\vec{\omega} \to \vec{m})$ and $\mathbf{\Sigma}^1_1(\vec{n} \to \vec{m})$ are true. What about the $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ levels? If we could find an idealized, proper forcing notion $\mathbb{P}_I$ such that $(\vec{\omega} \to \vec{m})$ or $(\vec{n} \to \vec{m})$ is (classwise) equivalent to I-regularity, then we could apply results from chapter 2 to prove characterization results and measure the strength of the hypotheses $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ and $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$. We were not able to find such a forcing, and there seem to be good arguments for why it cannot be done: the forcings most naturally related to the polarized partition properties belong to the class of *creature forcings*, and by an unpublished result of Zapletal such forcings cannot, in general, be represented as $\mathcal{B}(\omega^\omega) \setminus I$ for any σ-ideal I.

Even though we are not able to prove a characterization theorem, we can prove many non-trivial implications (and non-implications) locating the polarized partition property fairly accurately among other well-known regularity properties and transcendence statements.

In Section 3.2 we prove a connection with eventually different reals and in Section 3.3 we do the same for $\mathbb{E}_0$-measurability. In Section 3.4 we look at some non-implications, and in Section 3.5 we construct a forcing notion which forces $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ without adding unbounded or splitting reals.

3.2 Eventually different reals

Two reals x and y are called *eventually different* if $\forall^\infty n\ (x(n) \neq y(n))$, and x is *eventually different over* $L[r]$ if for every $y \in \omega^\omega \cap L[r]$, x is eventually different from y. We already mentioned eventually different reals in connection with the cardinal invariant $\mathrm{non}(\mathcal{M})$ in Section 1.3.3, and in connection with R-transcendence in Example 2.5.6 (4). Let us also call a real x *bounded eventually different over* $L[r]$ if it is eventually different over $L[r]$ and moreover there exists a $y \in \omega^\omega \cap L[r]$ such that $x \leq y$.

Theorem 3.2.1.

1. $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m}) \Longrightarrow \forall r\ \exists x$ (x is eventually different over $L[r]$).
2. $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m}) \Longrightarrow \forall r\ \exists x$ (x is bounded eventually different over $L[r]$).

Proof.

1. Suppose, towards contradiction, that there is an r such that for all x, there is a $y \in L[r]$ such that $\exists^\infty n\ (x(n) = y(n))$.

   Claim. *For all x, there is also a $y \in L[r]$ such that $\exists^\infty n\ (x(n) = y(n) \wedge x(n+1) = y(n+1))$.*

   Proof. Given x, let $x' := (\langle x(0), x(1)\rangle, \langle x(2), x(3)\rangle, \dots)$. Let $y' \in L[r]$ be such that $\exists^\infty n\ (x'(n) = y'(n))$. Now let y be such that $(\langle y(0), y(1)\rangle,$

$\langle y(2), y(3)\rangle, \dots) = y'$. Since we use *recursive* coding, y is also in $L[r]$. Now it is clear that y is as required. $\Box$(claim)

For each x, let y_x denote the $<_{L[r]}$-least real in $L[r]$ such that $\exists^\infty n\ (x(n) = y_x(n) \wedge x(n+1) = y_x(n+1))$. Now define the following set:

$$A := \{x \in \omega^\omega \mid \text{the least } n \text{ s.t. } x(n) = y_x(n) \text{ is even}\}.$$

To see that A is $\Delta^1_2(r)$, we use the same method as in the proofs of Fact 1.2.11 and Fact 1.3.8. Letting $\Phi(x, y, r)$ denote the statement "y is the $<_{L[r]}$-least real such that $\exists^\infty n(x(n) = y(n) \wedge x(n+1) = y(n+1))$ and the first n s.t. $x(n) = y(n)$ is even", and using the absolute definition of $<_{L[r]}$, we can write, for all $x \in \omega^\omega$: $x \in A$ iff $\exists(L_\delta[r])\exists y(x, y \in L_\delta[r] \wedge L_\delta[r] \models \Phi(x, y, r))$. Equivalently: there exists $E \subseteq \omega \times \omega$ and $y \in \omega^\omega$ such that

1. E is well-founded,
2. $\exists n\ \exists m\ \exists u\ (x = \pi_E(n),\ y = \pi_E(m),\ r = \pi_E(u)$ and $(\omega, E) \models \Theta(u) \wedge \Phi(n, m, u))$.

Here the sentence $\Theta(r)$ is just like the Θ from Fact 1.2.10 except that it asserts "$V = L[r]$" rather than $V = L$. It is clear that the above statement is $\Sigma^1_2(r)$.

Similarly, $x \notin A$ can be written in the same form but with "even" replaced by "odd", thus showing that A is $\Delta^1_2(r)$.

Next we show that A is indeed a counterexample to $(\vec{\omega} \to \vec{m})$. Suppose there is an H such that $[H] \subseteq A$ or $[H] \cap A = \varnothing$, without loss of generality the former. Let $x \in [H] \subseteq A$. Since x and y_x coincide on two consecutive digits somewhere, we can easily alter x to x' by changing only finitely many digits, so that still $x' \in [H]$ but the first n for which $x'(n) = y_x(n)$ is odd. Since x and x' are eventually equal, $y_x = y_{x'}$ and therefore $x' \notin A$, which is a contradiction.

2. Using an analogous proof, we will show that x can in fact be bounded by the real $\vec{n}' := (\langle n_0, n_1\rangle, \langle n_2, n_3\rangle, \dots)$ which is clearly in $L[r]$. Assume towards contradiction that for all x bounded by $\vec{n}'$ there is a $y \in L[r]$ infinitely equal to it. Using the same method as before, it follows that for every x bounded by $\vec{n}$, there is a $y \in L[r]$ infinitely equal on two consecutive digits. The rest of the proof proceeds analogously except that this time we define

$$A := \{x \in \prod_i n_i \mid \text{the least } n \text{ s.t. } x(n) = y_x(n) \text{ is even}\}$$

and use the fact that the H given by $(\vec{n} \to \vec{m})$ is contained within $\prod_i n_i$. $\Box$

3.3 $\mathbb{E}_0$-measurability

Next, we connect the polarized partition property to $\mathbb{E}_0$-measurability. Recall the definition of the I_{E_0}-ideal from Example 2.1.5 (6), and the corresponding dense forcing partial order $(\mathbb{E}_0, \leq)$, consisting of E_0-trees (Definition 2.1.6) ordered by inclusion. Notice that by the dense embedding, I_{E_0}-regularity is equivalent to $\mathbb{E}_0$-Marczewski measurability (which we simply call $\mathbb{E}_0$-measurability). As always, $\mathbf{\Gamma}(\mathbb{E}_0)$ abbreviates the statement "all sets of complexity $\mathbf{\Gamma}$ are $\mathbb{E}_0$-measurable".

Since $\mathbb{E}_0$ is a proper forcing and the ideal I_{E_0} is $\mathbf{\Sigma}^1_2$, by Theorem 2.3.7 and Corollary 2.3.8 we have that $\mathbf{\Delta}^1_2(\mathbb{E}_0)$ is equivalent to the statement "$\forall r\, \exists x$ (x is I_{E_0}-quasi-generic over $L[r]$)" and $\mathbf{\Sigma}^1_2(\mathbb{E}_0)$ to the statement "$\forall r\, \{x \in 2^\omega \mid x$ is not I_{E_0}-quasi-generic over $L[r]\} \in \mathcal{N}_{I_{E_0}}$".

We show that $(\vec{\omega} \to \vec{m})$ implies $\mathbb{E}_0$-measurability for all complexity classes $\mathbf{\Gamma}$, so in particular, for $\mathbf{\Gamma} = \mathbf{\Delta}^1_2$ and $\mathbf{\Gamma} = \mathbf{\Sigma}^1_2$.

Theorem 3.3.1. $\mathbf{\Gamma}(\vec{\omega} \to \vec{m}) \Longrightarrow \mathbf{\Gamma}(\mathbb{E}_0)$.

Proof. First, we define an auxiliary equivalence relation E_0^ω, which is just like E_0 but defined on Baire space rather than Cantor space, i.e., for $x, y \in \omega^\omega$ we define $x E_0^\omega y$ iff $\forall^\infty n\ (x(n) = y(n))$. The notions of a partial E_0^ω-transversal as well as the σ-ideal $I_{E_0^\omega}$ are defined analogously. Let $A \subseteq 2^\omega$ be an arbitrary set of complexity $\mathbf{\Gamma}$, and let $T \in \mathbb{E}_0$ be an arbitrary E_0-tree.

Claim 1. *There is a bijection $f : 2^\omega \to [T]$ which preserves E_0.*

Proof. Recall that if T is an E_0-tree, then there is a stem s_0 with $|s_0| = k_0$, and there are numbers $k_0 < k_1 < k_2 < \dots$, and for each i exactly two sequences $s^i_0, s^i_1 \in {}^{[k_i, k_{i+1})}2$, such that

$$[T] = \{s_0 {}^\frown s^0_{x(0)} {}^\frown s^1_{x(1)} {}^\frown s^2_{x(2)} {}^\frown \cdots \mid x \in 2^\omega\}.$$

Define f by $f(x) := s_0 {}^\frown s^0_{x(0)} {}^\frown s^1_{x(1)} {}^\frown s^2_{x(2)} {}^\frown \dots$. Then f is a bijection between 2^ω and $[T]$; if $\forall^\infty n (x(n) = y(n))$ then clearly $\forall^\infty n (f(x)(n) = f(y)(n))$; and conversely, if $\exists^\infty n (x(n) \neq y(n))$, then $\exists^\infty n (s^n_{x(n)} \neq s^n_{y(n)})$, and it follows that $\exists^\infty n (f(x)(n) \neq f(y)(n))$. So f preserves E_0. □ (Claim 1)

Claim 2. *There is an injective function $g : \omega^\omega \to 2^\omega$ which preserves E_0^ω with respect to E_0.*

Proof. Define g as follows: for each $x \in \omega^\omega$, let

$$g(x)(n) := \begin{cases} 1 & \text{if } x((n)_0) = (n)_1 \\ 0 & \text{otherwise} \end{cases}$$

where $n = \langle (n)_0, (n)_1 \rangle$ is the canonical coding. In other words, g sends every $x \in \omega^\omega$ to the characteristic function of the (encoded) graph of x. Again, it is easy to verify that for all $x, y \in \omega^\omega$, $x E_0^\omega y$ iff $f(x) E_0 f(y)$. □ (Claim 2)

Let $A' := (f \circ g)^{-1}[A]$. It is clear that both f and g are continuous functions, so A' is also of complexity $\mathbf{\Gamma}$. By assumption, there is a sequence $H = \langle H_i \mid i < \omega \rangle$ such that $[H] \subseteq A'$ or $[H] \cap A' = \varnothing$. For each H_i, let k_i^0 and k_i^1 be the first two elements of H_i. Now define $h : 2^\omega \to [H]$ by $h(x)(i) := k_i(x(i))$. It is obvious that h is an injection and preserves E_0 with respect to E_0^ω, i.e., xE_0y iff $h(x)E_0^\omega h(y)$.

Putting everything together, let $B := (f \circ g \circ h)$"$[2^\omega]$. Clearly either $B \subseteq [T] \cap A$ or $B \subseteq [T] \setminus A$. Moreover, as $f \circ g \circ h$ preserves E_0 and 2^ω is obviously I_{E_0}-positive, the image B is also I_{E_0}-positive. This shows that A is I_{E_0}-regular as required. □

We do not know whether we can get any stronger implication from $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ and $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$.

3.4 Implications and non-implications.

Let us sum up everything we have proved so far in an implication diagram, in the same style as we have done before with Cichoń's diagram for regularity hypotheses (Figure 1.3). In addition to the properties already mentioned, we include $\mathbb{M}$- and $\mathbb{L}$-Marczewski measurability, the doughnut property (see Definition 1.3.6) and the *splitting property*, i.e., the I^{R}-regularity property for the transcendence relation defined by $x \mathrel{\mathsf{R}} y$ iff y splits x (see Example 2.5.6 (1)).

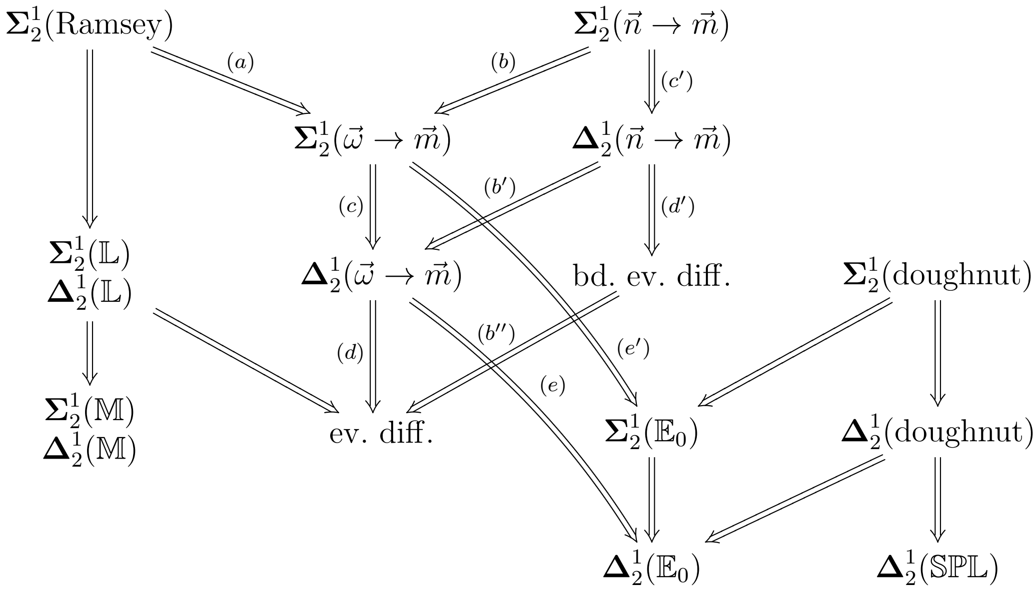


Figure 3.1: Implications between regularity hypotheses.

The latter two are included here because of their close relationship to the $\mathbb{E}_0$-measurability. This relationship was studied in [BHL05], and our results will have some relevance for that work. Recall that the doughnut property is equivalent to I_G-regularity and to $\mathbb{V}$-Marczewski measurability where $\mathbb{V}$ denotes the Silver forcing partial order (see Example 2.1.5 (7)). Similarly, the splitting property is equivalent to $\mathbb{SPL}$-Marczewski measurability where $\mathbb{SPL}$ denotes the partial order of splitting trees (Definition 2.5.7). Also, $\mathbf{\Delta}^1_2$(doughnut) is equivalent to the existence of I_G-quasi-generics over every $L[r]$, and $\mathbf{\Delta}^1_2(\mathbb{SPL})$ to the existence of splitting reals over every $L[r]$, by Theorem 2.3.7. Moreover, by Proposition 2.4 and Proposition 2.5 of [BHL05], $\mathbf{\Delta}^1_2$(doughnut) implies the existence of both I_{E_0}-quasi-generics and splitting reals over $L[r]$. In fact, since every Silver tree is an E_0-tree, the implication $\mathbf{\Gamma}(\text{doughnut}) \Longrightarrow \mathbf{\Gamma}(\mathbb{E}_0)$ holds for all projective classes $\mathbf{\Gamma}$.

In Figure 3.1, we have labeled all new implications, i.e., the ones involving the polarized partitions, while the unlabeled implications were well-known prior to our work. From the new ones, (b), (b'), (b''), (c) and (c') are trivial, (a) is Lemma 3.1.1, (d) and (d') are Theorem 3.2.1, and (e) and (e') are Theorem 3.3.1.

Note also that, although not included in the diagram, the strongest regularity hypothesis $\forall r\ (\aleph^{L[r]} < \aleph_1)$ implies all other properties considered here, and likewise, $\mathbf{\Sigma}^1_2(\mathbb{S})$ is implied by all of them. The only non-trivial case is showing that if $\forall r\ (\aleph^{L[r]} < \aleph_1)$ then $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ holds. To prove this, we can use the forcing from [ZS10] (or the forcing $\mathbb{P}$ constructed in the next section). Although this is not an idealized forcing, the methods of Proposition 2.2.6 and Corollary 2.2.7 can still be applied, and the result follows in the same way as in [ZS10, Section 3].

We are now interested whether the implications in this diagram are the only possible ones. In particular, we would like to prove that all the new implications are strict and cannot be reversed (i.e., they are not equivalences). We start by looking at (e) and (e').

Lemma 3.4.1. *In the Cohen model, i.e., the model obtained by an $\aleph_1$-iteration of Cohen forcing, $\mathbf{\Sigma}^1_2$(doughnut), and everything that follows from it, holds, whereas the existence of eventually different reals, and hence $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ and $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$, fail. In particular, implications (e) and (e') cannot be reversed.*

Proof. It is well-known that Cohen forcing does not add eventually different reals. On the other hand, by [BHL05, Proposition 3.7] all $\mathbf{\Sigma}^1_2$ sets (in fact all projective sets) have the doughnut property in the iterated Cohen model. □

Next, we turn to the arrows (b), (b') and (b''): is the bounded partition property really stronger then the unbounded one? The following terminology is well-known:

Definition 3.4.2. *A forcing $\mathbb{P}$ has the*

1. Laver property *if for every $p \in \mathbb{P}$ and every name for a real $\dot{x}$ such that for some y we have $p \Vdash \dot{x} \leq \check{y}$, there is an infinite sequence $S = \langle S_n \mid n < \omega \rangle$ with $\forall n\ (|S_n| \leq 2^n)$, and some $q \leq p$ such that $q \Vdash \dot{x} \in [\check{S}]$.*

2. weak Laver property *if for every $p \in \mathbb{P}$ and every name for a real $\dot{x}$ such that for some y we have $p \Vdash \dot{x} \leq \check{y}$, there is an infinite sequence $S = \langle S_n \mid n < \omega \rangle$ with $\forall n\ (|S_n| \leq 2^n)$, and some $q \leq p$ such that $q \Vdash \exists^\infty n\ (\dot{x}(n) \in \check{S}_n)$.*

In fact the weak Laver property has a simpler characterization:

Lemma 3.4.3. A forcing $\mathbb{P}$ has the weak Laver property iff it does not add a bounded eventually different real.

Proof. Let V be the ground model and $V[G]$ the generic extension by $\mathbb{P}$. Clearly, if for every bounded real x in $V[G]$ there is $y \in V$ infinitely equal to x, then there is also a product $S \in V$ with the same property—any S containing y will do. So it remains to prove the converse: let $x \in V[G]$ be a real bounded by $y \in V$. Partition ω into $\{B^n \mid n \in \omega\}$ by letting $B^0 := \{0\}, B^1 := \{1,2\}, B^2 := \{3,4,5,6\}$ and so on with $|B^n| = 2^n$. For convenience enumerate $B^n = \{b^n_0, \dots, b^n_{2^n-1}\}$. Let φ be the continuous function defined by $\varphi(x)(n) = \left\langle x(b^n_0), \dots, x(b^n_{2^n-1}) \right\rangle$.

Clearly $x' := \varphi(x)$ is bounded by $\varphi(y) \in V$. Let $S \in V$ be a product satisfying $\forall n\ (|S_n| \leq 2^n)$ and $\exists^\infty n\ (x'(n) \in S_n)$. Enumerate every S_n as $\{a^n_0, \dots, a^n_{2^n-1}\}$. Now, let $\{s^n_0, \dots, s^n_{2^n-1}\}$ be members of ${}^{B^n}\omega$ such that $\left\langle s^n_j(b^n_0), \dots, s^n_j(b^n_{2^n-1}) \right\rangle = a^n_j$ for every j. Then from the definition of φ it follows that for every n, if $x'(n) \in S_n$ then $x \restriction B^n = s^n_j$ for one of the j's. Hence $\exists^\infty n\ (x \restriction B^n = s^n_j$ for some $j)$. But then we can define a new real z by "diagonalizing" all the possible s^n_j's, that is, $z(b^n_i) := s^n_i(b^n_i)$. Then x is infinitely equal to z, and since z has been explicitly constructed from S, it follows that $z \in V$. This completes the proof. □

Corollary 3.4.4. *The Mathias model, i.e., the model obtained by an $\aleph_1$-iteration of Mathias forcing, $\mathbf{\Sigma}^1_2$(Ramsey) (and everything that follows from it) holds, while the existence of bounded eventually different reals, and hence $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$, fails. In particular, implications (b), (b') and (b'') cannot be reversed.*

Proof. It is well-known that $\mathbf{\Sigma}^1_2$(Ramsey) holds in the iterated Mathias model. However, it is also known that Mathias forcing satisfies the Laver property (cf. [BJ95, Section 7.4]), and that this is preserved by the $\aleph_1$-iteration. Therefore the iteration certainly also has the weak Laver property. By Lemma 3.4.3, this implies that in the Mathias model there are no bounded eventually different reals. □

We note that the nature of implications (c), (c'), (d) and (d') is still unknown. We conjecture that (d) and (d') are strict implications but efforts to prove this have so far been unsuccessful.

In the next section we prove a strong result which, in particular, will show that (a) is irreversible.

3.5 A fat creature forcing

We will now construct a forcing notion $\mathbb{P}$ which yields $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ without adding unbounded or splitting reals. As a result, we will have a proof of the consistency of $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ without $\mathbf{\Sigma}^1_2(\mathbb{M})$ or $\mathbf{\Sigma}^1_2(\mathbb{SPL})$. Most of this section will be devoted to the construction of this forcing and the study of its forcing-theoretic properties. The results proved here seem interesting in their own right, and the forcing we construct may have potential applications in other areas.

Our forcing $\mathbb{P}$ can be seen as a hybrid of two forcing notions already existing in the literature: the one used by DiPrisco and Todorčević in [DPT03] to prove the original result $\mathbf{\Sigma}^1_1(\vec{n} \to \vec{m})$ in ZFC, and a creature forcing developed by Shelah and Zapletal in [ZS10] and Kellner and Shelah in [KS09]. The latter forcing does not add unbounded or splitting reals by [ZS10] and can be applied directly to yield $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$, but seems insufficient for $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$. The DiPrisco-Todorčević forcing, on the other hand, does yield $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ but it is so combinatorially complex that it is difficult to prove preservation theorems about it, such as being ω^ω-bounding or not adding splitting reals. That is why we choose a "hybrid" solution.

We start with the following consideration: it is easy to compute integers $M_0, M_1, \dots$ such that the partition

$$\begin{pmatrix} M_0 \\ M_2 \\ \dots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \dots \end{pmatrix}$$

holds for *closed* partitions. For a proof, see [DPLT01, Theorem 1] or use an argument like in the proof of Theorem 3.5.7 (1). We fix such integers M_i for the rest of this section. The next definition and the lemma following it are instrumental in our approach to constructing a model of $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$.

Definition 3.5.1. *Let M be a model of set theory and H an infinite sequence of finite subsets of ω^ω. We say that H has the* clopification property *with respect to M if for every Borel set B with Borel code in M, the set $B \cap [H]$ is clopen relative to $[H]$ (i.e., in the subset topology on $[H]$ inherited from the standard topology on ω^ω).*

Lemma 3.5.2. *If for every $r \in \omega^\omega$ there is an H with $|H_i| = M_i$ having the clopification property with respect to $L[r]$, then $\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{2})$ holds. If, moreover, H is bounded by some recursive $\langle n_i \mid i < \omega \rangle$, then $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{2})$ holds.*

Proof. We may assume that for some r, $\aleph_1^{L[r]} = \aleph_1$, since otherwise $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{2})$ holds anyway. Using Fact 1.2.15 we can write $A = \bigcup_{\alpha<\omega_1} B_\alpha$ where each B_α is a Borel set coded in $L[r]$. Let H be the product with the clopification property. Then for each α, $B_\alpha \cap [H]$ is clopen relative to $[H]$, so $A \cap [H] = \bigcup_{\alpha<\omega_1}(B_\alpha \cap [H])$ is open relative to $[H]$, and $[H] \setminus A$ is closed relative to $[H]$, so the result follows.

The second statement of the theorem is also clear. □

We will construct a forcing $\mathbb{P}$ satisfying the following three properties:

1. $\mathbb{P}$ adds a *generic product* H_G, such that $\Vdash_{\mathbb{P}}$ "$[\dot{H}_G]$ has the clopification property with respect to the ground model, and is bounded by a recursive sequence $\vec{n}$",
2. $\mathbb{P}$ is proper and ω^ω-bounding (i.e., does not add an unbounded real), and
3. $\mathbb{P}$ does not add splitting reals.

It is well-known that being proper and ω^ω-bounding are properties preserved by $\aleph_1$-iterations with countable support (see [BJ95, Theorem 6.1.4, Theorem 6.3.6]). The property of not adding splitting reals may not be preserved, however its conjunction with being ω^ω-bounding is, by [Zap08, Corollary 6.3.8., p 290]. So, assuming we are able to construct such a $\mathbb{P}$ we have the following main result of this section:

Theorem 3.5.3. *It is consistent that $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ holds whereas both $\mathbf{\Delta}^1_2(\mathbb{M})$ and $\mathbf{\Delta}^1_2(\text{doughnut})$ fail.*

This is witnessed by the $\aleph_1$-iteration of the forcing $\mathbb{P}$ and follows from Lemma 3.5.2. In particular, implication (a) in the diagram cannot be reversed.

We now proceed with the definition of $\mathbb{P}$. We start by defining, for each n, a local partial order $(\mathbb{P}_n, \leq_n)$. After that $\mathbb{P}$ will be constructed roughly as a product of the $\mathbb{P}_n$.

Definition 3.5.4.

- *For n, let ϵ_n be a given "small" positive real number, and let X_n be a "large" integer. The precise nature of these two numbers will be determined later. Let* $\text{prenorm}_n : \mathscr{P}(X_n) \longrightarrow \omega$ *be a function satisfying the following condition:*

  > *For every $c \subseteq X_n$, if* $\text{prenorm}_n(c) \geq 1$ *then for every partition of $[c]^{M_n}$ into two parts A_0 and A_1, there exists a $d \subseteq c$ such that* $\text{prenorm}_n(d) \geq \text{prenorm}_n(c) - 1$ *and $[d]^{M_n}$ is completely contained in A_0 or A_1.*

- $\mathbb{P}_n$ *consists of tuples* (c,k)*, where* $c \subseteq X_n$ *and* k *is a natural number, such that* $\text{prenorm}_n(c) \geq k+1$*. The ordering is given by* $(c',k') \leq_n (c,k)$ *iff* $c' \subseteq c$ *and* $k \leq k'$*.*

- *Let* $\text{norm}_n : \mathbb{P}_n \longrightarrow \mathbb{R}$ *be any function such that for any* (c,k)*, whenever* $\text{norm}_n(c,k) \geq \epsilon_n$ *and* (d,ℓ) *is such that* $\text{prenorm}_n(d) - \ell \geq \frac{1}{2}(\text{prenorm}_n(c) - k)$*, then* $\text{norm}_n(d,\ell) \geq \text{norm}_n(c,k) - \epsilon_n$*. For convenience of later computations, let us fix one particular such function:*

$$\text{norm}_n(c,k) := \epsilon_n \cdot \log_2(\text{prenorm}_n(c) - k).$$

  Any other function with this property would suffice too, with the corresponding change in computations.

Note that one can have trivial partial orders satisfying the above conditions, for example, by choosing the X_n small and the function prenorm_n to be constantly 0. So we put an additional requirement: for each n, there must be at least one condition $(c,k) \in \mathbb{P}_n$ such that $\text{norm}_n(c,k) \geq n$. This can be accomplished by picking the X_n sufficiently large and using the finite Ramsey theorem to define prenorm_n. In general the value of X_n will depend on ϵ_n, i.e., the smaller the latter is the larger the former must be. Using the explicit definition of norm_n above, X_n must be so large that $\text{prenorm}_n(X_n) \geq 2^{(n/\epsilon_n)}$.

Definition 3.5.5. *The forcing notion* $\mathbb{P}$ *contains conditions* p *which are functions with domain* ω*, such that for some* $K \in \omega$*:*

- $\forall n < K : p(n) \subseteq X_n$ *and* $|p(n)| = M_n$*,*

- $\forall n \geq K : p(n) \in \mathbb{P}_n$*, and*

- *the function mapping* n *to* $\text{norm}_n(p(n))$ *converges to infinity.*

K *is the* stem-length *of* p *and* $p{\restriction}K$ *is the* stem *of* p*. For two conditions* p *and* p' *with stem-length* K *and* K'*, the ordering is given by* $p' \leq p$ *iff*

- $\text{stem}(p) \subseteq \text{stem}(p')$*,*

- $\forall n \in [K,K') : p'(n) \subseteq c$*, where* $p(n) = (c,k)$*, and*

- $\forall n \geq K' : p'(n) \leq_n p(n)$*.*

This forcing is very similar to the creature forcing used in [KS09] and [ZS10] and we refer the reader to these papers for some additional discussion about its properties. The main difference is that our forcing notion $\mathbb{P}$ does not just add one generic real, but a whole *generic product* of finite subsets of ω, defined from the generic filter G by

$$H_G := \bigcup\{\text{stem}(p) \mid p \in G\}.$$

By construction $H_G(n) \subseteq X_n$ and $|H_G(n)| = M_n$. Each forcing condition in G contains an initial segment of this generic product, namely the stem, concatenated with a sequence of $\mathbb{P}_n$-conditions with norms converging to infinity. Note that this is only possible because we have chosen X_n to be increasing sufficiently fast.

Next, let us introduce some notation.

Notation 3.5.6.

1. If $(c, k) \in \mathbb{P}_n$, we refer to the first coordinate c by "val", i.e., $\mathrm{val}(c, k) = c$. By a slight abuse of notation, if p is a condition with stem-length K we define $\mathrm{val}(p(n)) = p(n)$ for all $n < K$.

2. For $p \in \mathbb{P}$, let $\mathcal{T}(p) := \{s \in \omega^{<\omega} \mid \forall n : \; s(n) \in \mathrm{val}(p(n))\}$.

3. Let $\mathcal{S}eq$ denote the set of all finite initial segments potentially in the generic product, i.e.:

$$\mathcal{S}eq := \{\sigma : m \to \mathscr{P}(\omega) \mid \forall n < m \; (\sigma(n) \subseteq X_n \text{ and } |\sigma(n)| = M_n)\}.$$

   For n, let $\mathcal{S}eq_n := \{\sigma \in \mathcal{S}eq \mid |\sigma| = n\}$.

4. For $p \in \mathbb{P}$, let $\mathcal{S}eq(p) := \{\sigma \in \mathcal{S}eq \mid \forall n : \; \sigma(n) \subseteq \mathrm{val}(p(n))\}$ and $\mathcal{S}eq_n(p) := \{\sigma \in \mathcal{S}eq(p) \mid |\sigma| = n\}$.

5. For $\sigma \in \mathcal{S}eq(p)$, let $p \uparrow \sigma$ be the $\mathbb{P}$-condition defined by

$$(p \uparrow \sigma)(n) := \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ p(n) & \text{otherwise} \end{cases}.$$

We will use the letters $s, t, \ldots$ for elements of $\omega^{<\omega}$ and $\sigma, \tau, \ldots$ for elements of $\mathcal{S}eq$.

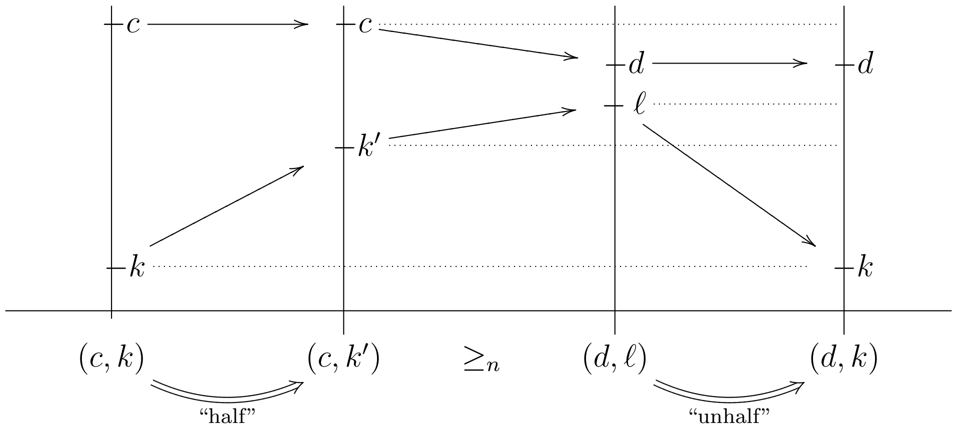
It is important to note that the forcing $\mathbb{P}$ is not separative. In particular $\mathcal{T}(q) \subseteq \mathcal{T}(p)$ does not imply $q \leq p$. However, if there exists a K such that $\mathcal{T}(q) \restriction K \subseteq \mathcal{T}(p) \restriction K$ and $\forall n \geq K : q(n) \leq_n p(n)$, then q is inseparable from p, and hence forces whatever p forces. We shall need this fact several times in the proofs.

In [KS09, ZS10], the main tools for proving results about the forcing notion were so-called ϵ_n*-bigness* and ϵ_n*-halving*. In our setting, the former is significantly stronger although the latter is essentially the same.

- "ϵ_n-bigness" is essentially a re-statement of the definition of prenorm. If $(c, k) \in \mathbb{P}_n$ is any condition with $\mathrm{norm}_n(c, k) \geq \epsilon_n$, then $\mathrm{prenorm}_n(c) - k \geq 2$. In particular, if $[c]^{M_n}$ is partitioned into two pieces A_0 and A_1, then, by the definition of prenorm, there is a $d \subseteq c$ such that $[d]^{M_n}$ is completely contained in A_0 or A_1 and $\mathrm{prenorm}_n(d) \geq \mathrm{prenorm}_n(c) - 1$. In particular, $\mathrm{prenorm}_n(d) - k \geq \mathrm{prenorm}_n(c) - k - 1 \geq \frac{1}{2}(\mathrm{prenorm}_n(c) - k)$, therefore $(d, k) \leq_n (c, k)$ is a valid $\mathbb{P}_n$-condition with $\mathrm{norm}_n(d, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n$.

- By "ϵ_n-halving" we mean the following phenomenon: if $(c,k) \in \mathbb{P}_n$ is any condition with $\mathrm{norm}_n(c,k) \geq \epsilon_n$, then let $k' := \lfloor \frac{1}{2}(\mathrm{prenorm}_n(c) + k) \rfloor$. The condition $(c,k') \leq_n (c,k)$ is called the *half of* (c,k), denoted by $\mathrm{half}(c,k)$. It satisfies the following conditions:
  - $\mathrm{norm}_n(c,k') \geq \mathrm{norm}_n(c,k) - \epsilon_n$, and
  - every $(d,\ell) \leq_n (c,k')$ can be "un-halved" to $(d,k) \leq_n (c,k)$ with $\mathrm{norm}_n(d,k) \geq \mathrm{norm}_n(c,k) - \epsilon_n$.

  The last inequality holds because $\mathrm{prenorm}_n(d) - k \geq \frac{1}{2}(\mathrm{prenorm}_n(c) - k)$.



Theorem 3.5.7.

1. Let $\mathbb{P}$ be the forcing described above, and assume that for all n, $\epsilon_n \leq 1/\left(\prod_{i<n} X_i\right)$. Then $\Vdash_{\mathbb{P}}$"$[\dot{H}_G]$ has the clopification property w.r.t. the ground model and is bounded".
2. Assume that, additionally, for all n, $\epsilon_n \leq 1/\left(\prod_{i<n} \binom{X_i}{M_i}\right)$. Then $\mathbb{P}$ is proper and ω^ω-bounding.
3. Assume that, additionally, for all n, $\epsilon_n \leq 1/(\prod_{i<n} \mathrm{prenorm}_i(X_i) \cdot 2^{X_i})$. Then $\mathbb{P}$ does not add splitting reals.

Recall that the numbers X_n depend on the value of ϵ_n. In this theorem, we require that ϵ_n depends on the previous values of X_i. The combination of these two requirements gives an inductive computation of the numbers X_n which eventually form the upper bound $\vec{n}$ in the partition property $(\vec{n} \to \vec{m})$.

Part 1 of this theorem is loosely based on [DPT03] and parts 2 and 3 are modifications of [ZS10, Proposition 2.6]. The rest of this section is devoted to the proof of these three claims.

Before starting on the proofs, let us stipulate how *fusion* works in the case of $\mathbb{P}$: for two conditions p and q and $k \in \omega$, say that $q \leq_{(k)} p$ iff $q \leq p$ and there is a K such that $p{\restriction}K = q{\restriction}K$ and for all $n \geq K : \text{norm}_n(q(n)) \geq k$. It is easy to verify that if $p_0 \geq_{(0)} p_1 \geq_{(1)} p_2 \geq_{(2)} \dots$ is a fusion sequence, then the natural (pointwise) limit q of this sequence is a $\mathbb{P}$-condition below every p_i.

Proof of 1. For every Borel set B, define $D_B := \{p \in \mathbb{P} \mid B \cap [\mathcal{T}(p)]$ is clopen in $[\mathcal{T}(p)]\}$. Since every $p \in \mathbb{P}$ forces "$[\dot{H}_G] \subseteq [\mathcal{T}(p)]$" it is sufficient to show that every D_B is dense. Define

$$\text{CL} := \{A \subseteq \prod_i X_i \mid \forall p \in \mathbb{P}\ \forall k\ \exists q \leq_{(k)} p\ (A \cap [\mathcal{T}(q)] \text{ is clopen in } [\mathcal{T}(q)])\}.$$

We claim that:

1. if A is closed then $A \in \text{CL}$,
2. if $A \in \text{CL}$ then $(\prod_i X_i \setminus A) \in \text{CL}$, and
3. if $A_n \in \text{CL}$ for every n, then $\bigcap_n A_n \in \text{CL}$.

In particular, all Borel sets are in CL and hence every D_B is dense.

Point 2 of the claim follows trivially from the definition of CL. Also, once we have proven point 1, point 3 will follow more or less immediately: by a standard fusion construction $\bigcap_n A_n$ can be rendered relatively closed, and by an application of point 1, it can then be rendered relatively clopen. We leave the details of this construction to the reader and instead focus our efforts on the proof of point 1.

First we need to fix some terminology: let T be any tree, and $X \subseteq T$. For $t \in T$ we say that "the membership of t in X depends only on $t{\restriction}m$" if

$$t \in X \iff \forall s \in T\ (t{\restriction}m \subseteq s \ \rightarrow\ s \in X) \text{ and}$$

$$t \notin X \iff \forall s \in T\ (t{\restriction}m \subseteq s \ \rightarrow\ s \notin X).$$

Let $\mathbb{P}{\restriction}m := \{p{\restriction}m \mid p \in \mathbb{P}\}$. If $h \in \mathbb{P}{\restriction}m$ is such that $h = p{\restriction}m$, we define $\mathcal{T}(h) := \mathcal{T}(p){\restriction}m$, i.e., the tree of finite sequences through h.

Now suppose C is a closed subset of $\prod_i X_i$ and let T_C be the tree of C. Let $p \in \mathbb{P}$ be a condition and $k \in \omega$. Find K such that $\forall n \geq K : \text{norm}_n(p(n)) \geq k + 1$. We claim the following:

Subclaim. *For all $m > K$, there is $h \in \mathbb{P}{\restriction}m$ such that $h{\restriction}K = p{\restriction}K$, $\forall n \in [K, m) : \text{norm}_n(h(n)) \geq \text{norm}_n(p(n)) - 1$, and for every $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t{\restriction}K$.*

Proof. The proof works by backwards-induction, from m down to K. First, we set $n := m - 1$. Let $\{s_0, \dots, s_{\ell-1}\}$ enumerate $\mathcal{T}(p){\restriction}n$. Suppose $p(n) = (c, k)$. We partition c into two parts: $A_0 := \{i \in c \mid s_0{}^\frown\langle i\rangle \in T_C\}$ and $A_1 := c \setminus A_0$. Note that this can be viewed as a partition of $[c]^1$. Our version of "ϵ_n-bigness" is meant to take care of partitions of $[c]^{M_n}$, so it certainly takes care of partitions of $[c]^1$. Therefore, there exists a $(c_0, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_0, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n$ and $c_0 \subseteq A_0$ or $c_0 \subseteq A_1$. Now, partition c_0 again into two parts: $A'_0 := \{i \in c_0 \mid s_1{}^\frown\langle i\rangle \in T_C\}$ and $A'_1 := c_0 \setminus A'_0$. Again, ϵ_n-bigness allows us to shrink to a condition $(c_1, k) \leq_n (c_0, k)$ such that $\mathrm{norm}_n(c_1, k) \geq \mathrm{norm}_n(c_0, k) - \epsilon_n$ and $c_1 \subseteq A'_0$ or $c_1 \subseteq A'_1$. We can continue this procedure until we have dealt with all of the s_i. So in the end we have a condition $(c_{\ell-1}, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_{\ell-1}, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n \cdot \ell$ and, if we define $h := p{\restriction}n^\frown \langle (c_{\ell-1}, k)\rangle$, then for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t{\restriction}n$. Notice that $\ell \leq \prod_{i<n} X_i$, so by the assumption on the size of ϵ_n it follows that $\mathrm{norm}_n(c_{\ell-1}, k) \geq \mathrm{norm}_n(c, k) - 1$.

Now we go one step back, set $n := m - 2$, let $\{s_0, \ldots, s_{\ell-1}\}$ enumerate $\mathcal{T}(p){\restriction}n$, and repeat exactly the same procedure. Again, we apply ϵ_n-bigness ℓ times (for the new value of ℓ) and in the end get a new condition, say $h(n)$, such that $\mathrm{norm}_n(h(n)) \geq \mathrm{norm}_n(p(n)) - 1$ and for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t{\restriction}n$.

Finally we reach K, and see that we have constructed a partial condition $h \in \mathbb{P}{\restriction}m$, such that $h{\restriction}K = p{\restriction}K$, $\forall n \in [K, m) : \mathrm{norm}_n(h(n)) \geq \mathrm{norm}(p(n)) - 1$ and for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t{\restriction}K$. □ (Subclaim)

Let $\mathfrak{T}$ be the collection of all h that satisfy the statement of the subclaim for some $m > K$, i.e., $\mathfrak{T} := \{h \mid h \in \mathbb{P}{\restriction}m$ for some $m > K$, $h{\restriction}K = p{\restriction}K$, $\forall n \in [K, m) : \mathrm{norm}_n(h(n)) \geq \mathrm{norm}_n(p(n)) - 1$, and for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t{\restriction}K\}$. Notice that if $h \in \mathfrak{T}$ and j is an initial segment of h with $|j| > K$, then $j \in \mathfrak{T}$. Therefore $\mathfrak{T}$ is a tree with respect to the ordering of initial segments. It is clearly a *finitely branching* tree, but it is also an *infinite* tree by the subclaim. Therefore, by König's Lemma, $\mathfrak{T}$ has an infinite branch, which we call q. It is now straightforward to verify that $q{\restriction}K = p{\restriction}K$, that $\forall n > K : \mathrm{norm}_n(q(n)) \geq \mathrm{norm}_n(p(n)) - 1 \geq k$, and that for every $x \in [\mathcal{T}(q)]$, the membership of x in C depends only on $x{\restriction}K$. But this is exactly to say that $q \leq_{(k)} p$ and $C \cap [\mathcal{T}(q)]$ is clopen in $[\mathcal{T}(q)]$, thus completing the proof. □ (part 1)

Now we look at part 2 of Theorem 3.5.7.

Proof of 2. Let $\dot\alpha$ be a name for an ordinal. If $p \in \mathbb{P}$ is a condition, we say that p *essentially decides* $\dot\alpha$ if there is m such that $\forall \sigma \in \mathcal{S}eq_m(p) : p \uparrow \sigma$ decides $\dot\alpha$. It is clear that if p essentially decides $\dot\alpha$ then p forces $\dot\alpha$ into a finite set in the ground model. Therefore, what we must prove is that for each $p \in \mathbb{P}$ and k there is a $q \leq_{(k)} p$ which essentially decides $\dot\alpha$—by standard techniques this will allow us to build a fusion sequence showing that $\mathbb{P}$ is proper and ω^ω-bounding.

For a $p \in \mathbb{P}$ and $\sigma \in \mathcal{S}eq(p)$, we call σ *deciding (in p)* if $p \uparrow \sigma$ essentially decides $\dot\alpha$, and *bad (in p)* if there is no $p' \leq p \uparrow \sigma$ with $\mathrm{stem}(p') = \sigma$ which essentially decides $\dot\alpha$.

Lemma 3.5.8. *Let $p \in \mathbb{P}$ and $K \in \omega$ be such $\forall n > K : \mathrm{norm}_n(p(n)) \geq N$ for some $N \geq 1$. Then there is a $q \leq p$ such that $q{\restriction}K = p{\restriction}K$, $\forall n \geq K$: $\mathrm{norm}_n(q(n)) \geq N - 1$, and every $\sigma \in \mathcal{S}eq_K(q)$ is either deciding or bad (in q).*

Proof. Let $\{\sigma_0, \dots, \sigma_{\ell-1}\}$ enumerate $\mathcal{S}eq_K(p)$. Let $p_{-1} := p$ and, by induction, do the following construction: for each i, suppose p_{i-1} has been defined and for all $n \geq K : \mathrm{norm}_n(p_{i-1}(n)) \geq N - \epsilon_n \cdot i$. Then there are two cases:

- Case 1: there is a $p' \leq p_{i-1} \uparrow \sigma_i$ such that $\forall n \geq K : \mathrm{norm}_n(p'(n)) \geq N - \epsilon_n \cdot (i+1)$ and p' essentially decides $\dot{\alpha}$. Let $p_i := p{\restriction}K^\frown(p'{\restriction}[K, \infty))$.

- Case 2: it is not possible to find such a p'. Then, define p_i by $p_i{\restriction}K := p{\restriction}K$ and $\forall n \geq K : p_i(n) := \mathrm{half}(p_{i-1}(n))$.

Finally let $q := p{\restriction}K^\frown(p_{\ell-1}{\restriction}[K, \infty))$. Clearly $q \leq p$ and for $n \geq K$ we have $\mathrm{norm}_n(q(n)) \geq N - \epsilon_n \cdot \ell$. Since $\ell \leq \prod_{i<K} \binom{X_i}{M_i}$, the assumption on the size of ϵ_n implies that $\mathrm{norm}_n(q(n)) \geq N - 1$.

Every σ_i for which Case 1 occurred is clearly deciding (in q). If Case 2 occurred, we will show that σ_i is bad. Suppose not, i.e., suppose there is a $q' \leq q \uparrow \sigma_i$ such that $\mathrm{stem}(q') = \sigma_i$ and q' essentially decides $\dot{\alpha}$. Let $L > K$ be such that $\forall n > L$: $\mathrm{norm}_n(q'(n)) \geq N - \epsilon_n \cdot (i+1)$. For every $n \in [K, L)$, by assumption $p_i(n) = \mathrm{half}(p_{i-1}(n))$. Since $q'(n) \leq q(n) \leq p_i(n)$, by the property called "ϵ_n-halving" there exists a condition $r(n) \leq p_{i-1}(n)$ such that $\mathrm{norm}_n(r(n)) \geq \mathrm{norm}_n(p_{i-1}(n)) - \epsilon_n$ and $\mathrm{val}(r(n)) = \mathrm{val}(q'(n))$. Define $r' := \sigma_i{}^\frown(r{\restriction}[K, L))^\frown(q'{\restriction}[L, \infty))$. Then for all $n \geq K$ we have $\mathrm{norm}_n(r'(n)) \geq N - \epsilon_n \cdot (i+1)$. Moreover, $\forall n \leq L$ we know that $\mathrm{val}(r'(n)) = \mathrm{val}(q'(n))$ and $\forall n > L : r'(n) = q'(n)$. As we mentioned before, this implies that r' is inseparable from q', and since q' essentially decides $\dot{\alpha}$, so does r'. But now the condition r' satisfies all the requirements for Case 1 to occur at step i of the induction, which is a contradiction. □ (Lemma 3.5.8)

For the next lemma, we fix the following terminology: let $T \subseteq \mathcal{S}eq$ be a set closed under initial segments and $X \subseteq T$. For $\sigma \in T$ we say that "the membership of σ in X depends only on $\sigma{\restriction}m$" if

$$\sigma \in X \iff \forall \tau \in T\ (\sigma{\restriction}m \subseteq \tau \ \rightarrow\ \tau \in X) \text{ and}$$

$$\sigma \notin X \iff \forall \tau \in T\ (\sigma{\restriction}m \subseteq \tau \ \rightarrow\ \tau \notin X).$$

Lemma 3.5.9. *Let $p \in \mathbb{P}$ and $K < K'$ be such that $\forall n \in [K, K') : \mathrm{norm}_n(p(n)) \geq 1$. Let $X \subseteq \mathcal{S}eq_{K'}(p)$. Then there exists a $q \leq p$ such that $q{\restriction}K = p{\restriction}K$, $q{\restriction}[K', \infty) = p{\restriction}[K', \infty)$, for all $n \in [K, K') : \mathrm{norm}_n(q(n)) \geq \mathrm{norm}_n(p(n)) - 1$, and for all $\sigma \in \mathcal{S}eq_{K'}(q)$, the membership of σ in X depends only on $\sigma{\restriction}K$.*

Proof. This proof works by backwards-induction, analogously to the proof of the subclaim in the proof of Theorem 3.5.7 (1) above. First we set $n := K' - 1$. Let $\{\sigma_0, \dots, \sigma_{\ell-1}\}$ enumerate $\mathcal{S}eq_n(p)$. Suppose $p(n) = (c, k)$. We partition $[c]^{M_n}$ into two parts: $A_0 := \{b \subseteq c \mid |b| = M_n \text{ and } \sigma_0{}^\frown \langle b \rangle \in X\}$, and $A_1 := [c]^{M_n} \setminus A_0$. By ϵ_n-bigness, there exists a condition $(c_0, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_0, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n$ and $[c_0]^{M_n} \subseteq A_0$ or $[c_0]^{M_n} \subseteq A_1$. Now, partition $[c_0]^{M_n}$ again into two parts: $A_0' := \{b \subseteq c_0 \mid |b| = M_n \text{ and } \sigma_1{}^\frown \langle b \rangle \in X\}$, and $A_1' := [c_0]^{M_n} \setminus A_0'$. Again, ϵ_n-bigness allows us to shrink to a condition $(c_1, k) \leq_n (c_0, k)$ such that $\mathrm{norm}_n(c_1, k) \geq \mathrm{norm}_n(c_0, k) - \epsilon_n$ and $[c_1]^{M_n} \subseteq A_0'$ or $[c_1]^{M_n} \subseteq A_1'$, etc. Finally we get a condition $(c_{\ell-1}, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_{\ell-1}, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n \cdot \ell$. If we define $p_{K'-1} := p{\restriction}(K'-1)^\frown \langle (c_{\ell-1}, k) \rangle^\frown (p{\restriction}[K', \infty))$, then for all $\tau \in \mathcal{S}eq_{K'}(p_{K'-1})$, the membership of τ in X depends only on $\tau{\restriction}(K'-1)$. Moreover, $\ell \leq \prod_{i<K} \binom{X_i}{M_i}$, so by the assumption on the size of ϵ_n it follows that $\mathrm{norm}_n(c_{\ell-1}, k) \geq \mathrm{norm}_n(c, k) - 1$.

Now we repeat the same procedure for $n := K'-2$ and find a new condition $p_{K'-2}$, such that $p_{K'-2}{\restriction}(K'-2) = p{\restriction}(K'-2)$, $p_{K'-2}{\restriction}[K', \infty) = p{\restriction}[K', \infty)$, $\forall n \in \{K'-2, K'-1\} : \mathrm{norm}_n(p_{K'-2}(n)) \geq \mathrm{norm}_n(p(n)) - 1$, and for all $\tau \in \mathcal{S}eq_{K'}(p_{K'-2})$, the membership of τ in X depends only on $\tau{\restriction}(K'-2)$.

Finally we reach K, and see that we have constructed a condition $q := p_K$ such that $q{\restriction}K = p{\restriction}K$, $q{\restriction}[K', \infty) = p{\restriction}[K', \infty)$, $\forall n \in [K, K') : \mathrm{norm}_n(q(n)) \geq \mathrm{norm}(p(n)) - 1$, and for all $\tau \in \mathcal{S}eq_{K'}(q)$, the membership of τ in X depends only on $\tau{\restriction}K$. □ (Lemma 3.5.9)

We are ready to prove the main result. Let $p \in \mathbb{P}$ and k be given. We must find a $q \leq_{(k)} p$ which essentially decides $\dot{\alpha}$. Find K such that $\forall n \geq K : \mathrm{norm}_n(p(n)) \geq k + 2$. Apply Lemma 3.5.8 with p and K to get a condition $q \leq p$ such that $q{\restriction}K = p{\restriction}K$, $\forall n \geq K : \mathrm{norm}_n(q(n)) \geq k + 1$ and every $\sigma \in \mathcal{S}eq_K(q)$ is either deciding or bad. If every σ is deciding then q essentially decides $\dot{\alpha}$, and $q \leq_{(k)} p$ holds, so the proof is complete. We will show that this is the only possibility, i.e., that no $\sigma \in \mathcal{S}eq_K(q)$ can be bad.

Towards contradiction, fix some $\sigma \in \mathcal{S}eq_K(q)$ which is bad. By induction, we will construct an increasing sequence of integers $K_0 < K_1 < K_2 < \dots$ and conditions $q_0 \geq q_1 \geq \dots$. We start by setting $K_0 := K$ and $q_0 := q \uparrow \sigma$. The induction hypothesis for stage i says that

1. $\forall n \geq K_i : \mathrm{norm}_n(q_i(n)) \geq k + i + 1$, and
2. all $\tau \in \mathcal{S}eq_{K_i}(q_i)$ are bad.

We will also guarantee that $\forall i, \forall j \geq i + 1, \forall n \geq K_i : \mathrm{norm}_n(q_j(n)) \geq k + i$.

Clearly, q_0 satisfies the conditions since the only $\tau \in \mathcal{S}eq_K(q)$ is σ. Suppose K_j and q_j have been defined for $j < i$. We describe the i-th induction step. Let K_i

be such that $\forall n \geq K_i : \text{norm}_n(q_{i-1}(n)) \geq k + i + 2$. Apply Lemma 3.5.8 with parameters q_{i-1} and K_i to find a condition $q'_i \leq q_{i-1}$ such that $q'_i \restriction K_i = q_{i-1} \restriction K_i$, $\forall n \geq K_i : \text{norm}_n(q'_i(n)) \geq k + i + 1$ and every $\tau \in \mathcal{S}eq_{K_i}(q'_i)$ is either deciding or bad. Now apply Lemma 3.5.9 on the condition q'_i and the interval $[K_{i-1}, K_i)$ to find a condition $q_i \leq q'_i$ such that $q_i \restriction K_{i-1} = q'_i \restriction K_{i-1}$, $q_i \restriction [K_i, \infty) = q'_i \restriction [K_i, \infty)$, for all $n \in [K_{i-1}, K_i) : \text{norm}_n(q(i)) \geq \text{norm}_n(q'_i(n)) - 1 \geq k + (i - 1)$, and for all $\tau \in \mathcal{S}eq_{K_i}(q_i)$, whether τ is deciding or bad depends only on $\tau \restriction K_{i-1}$.

If there is any $\tau' \in \mathcal{S}eq_{K_{i-1}}(q_i)$ such that all $\tau \in \mathcal{S}eq_{K_i}(q_i)$ extending τ' are deciding, then τ' itself would be deciding (in q_i), and hence τ' could not be bad in q_{i-1}, contradicting the induction hypothesis. Thus, in fact all $\tau \in \mathcal{S}eq_{K_i}(q_i)$ must be bad, which completes the i-th induction step.

In the end, let q_ω be the limit of this sequence. It is clear that $\forall i\, \forall n \in [K_i, K_{i+1}) :$ $\text{norm}_n(q_\omega(n)) \geq k + i$ and hence q_ω is a valid $\mathbb{P}$-condition. By construction, all $\tau \in \mathcal{S}eq(q_\omega)$ are bad. But there must be some $r \leq q_\omega$ deciding $\dot{\alpha}$, and then stem(r) cannot be bad. This contradiction completes the proof. □ (part 2)

Finally, we turn to the splitting reals. Here, the proof is almost exactly the same as [ZS10, Thm X].

Proof of 3. Let $\dot{x}$ be a name for an element of 2^ω and p a condition. To show that $\mathbb{P}$ does not add splitting reals, it suffices to find a condition $q \leq p$ such that for infinitely many n, q decides $\dot{x}(n)$. By the previous argument, we can assume, without loss of generality, that p essentially decides $\dot{x}(i)$ for every i.

Here we need to introduce new notation. For two partial conditions $h, j \in \mathbb{P} \restriction K$, $h \leq j$ is defined as for conditions in $\mathbb{P}$. For every $p \in \mathbb{P}$, let $\mathcal{S}ub_K(p) := \{h \in (\mathbb{P} \restriction K) \mid h \leq p\}$. Consider any $h \in \mathcal{S}ub_K(p)$, where $K > |\text{stem}(p)|$. Call such an h *i-deciding* (in p) if $h^\frown(p \restriction [K, \infty))$ decides $\dot{x}(j)$ for some $j > i$, and *i-bad* (in p) if there is no $p' \leq p$ such that $p' \restriction K = h$ which decides $\dot{x}(j)$ for any $j > i$.

Lemma 3.5.10. Let $p \in \mathbb{P}$ and $K \in \omega$ be such $\forall n > K : \text{norm}_n(p(n)) \geq N$ for some $N \geq 1$. Then for all i, there is a $q \leq p$ such that $q \restriction K = p \restriction K$, $\forall n \geq K :$ $\text{norm}_n(q(n)) \geq N - 1$, and every $h \in \mathcal{S}ub_K(q)$ is either i-deciding or i-bad (in q).

Proof. This is proved exactly as Lemma 3.5.8. The only difference is that we iterate over $\mathcal{S}ub_K(p)$ instead of $\mathcal{S}eq_K(p)$. Note that for each p and each n, if $p(n) = (c, k)$ then there are at most 2^{X_n} possibilities for values of c and at most $\text{prenorm}_n(X_n)$ possibilities for values of k. Therefore, for each p and each K, there are at most $\prod_{i<K} \text{prenorm}_i(X_i) \cdot 2^{X_i}$ members of $\mathcal{S}ub_K(p)$. The definition of ϵ_n compensates for this precisely. □ (Lemma 3.5.10)

Now we construct a sequence $p_0 \geq p_1 \geq \ldots$ of conditions and a sequence $K_0 < K_1 < \ldots$ of integers by the following induction. Let $p_{-1} := p$. For each i, if p_{i-1} has been defined, pick K_i such that $\forall n \geq K_i : \text{norm}_n(p_i(n)) \geq i + 2$. Apply Lemma 3.5.10 with p_{i-1}, K_i and i-decision/badness, and let p_i be the new condition. It is clear that in this way we get a fusion sequence whose limit $q \leq p$ has the following property: $\forall i\, \forall h \in \mathcal{S}ub_{K_i}(q) : h$ is i-deciding or i-bad. Also note that $\forall n \geq K_0 : \text{norm}_n(q(n)) \geq 1$.

Claim. *For each i, there is a condition $q_i \leq q$ such that $\forall n \geq K_0$ such that $\text{norm}_n(q_i(n)) \geq \text{norm}_n(q(n)) - 1$ and q_i decides $\dot{x}(i)$.*

Proof. Recall that q essentially decides $\dot{x}(i)$, so let m be such that $\forall \sigma \in \mathcal{S}eq_m(q)$: $q \uparrow \sigma$ decides $\dot{x}(i)$. Label each such σ "positive" or "negative" depending on whether $q \uparrow \sigma \Vdash \dot{x}(i) = 1$ or $q \uparrow \sigma \Vdash \dot{x}(i) = 0$. Apply Lemma 3.5.9 on the condition q and the interval $[K_0, m)$ to form a new condition q'_i such that $\forall n \in [K_0, m) : \text{norm}_n(q'_i(n)) \geq \text{norm}_n(q(n)) - 1$ and for all $\sigma \in \mathcal{S}eq_m(q'_i)$, whether σ is positive or negative depends only on $\sigma{\restriction}K_0$ (if $m \leq K_0$, skip this step). Now shrink q'_i further down to q_i on the digits $n < K_0$, by whatever means necessary, to make sure that $q_i \Vdash \dot{x}(i) = 0$ or $q_i \Vdash \dot{x}(i) = 1$. □ (claim)

Each forcing condition $p \in \mathbb{P}$ can be viewed as an element in the compact topological space $\mathcal{X} := \prod_n (\mathscr{P}(X_n) \times \text{prenorm}_n(X_n))$. In such a space every infinite sequence has an infinite convergent subsequence, in particular this applies to the sequence $\langle q_i \mid i \in \omega \rangle$. Let $a \subseteq \omega$ be an infinite set such that $\langle q_i \mid i \in a \rangle$ converges to some $r \in \mathcal{X}$. Since for all $n \geq K_0$, $\text{norm}_n(q_i(n))$ is bounded from below by $\text{norm}_n(q(n)) - 1$, the same is true for $r(n)$ which shows that r is a valid $\mathbb{P}$-condition.

But now we see that r decides infinitely many values of $\dot{x}$: for any given i, pick $j \in a$ with $j > i$ so that $q_j{\restriction}K_i = r{\restriction}K_i$. Let $h := r{\restriction}K_i$. Since $q_j \leq q$, $q_j{\restriction}K_i = h$, and q_j decides $\dot{x}(j)$, h certainly cannot be i-bad in q. So then it must be i-deciding in q, i.e., $h^\frown(q{\restriction}[K_i, \infty))$ must decide $\dot{x}(k)$ for some $k > i$. But then $r \leq h^\frown(q{\restriction}[K_i, \infty))$ must do so, too. □ (part 3)

We have proved Theorem 3.5.7, and with that, also the main Theorem 3.5.3.

3.6 Conclusion and open questions.

We have not been able to understand the nature of the arrows $(c), (c'), (d)$ and (d') in Figure 3.1. Recall that for some regularity properties (e.g. $\mathbb{M}$-, $\mathbb{L}$- and $\mathbb{S}$-measurability), the $\mathbf{\Delta}^1_2$ hypothesis is equivalent to the $\mathbf{\Sigma}^1_2$ hypothesis. For others, this is not the case (e.g. Lebesgue measure, Baire property, doughnut property). We currently have no intuition as to what the situation is in the case of the polarized partition properties.

Concerning eventually different reals, we believe that the arrows (d) and (d') are irreversible, i.e., that $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ is stronger than the existence of eventually different reals. Indeed, we conjecture the following:

Conjecture 3.6.1. *In the random model, i.e., the $\aleph_1$-iteration of random forcing, $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ fails.*

An alternative way to go about this problem would be to search for a forcing notion that adds eventually different reals without adding I_{E_0}-quasi-generics (and the latter is preserved in $\aleph_1$-iterations). Random forcing is not one of them, since it does add I_{E_0}-quasi-generics (see [BHL05, Corollary 2.3]), but perhaps a more sophisticated partial order can be found to do the job.

Finally, we would like to mention that our result have some relevance for the relationship between the $\mathbb{E}_0$-measurability, the doughnut property and the splitting property. In [BHL05, Question 6], the authors asked whether the existence of I_{E_0}-quasi-generics over $L[r]$ implied the existence of Silver-quasi-generics over $L[r]$. In the current setting, this is asking whether $\mathbf{\Delta}^1_2(\mathbb{E}_0)$ and $\mathbf{\Delta}^1_2(\text{doughnut})$ are equivalent. By Theorem 3.5.3, they are not.

In [BHL05, Question 5], they also asked whether the existence of splitting reals over $L[r]$ implied the existence of Silver-quasi-generics over $L[r]$. In the current setting, this is asking whether $\mathbf{\Delta}^1_2(\mathbb{SPL})$ and $\mathbf{\Delta}^1_2(\text{doughtnut})$ are equivalent. Our results do not answer this question, but it seems likely that the answer is negative. A method for a proof would be showing that an iteration of the $\mathbb{SPL}$-forcing does not add Silver-quasi-generic reals.

Chapter 4

Hausdorff gaps

Now we are going to depart somewhat from the theme of regularity properties, instead focusing on the definability of special objects. We mentioned some of these in Section 1.3: a Bernstein set, an ultrafilter, a maximal almost disjoint (mad) family, etc. In this chapter, we look at *Hausdorff gaps*, another kind of combinatorial object known since the beginning of the 20th century. Hausdorff's construction [Hau36] of an (ω_1, ω_1) gap in $\mathscr{P}(\omega)/\text{fin}$ was widely celebrated as an early success of the techniques and methods of set theory in mathematics. Many aspects of Hausdorff gaps, and other kinds of gaps, have been studied since then, such as extensions to higher cardinals, more general algebras, and the way forcing can destroy or create gaps. Surprisingly enough, the definability question for Hausdorff gaps has only been considered recently, in the work of Stevo Todorčević [Tod96] who showed, among other things, that there are no analytic Hausdorff gaps in $\mathscr{P}(\omega)/\text{fin}$. We shall continue this line of research, investigating what happens on higher projective levels, as well as the Solovay model, and under suitable axioms of determinacy.

4.1 Introduction

The underlying space in this chapter will be $[\omega]^\omega$, the collection of infinite subsets of ω. The notations $=^*$ and $\subseteq^*$ will be used throughout to represent the equality or subset relation between two elements of $[\omega]^\omega$ modulo finite. The following terminology has been established in the parlance of Hausdorff gaps: two sets $a, b \in [\omega]^\omega$ are *orthogonal* (notation $a \perp b$) if $a \cap b$ is finite. If B is a set, then a is *orthogonal to* B (notation $a \perp B$) if $a \perp b$ for every $b \in B$. Finally, $A, B \subseteq [\omega]^\omega$ are *orthogonal* (notation $A \perp B$) if $a \perp b$ holds for every $a \in A$ and every $b \in B$.

A pair (A, B) of orthogonal subsets of $[\omega]^\omega$ is called a *pre-gap*. There is one very simple way of constructing a pre-gap: take any infinite, co-infinite set $c \in [\omega]^\omega$ and pick any A and B so that $\forall a \in A : a \subseteq^* c$ and $\forall b \in B : c \cap b =^* \varnothing$. Such a set c is said to *separate*, or *interpolate*, the pre-gap (A, B). Of course, the interesting object is a pre-gap (A, B) which is *not* constructed in this trivial fashion.

Definition 4.1.1. *A pre-gap (A, B) is called a* gap *if there is no c which separates A from B.*

An early result of Hadamard [Had94] already established that there cannot be a gap (A, B) if both A and B are countable, although this is most widely known from [Hau36]. On the other hand, a gap (A, B) where $|A| = |B| = 2^{\aleph_0}$ can be explicitly constructed. For example, in [Tod96, p 56–57] Todorčević gives a very simple construction of a gap (A, B) where A and B are perfect sets: for $x \in 2^\omega$, define $a_x := \{x{\restriction}n \mid x(n) = 0\}$ and $b_x := \{x{\restriction}n \mid x(n) = 1\}$. Identifying $2^{<\omega}$ with ω, it is not hard to see that $(\{a_x \mid x \in 2^\omega\}, \{b_x \mid x \in 2^\omega\})$ is a gap.

Hausdorff's classical construction [Hau36] was very different. His gap (A, B) was such that $|A| = |B| = \aleph_1$, regardless of the size of the continuum; moreover, A and B were *σ-directed.*

Definition 4.1.2.

1. *A set $A \subseteq [\omega]^\omega$ is* σ-directed *if for every countable collection $\{a_n \in A \mid n \in \omega\}$, there exists $a \in A$ such that $a_n \subseteq^* a$ for all n.*

2. *A pair (A, B) is called a* Hausdorff gap *if it is a gap and both A and B are σ-directed.*

In the literature, the definition of a Hausdorff gap usually requires that A and B are well-ordered by $\subseteq^*$, as the original construction from [Hau36] in fact was, but for our purposes σ-directedness is sufficient.

That the perfect gap created by Todorčević in [Tod96, p 56–57] cannot be a Hausdorff gap follows from the following result of the same paper:

Theorem 4.1.3 (Todorčević)**.** *Let (A, B) be a pre-gap such that both A and B are σ-directed and A is analytic. Then (A, B) is not a gap.*

Proof. See [Tod96, Corollary 1]. □

This is, to our knowledge, the only result that deals with Hausdorff gaps from the definable point of view. We are interested in extending Todorčević's result in several directions and looking at Hausdorff gaps on definability levels beyond the analytic. We shall use the following notation: if $\mathbf{\Gamma}$ is a projective pointclass, we say that (A, B) is a $(\mathbf{\Gamma}, \mathbf{\Gamma})$-*Hausdorff gap* if both A and B are in $\mathbf{\Gamma}$, and a $(\mathbf{\Gamma}, \cdot)$-*Hausdorff gap* if $A \in \mathbf{\Gamma}$ and B is arbitrary. The theorem above says that there are no $(\mathbf{\Sigma}^1_1, \cdot)$-Hausdorff gaps. Our main result from Sections 4.2 and 4.3 (Corollary 4.3.10) will show that the following are equivalent:

1. there is no $(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap,
2. there is no $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$-Hausdorff gap,
3. there is no $(\mathbf{\Pi}^1_1, \cdot)$-Hausdorff gap,
4. there is no $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap,
5. $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial; (5) $\Rightarrow$ (1) will be proved in the next section, using a variation of the argument from [Tod96]; and (4) $\Rightarrow$ (5) will be proved in Section 4.3 using the method of Miller [Mil89] for the inductive construction of $\mathbf{\Pi}^1_1$ sets in L.

In Section 4.4 we show that in the Solovay model, there are no Hausdorff gaps whatsoever, and in Section 4.5 we show that the same is true under $\mathsf{AD}_{\mathbb{R}}$ (the axiom of real determinacy). In Section 4.6 we briefly look at non-Hausdorff gaps and generalize a dichotomy proved in [Tod96, Theorem 2].

4.2 Hausdorff gaps on the second level

Because of the equivalence mentioned above, the statement "there is no $(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap" has large cardinal strength, so it cannot be obtained by iterated forcing over L. For the same reason, we cannot hope to have a forcing-style proof of the implication "$\forall r\ (\aleph_1^{L[r]} < \aleph_1) \Longrightarrow \nexists(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap", as we did, say, in Corollary 2.2.7. Indeed, Todorčević's proof that there are no $(\mathbf{\Sigma}^1_1, \cdot)$-Hausdorff gaps was forcing-free, relying instead on a classical construction similar to the Cantor-Bendixson method. We will extend this method to prove the result about $(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gaps. The way our proof is derived from Todorčević's original proof is similar to the way the Mansfield-Solovay theorem (Theorem 1.3.14) is derived from the theorem that all analytic sets satisfy the perfect set property (compare, e.g., [Jec03, Theorem 25.23] and [Jec03, Theorem 11.17 (iii)]).

Here and in the future, it will be useful to look at the space $\omega^{\uparrow\omega}$ of strictly increasing functions from ω to ω and, as usual, to identify elements of $[\omega]^\omega$ with their increasing enumerations. For the proof we need several definitions.

Definition 4.2.1. *Let (A, B) be a pre-gap (not necessarily σ-directed).*

1. *Let C be a set. We say that A and B are C-separated if $C \perp B$ and for every $a \in A$ there is $c \in C$ such that $a \subseteq^* c$.*
2. *We say that A and B are* σ-separated *if they are C-separated by some countable C.*

3. *Let S be a tree on $\omega^{\uparrow\omega}$. We call S an (A,B)-*tree *if*
   (a) *$\forall\sigma \in S : \{i \in \omega \mid \sigma^\frown\langle i\rangle \in S\}$ has infinite intersection with some $b \in B$, and*
   (b) *$\forall x \in [S]$, $\mathrm{ran}(x) \subseteq^* a$ for some $a \in A$.*

If (A,B) is not a gap, then it is σ-separated, but the converse need not be true in general. It is, however, true whenever A is σ-directed. On the other hand, the existence of an (A,B)-tree contradicts B being σ-directed.

Lemma 4.2.2. *Let (A,B) be a pre-gap. If B is σ-directed, then there is no (A,B)-tree.*

Proof. Suppose, towards contradiction, that S is an (A,B)-tree. For each $\sigma \in S$, fix some $b_\sigma \in B$ such that $\{i \mid \sigma^\frown\langle i\rangle \in S\} \cap b_\sigma$ is infinite. By σ-directedness, there is a $b \in B$ which almost contains every b_σ. In particular, for each σ, the set $\{i \mid \sigma^\frown\langle i\rangle \in S\} \cap b$ is infinite. Therefore we can inductively pick $i_0, i_1, i_2 \in b$ in such a way that $\langle i_0, i_1, i_2, \dots\rangle$ is a branch through S. Then by definition of an (A,B)-tree $\{i_0, i_1, i_2 \dots\} \subseteq^* a$ for some $a \in A$. But that implies that $a \cap b$ is infinite, contradicting the orthogonality of A and B. □

Todorčević's proof in fact shows the following dichotomy: if (A,B) is a pre-gap and A is analytic, then either A and B are σ-separated or there exists an (A,B)-tree. We prove a similar dichotomy for $\boldsymbol{\Sigma}^1_2$ sets, with separation by a subset of $L[r]$ replacing σ-separation.

Theorem 4.2.3. *Let (A,B) be a pre-gap such that A is $\Sigma^1_2(r)$. Then:*

1. *either there is a $C \subseteq L[r]$ which separates A from B, or*
2. *there exists an (A,B)-tree.*

Proof. Let $A^* \subseteq \omega^{\uparrow\omega}$ be such that $x \in A^*$ iff $\mathrm{ran}(x) \in A$. Let T be a tree on $\omega \times \omega_1$, increasing in the first coordinate, such that $A^* = p[T]$ and $T \in L[r]$. Define an operation on such trees T as follows

- for $(s,h) \in T$, let

$$c_{(s,h)} := \{i > \max(\mathrm{ran}(s)) \mid \exists (s',h') \in T \text{ extending } (s,h) \text{ s.t. } i \in \mathrm{ran}(s')\}$$

- let $T' := \{(s,h) \in T \mid c_{(s,h)}$ has infinite intersection with some $b \in B\}$.

Now let $T_0 := T$, $T_{\alpha+1} := T'_\alpha$ and $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$ for limit λ. Note that this definition is absolute for $L[r]$ so all the trees T_α are in $L[r]$.

Let α be least such that $T_\alpha = T_{\alpha+1}$. We distinguish two cases:

- **Case 1:** $T_\alpha = \varnothing$. Let $x \in A^*$ be given. Let $f \in \omega_1^\omega$ be such that $(x, f) \in [T_0]$. Let $\gamma < \alpha$ be such that $(x, f) \in [T_\gamma] \setminus [T_{\gamma+1}]$, and let $(s, h) \subseteq (x, f)$ be such that $(s, h) \in T_\gamma \setminus T_{\gamma+1}$. Now let $c_x := c_{(s,h)}$ and note that this set is in $L[r]$ since it is constructible from T_γ and (s, h) both of which are in $L[r]$. By assumption $c_x \perp B$, and it is also clear that $\operatorname{ran}(x) \subseteq^* c_x$. It follows that the collection $C := \{c_x \mid x \in A^*\}$, with each c_x defined as above, forms a subset of $L[r]$ which separates A from B.

- **Case 2:** $T_\alpha \neq \varnothing$. In this case we will use the tree T_α to construct an (A, B)-tree S. By induction, we will construct S and to each $\sigma \in S$ associate $(s_\sigma, h_\sigma) \in T_\alpha$, satisfying the following conditions:

  - $\sigma \subseteq \tau \implies (s_\sigma, h_\sigma) \subseteq (s_\tau, h_\tau)$, and
  - $\operatorname{ran}(\sigma) \subseteq \operatorname{ran}(s_\sigma)$.

  First $\varnothing \in S$, and we associate to it $(s_\varnothing, h_\varnothing) := (\varnothing, \varnothing)$. Next, suppose $\sigma \in S$ has already been defined and $(s_\sigma, h_\sigma) \in T_\alpha$ associated to it. By assumption, $(s_\sigma, h_\sigma) \in T'_\alpha$, so $c_{(s_\sigma,h_\sigma)}$ has infinite intersection with some $b \in B$. For each $i \in c_{(s_\sigma,h_\sigma)}$ we add $\sigma^\frown \langle i \rangle$ to S. Moreover, by assumption, for each $i \in c_{(s_\sigma,h_\sigma)}$ there exists $(s', h') \in T_\alpha$ extending (s, h) such that $i \in \operatorname{ran}(s')$. Now associate precisely these (s', h') to $\sigma^\frown \langle i \rangle$, i.e., let $s_{\sigma^\frown \langle i \rangle} := s'$ and $h_{\sigma^\frown \langle i \rangle} := h'$. By induction, it follows that the condition $\operatorname{ran}(\sigma^\frown \langle i \rangle) \subseteq \operatorname{ran}(s_{\sigma^\frown \langle i \rangle})$ is satisfied.

  Now we have a tree S on $\omega^{\uparrow\omega}$. By definition, for every $\sigma \in S$ the set of its successors $c_{(s_\sigma,h_\sigma)}$ has infinite intersection with some $b \in B$. Now let $x \in [S]$. By construction, $\bigcup\{(s_\sigma, h_\sigma) \mid \sigma \subseteq x\}$ forms an infinite branch through T_α, whose projection $a := \bigcup\{s_\sigma \mid \sigma \subseteq x\}$ is a member of $p[T_\alpha] \subseteq p[T_0] = A^*$. Since by assumption $\operatorname{ran}(\sigma) \subseteq \operatorname{ran}(s_\sigma)$ holds for all $\sigma \subseteq x$, it follows that $\operatorname{ran}(x) \subseteq \operatorname{ran}(a)$. This proves that S is an (A, B)-tree. $\square$

Corollary 4.2.4. *If* $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$ *then there is no* $(\mathbf{\Sigma}^1_2, \cdot)$*-Hausdorff gap.*

Proof. Let (A, B) be a pre-gap such that A and B are σ-directed and A is $\Sigma^1_2(r)$. By Lemma 4.2.2, the second alternative of Theorem 4.2.3 is impossible, hence there is a $C \subseteq L[r]$ which separates A from B. Since the reals of $L[r]$ are countable, C is countable, so A and B are σ-separated. Since A is also σ-directed, (A, B) cannot be a gap. $\square$

4.3 Inaccessibility by reals

It was already mentioned in [Tod96] that if $V = L$ then there exists a $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap, though a proof of this fact was not provided. In this section we give a proof of this result and, moreover, prove the stronger statement that the non-existence of a $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap implies that $\aleph_1$ is inaccessible in L. This complements the result of the previous section, i.e., Corollary 4.2.4.

Since the argument will involve a modification of Hausdorff's original construction, let us briefly review it.

The (ω_1, ω_1)-gap Hausdorff constructed in [Hau36] had the form (A, B) where $A = \{a_\alpha \mid \alpha < \aleph_1\}$, $B = \{b_\alpha \mid \alpha < \aleph_1\}$, both A and B are well-ordered by $\subseteq^*$, and, additionally, the following condition is satisfied:

$$\forall \alpha < \aleph_1 \, \forall k \in \omega \, (\{\gamma < \alpha \mid a_\alpha \cap b_\gamma \subseteq k\} \text{ is finite}). \qquad \text{(HC)}$$

We refer to this as *Hausdorff's condition* (HC).

Lemma 4.3.1 (Hausdorff)**.** *Any pre-gap (A, B) satisfying HC is a gap.*

Proof. Towards contradiction, suppose c separates A from B. For each $\alpha < \aleph_1$, let n_α be such that $a_\alpha \setminus c \subseteq n_\alpha$ and $b_\alpha \cap c \subseteq n_\alpha$. The values n_α must be constant on some uncountable set $X \subseteq \omega_1$, i.e., there is n such that $n_\alpha = n$ for all $\alpha \in X$. Pick any $\alpha \in X$ such that there are infinitely many γ below α in X. For all of these γ, we have $a_\alpha \cap b_\gamma \subseteq (a_\alpha \setminus c) \cup (b_\gamma \cap c) \subseteq n$, contradicting HC. □

The point of Hausdorff's condition is that it provides an *absolute* way to prove that (A, B) cannot be separated. In general, the notion of a gap is not absolute, i.e., a gap existing in some model could become a non-gap if a real c separating A from B is generically added to the model. However, if the original gap satisfies Hausdorff's condition, then this cannot happen as long as $\aleph_1$ is preserved.

Lemma 4.3.2. *Let (A, B) be a pre-gap in V, satisfying HC. Let W be a larger model with $\aleph_1^W = \aleph_1^V$. Then in W, (A, B) is still a gap.*

Proof. Apply the same argument as before. □

In particular, if (A, B) is a pre-gap in $L[r]$ and $\aleph_1^{L[r]} = \aleph_1$, then (A, B) is still a gap in V. Therefore our goal is to construct a pre-gap (A, B) in $L[r]$ satisfying HC, with both A and B being $\mathbf{\Pi}^1_1$, and in such a way that the same $\mathbf{\Pi}^1_1$ definition can work in V, too.

For starters, let us see how to construct a $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$-gap satisfying HC in L. The $\mathbf{\Pi}^1_1$ construction will then be a subtle modification of it using a method developed by Arnold Miller in [Mil89]. Hausdorff constructed his gap by induction on $\alpha < \aleph_1$, using the following instrumental Lemma at each induction step.

Lemma 4.3.3 (Hausdorff)**.** *Let α be some countable ordinal, and let $(\{a_\gamma \mid \gamma < \alpha\}, \{b_\gamma \mid \gamma < \alpha\})$ be a pre-gap well-ordered by $\subseteq^*$ and satisfying HC. Then there exist sets c, d such that $(\{a_\gamma \mid \gamma < \alpha\} \cup \{c\}, \{b_\gamma \mid \gamma < \alpha\} \cup \{d\})$ is still a pre-gap, is well-ordered by $\subseteq^*$, and satisfies HC.*

Proof. See [Hau36] or [Sch93, Theorem 10]. □

An (ω_1, ω_1)-gap satisfying HC can now inductively be constructed using this lemma. And just as we have already seen many times (cf. Fact 1.2.11, Fact 1.3.8, Theorem 3.2.1 etc.), in L this construction can be modified to produce a $\mathbf{\Sigma}^1_2$ definable gap. So, at step α, instead of just picking an arbitrary pair (c, d) given by Lemma 4.3.3, pick the $<_L$-least such. Let $A = \{a_\alpha \mid \alpha < \aleph_1\}$ and $B = \{a_\alpha \mid \alpha < \aleph_1\}$ be the resulting sets. Now, as before, we may write $a \in A$ iff $\exists L_\delta \; (a \in L_\delta \;\wedge\; L_\delta \models a \in A)$, or equivalently: there is $E \subseteq \omega \times \omega$ such that

1. E is well-founded,
2. $(\omega, E) \models \Theta$,
3. $\exists n \; (a = \pi_E(n)$ and $(\omega, E) \models n \in \pi_E^{-1}[A])$.

This statement is $\mathbf{\Sigma}^1_2$. Clearly, the same can be done for the set B. Notice also that the $\mathbf{\Sigma}^1_2$ definitions of A and B define the same sets in any larger model, i.e., even when $V \neq L$, the set of all a satisfying the sentence $\exists L_\delta \; (a \in L_\delta \;\wedge\; L_\delta \models a \in A)$ defines the same subset of L, and the same holds for B. It is also clear that we can replace L with an arbitrary $L[r]$ in this argument. We have now already proved the following:

Proposition 4.3.4. *If there is no $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$-Hausdorff gap, then $\forall r \; (\aleph_1^{L[r]} < \aleph_1)$.*

Proof. Assume r is such that $\aleph_1^{L[r]} = \aleph_1$. In $L[r]$, construct a $(\Sigma^1_2(r), \Sigma^1_2(r))$-Hausdorff gap, satisfying HC, as described above. By Lemma 4.3.2, it is still a gap in V. □

In [Mil89], Miller introduced a method by which many inductive constructions in L, like the one above, could be rendered not only $\mathbf{\Sigma}^1_2$ definable, but $\mathbf{\Pi}^1_1$ definable. The idea is to eliminate the existential quantifier in the sentence "$\exists L_\delta \dots$", or "$\exists E \subseteq \omega \times \omega \dots$", by coding E directly into the real a constructed at each stage. This would allow us to write "$a \in A \iff e(a)$ is well-founded, etc.", where e is a recursive "decoding" function recovering the relation $E \subseteq \omega \times \omega$ from a. Quoting Miller:

> "The general principle is that if a transfinite construction can be done so that at each stage an arbitrary real can be encoded into the real constructed at that stage then the set being constructed will be Π^1_1. The reason is basically that then each element of the set can encode the entire construction up to that point at which it itself is constructed."
>
> [Mil89, p. 194]

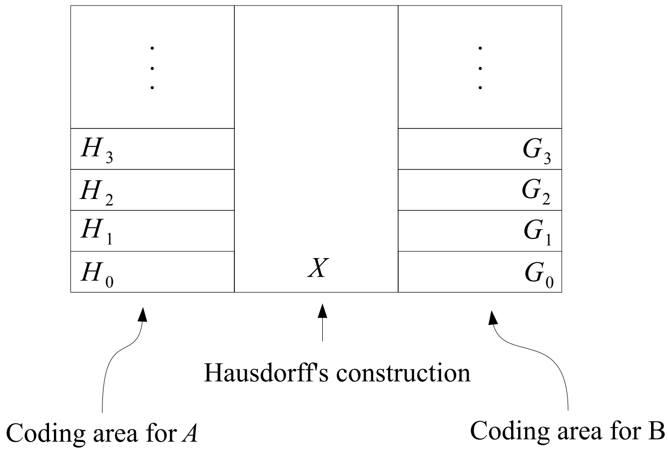


Figure 4.1: Partition of ω.

Miller himself applied this principle to show that in L there is a $\mathbf{\Pi}^1_1$ subset of $\mathbb{R}^2$ meeting every line in exactly two points, a $\mathbf{\Pi}^1_1$ mad family, and a $\mathbf{\Pi}^1_1$ Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. In [KSZ08, Theorem 3.1] the authors used the same method to show that in L there is a $\mathbf{\Pi}^1_1$ ω-mad family. Other applications exist in the literature, for instance the recent [FT10] showing that in L there is a $\mathbf{\Pi}^1_1$ maximal set of orthogonal measures on Cantor space.

To apply Miller's method, we need to prove a Coding Lemma: a stronger version of Lemma 4.3.3 stating that the c and d constructed at each induction step can encode an arbitrary relation $E \subseteq \omega \times \omega$. First, we recursively partition ω into three infinite sets: H, X and G. Further, we recursively partition H into infinitely many infinite sets H_n, and G into infinitely many infinite sets G_n. All the essential properties of Hausdorff's construction will take place within X, while the areas H and G will be used for coding purposes only. The plan is to encode an arbitrary real $z \in 2^\omega$ into a set $a \in [\omega]^\omega$ by making sure that $|H_n \cap a|$ is even if $z(n) = 1$ and odd if $z(n) = 0$, and the same for b and the G_n. A relation $E \subseteq \omega \times \omega$ can easily be encoded into a real $z \in 2^\omega$ (using a recursive bijection between ω and $\omega \times \omega$).

Lemma 4.3.5 (Coding Lemma)**.** *Let α be some countable ordinal and let $(\{a_\gamma \mid \gamma < \alpha\}, \{b_\gamma \mid \gamma < \alpha\})$ be a pre-gap with $a_\gamma \subseteq H \cup X$ and $b_\gamma \subseteq X \cup G$, which is well-ordered by $\subseteq^*$, satisfies HC, and also satisfies the following condition:*

$$(*) \qquad \forall n \in \omega \, \forall \gamma < \alpha \, (|a_\gamma \cap H_n| < \omega \ \text{ and } |b_\gamma \cap G_n| < \omega).$$

Let $E \subseteq \omega \times \omega$ be an arbitrary relation. Then there exist infinite sets c, d, with $c \subseteq H \cup X$, $d \subseteq X \cup G$, such that $(\{a_\gamma \mid \gamma < \alpha\} \cup \{c\}, \{b_\gamma \mid \gamma < \alpha\} \cup \{d\})$ is still a pre-gap, well-ordered by $\subseteq^*$, satisfies HC, satisfies condition $(*)$, and moreover both c and d recursively encode E.

Proof. First, we consider the restriction of the pre-gap to X: $(\{a_\gamma \cap X \mid \gamma < \alpha\}, \{b_\gamma \cap X \mid \gamma < \alpha\})$. Note that this is also a pre-gap well-ordered by $\subseteq^*$. Moreover, since for all γ, γ' we know that a_γ is disjoint from $b_{\gamma'}$ everywhere outside of X, the restricted pre-gap must satisfy HC, too. Using a bijection between X and ω we can apply Hausdorff's original Lemma 4.3.3 to the restricted pre-gap, and get new sets $c', d' \subseteq X$, such that $(\{a_\gamma \cap X \mid \gamma < \alpha\} \cup \{c'\}, \{b_\gamma \cap X \mid \gamma < \alpha\} \cup \{d'\})$ is a pre-gap, is well-ordered by $\subseteq^*$, and satisfies HC.

We now describe what happens inside H and G. Let $\{a'_n \mid n < \omega\}$ and $\{b'_n \mid n < \omega\}$ be a re-enumeration of the countable sets $\{a_\gamma \mid \gamma < \alpha\}$ and $\{b_\gamma \mid \gamma < \alpha\}$. Let $z \in 2^\omega$ be a real recursively coding the relation E. Now pick $c_n \subseteq H_n$ and $d_n \subseteq G_n$ such that

1. c_n and d_n are finite,
2. $\bigcup_{m \leq n}(a'_m \cap H_n) \subseteq c_n$,
3. $\bigcup_{m \leq n}(b'_m \cap G_n) \subseteq d_n$, and
4. $|c_n|$ and $|d_n|$ are even if $z(n) = 1$, and odd if $z(n) = 0$.

That this can always be done follows from condition $(*)$ of the induction hypothesis.

Now we set $c := c' \cup \bigcup_n c_n$ and $d := d' \cup \bigcup_n d_n$, and claim that the new pair of sequences $(\{a_\gamma \mid \gamma < \alpha\} \cup \{c\}, \{b_\gamma \mid \gamma < \alpha\} \cup \{d\})$ satisfies all the requirements of the lemma. It is obvious that it is a pre-gap and satisfies HC. Condition $(*)$ is also clear, since $c \cap H_n = c_n$ and $d \cap G_n = d_n$ and we defined these to be finite. To show that it is well-ordered by $\subseteq^*$, pick any a_γ. We must show that $a_\gamma \subseteq^* c$, and for that, we need to show that $a_\gamma \cap X \subseteq^* c \cap X$, and $a_\gamma \cap H \subseteq^* c \cap H$. The former is clear, because on X we have applied Lemma 4.3.3. For the latter, suppose $a_\gamma = a'_n$ in the re-enumeration used. For $k \geq n$, the definition implies that $a'_n \cap H_k \subseteq c_k$. Moreover, $\bigcup_{k<n}(a'_n \cap H_k)$ is finite by property $(*)$ of the induction hypothesis. This shows that indeed $a'_n \cap H \subseteq^* \bigcup_k c_k$ as had to be shown. Analogously, we can show $b_\gamma \subseteq^* d$.

Finally, it is clear that c and d recursively encode the relation E. □

Now define the two decoding functions $e_0, e_1 : [\omega]^\omega \to 2^\omega$ by

$$e_0(a)(n) := \begin{cases} 1 & \text{if } |H_n \cap a| \text{ is even} \\ 0 & \text{if } |H_n \cap a| \text{ is odd} \end{cases}$$

$$e_1(b)(n) := \begin{cases} 1 & \text{if } |G_n \cap b| \text{ is even} \\ 0 & \text{if } |G_n \cap b| \text{ is odd} \end{cases}$$

and by identifying z with the relation $E \subseteq \omega \times \omega$ that it recursively codes, we consider e_0 and e_1 as functions from $[\omega]^\omega$ to $\mathscr{P}(\omega \times \omega)$.

In order to use Miller's method, some special properties of the constructible hierarchy are needed, which we now present as black box results.

Definition 4.3.6 (Miller)**.** *For a countable limit ordinal α, an L_α is called* point-definable *if there exists $E \in L_{\alpha+\omega}$ such that $(\omega, E) \cong (L_\alpha, \in)$.*

Fact 4.3.7 (Miller)**.**

1. *There are unboundedly many $\alpha < \aleph_1$ such that L_α is point-definable.*
2. *Suppose L_α is point-definable. For any $\beta \leq \alpha$, if L_β is point-definable then $L_{\alpha+\omega} \models$ "L_β is point-definable".*
3. *If L_α is point-definable and E is such that $(\omega, E) \cong (L_\alpha, \in)$, then there is a recursive function mapping E to another relation, $E^{+\omega}$, such that $(\omega, E^{+\omega}) \cong (L_{\alpha+\omega}, \in)$.*

Proof. The point-definable L_α's are those levels of the constructible hierarchy whose closure under the definable Skolem functions of L is isomorphic to itself. For a detailed proof, see [KSZ08], specifically Lemmas 3.4, 3.5 and 3.6, and the relevant comments regarding absoluteness of the definitions. □

Fix an enumeration $\{\xi_\alpha \mid \alpha < \aleph_1\}$ of those countable limit ordinals for which L_{ξ_α} is point-definable. We may assume without loss of generality that $\xi_\alpha + \omega < \xi_{\alpha+1}$ for all α. By induction on $\alpha < \aleph_1$, we can now build our sets $\{a_\alpha \mid \alpha < \aleph_1\}$ and $\{b_\alpha \mid \alpha < \aleph_1\}$, with an induction hypothesis guaranteeing that a_α and b_α are members of $L_{\xi_\alpha+\omega}$.

Suppose $\{a_\gamma \mid \gamma < \alpha\}$ and $\{b_\gamma \mid \gamma < \alpha\}$ has been constructed and satisfies all the relevant conditions, i.e., is a pre-gap, is well-ordered by $\subseteq^*$, satisfies HC, and satisfies condition $(*)$ from the Coding Lemma (Lemma 4.3.5). Also, assume $a_\gamma, b_\gamma \in L_{\xi_\gamma+\omega}$ for each $\gamma < \alpha$. Then in fact $a_\gamma, b_\gamma \in L_{\xi_{\gamma+1}} \subseteq L_{\xi_\alpha}$, therefore the sets $\{a_\gamma \mid \gamma < \alpha\}$ and $\{b_\gamma \mid \gamma < \alpha\}$ are in $L_{\xi_\alpha+1}$. Moreover, since $L_{\xi_\alpha+\omega}$ contains an E satisfying $(\omega, E) \cong (L_{\xi_\alpha}, \in)$, it follows that $L_{\xi_\alpha+\omega} \models \alpha$ is countable. In particular, the two initial segments of the Hausdorff gap are countable in $L_{\xi_\alpha+\omega}$, so we can apply the Coding Lemma inside $L_{\xi_\alpha+\omega}$ to get two sets c, d in $L_{\xi_\alpha+\omega}$ which both recursively encode E. We choose a_α and b_α to be the $<_L$-least (or $<_{L_{\xi_\alpha+\omega}}$-least) such c and d. The Coding Lemma guarantees that all the requirements to proceed with the induction are satisfied by the extended initial segments $\{a_\gamma \mid \gamma \leq \alpha\}$ and $\{b_\gamma \mid \gamma \leq \alpha\}$.

Let $A := \{a_\alpha \mid \alpha < \aleph_1\}$ and $B := \{b_\alpha \mid \alpha < \aleph_1\}$ be the sets thus constructed. It is clear that (A, B) is a Hausdorff gap satisfying HC. Now recall the decoding functions e_0 and e_1. By Fact 4.3.7 (3), there are also recursive functions $e_0^{+\omega}$ and $e_1^{+\omega}$ such that if $(\omega, e_i(a)) \cong (L_{\xi_\alpha}, \in)$ for some ξ_α, then $(\omega, e_i^{+\omega}(a)) \cong (L_{\xi_\alpha+\omega}, \in)$. Now it only remains to prove the following:

Claim 4.3.8.

1. *For all* $a \in [\omega]^\omega$, $a \in A \iff$
   (a) $e_0(a)$ *is well-founded,*
   (b) $(\omega, e_0(a)) \models \Theta$,
   (c) $\exists n \in \omega\, (a = \pi_{e_0^{+\omega}(a)}(n)$ *and* $(\omega, e_0^{+\omega}(a)) \models n \in \pi^{-1}_{e_0^{+\omega}(a)}[A])$.
2. *For all* $b \in [\omega]^\omega$, $b \in B \iff$
   (a) $e_1(b)$ *is well-founded,*
   (b) $(\omega, e_1(b)) \models \Theta$,
   (c) $\exists n \in \omega\, (b = \pi_{e_1^{+\omega}(b)}(n)$ *and* $(\omega, e_1^{+\omega}(b)) \models n \in \pi^{-1}_{e_1^{+\omega}(b)}[B])$.

Proof. The two parts are obviously analogous so let us check the first one. If $a \in A$, then $a = a_\alpha$ for some α. Let $E := e_0(a)$. Then by construction $(\omega, E) \cong (L_{\xi_\alpha}, \in)$, so points (a) and (b) are satisfied. Moreover, note that the way we picked a_α in $L_{\xi_\alpha+\omega}$ using Lemma 4.3.5 was absolute between $L_{\xi_\alpha+\omega}$ and L because the relevant initial segment of the construction was in $L_{\xi_\alpha+\omega}$ and we picked the $<_L$-least such a_α. Therefore $L_{\xi_\alpha+\omega} \models a \in A$, so point (c) is satisfied.

Conversely, suppose a satisfies points (a), (b) and (c). Let $E := e_0(a)$. Then $(\omega, E) \cong (L_\delta, \in)$ for some countable limit ordinal δ, $a \in L_{\delta+\omega}$ and $L_{\delta+\omega} \models a \in A$. Then $L_{\delta+\omega} \models (\omega, e_0(a)) \cong (L_{\xi_\alpha}, \in)$ for some $\xi_\alpha < \delta + \omega$. But since this isomorphism must be absolute, in fact $\xi_\alpha = \delta$, so $L_{\xi_\alpha+\omega} \models a \in A$. Then the absoluteness of the definition of A implies that $a \in A$ holds in L, too (in fact $a = a_\alpha$). □

The claim gives us a $\mathbf{\Pi}^1_1$ definition of both A and B, since part (a) is a $\mathbf{\Pi}^1_1$ statement and the others are arithmetical. As before, it is also clear that in any larger model V, the sets A and B defined as in the claim are still exactly the same subsets of L. Also, L can be replaced by an arbitrary $L[r]$ in all the above arguments. As a result, we have shown the following:

Theorem 4.3.9. *If there is no* $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$*-Hausdorff gap, then* $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$.

Combining this result with what we proved in the last section, we get, as promised, the following corollary:

Corollary 4.3.10. *The following are equivalent:*

1. *there is no* $(\mathbf{\Sigma}^1_2, \cdot)$*-Hausdorff gap,*
2. *there is no* $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$*-Hausdorff gap,*
3. *there is no* $(\mathbf{\Pi}^1_1, \cdot)$*-Hausdorff gap,*

4. *there is no* $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$*-Hausdorff gap,*

5. $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$.

Proof. The direction (5) ⇒ (1) is Corollary 4.2.4 and (4) ⇒ (5) is Theorem 4.3.9. The other implications are obvious. □

4.4 Solovay model

We now turn our attention to the Solovay model, and the question whether Hausdorff gaps have to exist at all, not assuming AC.

Theorem 4.4.1. *Let* V *be a model with an inaccessible cardinal* κ *and* $V[G]$ *the Lévy collapse of* κ*. In* $V[G]$*, let* (A, B) *be a pre-gap with* A *and* B *definable from a countable sequence of ordinals. Then*

1. *either* A *and* B *are* σ*-separated, or*

2. *there exists an* (A, B)*-tree.*

Proof. In $V[G]$, let $s \in \mathrm{Ord}^\omega$ be such that A is definable by $\varphi(s, x)$ and B is definable by $\psi(s, x)$. By standard properties of the Lévy collapse (Lemma 1.2.20), there are formulas $\tilde{\varphi}$ and $\tilde{\psi}$ such that for all x, $V[G] \models \varphi(s, x)$ iff $V[s][x] \models \tilde{\varphi}(s, x)$, and $V[G] \models \psi(s, x)$ iff $V[s][x] \models \tilde{\psi}(s, x)$.

Assume that A and B are not σ-separated. Since κ is inaccessible in $V[s]$, there are countably many reals in $V[s]$, so A and B are not $V[s]$-separated (in the sense of Definition 4.2.1 (1)). Hence, there exists an $a \in A$ such that for all $c \in V[s]$, if $a \subseteq^* c$ then $c \not\perp B$. Let $x \in \omega^{\uparrow\omega}$ be the increasing enumeration of a. The sentence

$$\Phi(x) \;\equiv\; \forall c \in V[s]\ (\mathrm{ran}(x) \subseteq^* c \;\rightarrow\; \exists b\ (V[s][b] \models \tilde{\psi}(s, b) \wedge |c \cap b| = \aleph_0))$$

is true in $V[G]$. By another standard property of the Lévy collapse, there is a generic H such that $V[s][H] = V[G]$, and moreover there is a partial suborder Q of the Lévy collapse (depending on x), such that $Q \in V[s]$, $|Q| < \kappa$ and $x \in V[s][Q \cap H]$. Then in $V[s]$ there is a name $\dot{x}$ and a condition $p \in Q$ such that

$$p \Vdash \Phi(\dot{x}) \wedge V[s][\dot{x}] \models \tilde{\varphi}(s, \dot{x}).$$

Let $\{D_i \mid i \in \omega\}$ enumerate all the Q-dense sets in $V[s]$ (there are only countably many because κ is inaccessible in $V[s]$).

Now we inductively construct a tree $S \subseteq \omega^{<\uparrow\omega}$, and for every $t \in S$ a condition $p_t \leq p$ and an infinite set $c_t \in V[s]$, such that the following conditions are satisfied:

1. $s \subseteq t \iff p_t \leq p_s$,

2. for every t, $p_t \in D_{|t|}$,

3. for every t, $p_t \Vdash \mathrm{ran}(t) \subseteq \mathrm{ran}(\dot{x})$, and

4. for every t, $p_t \Vdash \mathrm{ran}(\dot{x}) \subseteq^* \check{c}_t$.

Let $p_\varnothing \le p$ be any condition in D_0. Clearly conditions (2) and (3) are satisfied. Assume p_t is already defined for $t \in S$, and satisfies condition (3). Let

$$c_t := \{i \mid i > \max(t) \text{ and } \exists q \le p_t \ (q \Vdash i \in \mathrm{ran}(\dot{x}))\}.$$

Then c_t is in $V[s]$ and condition (4) is satisfied (at stage t). For every $i \in c_t$, let $t^\frown \langle i \rangle$ also be an element of the tree S, and let $p_{t^\frown\langle i\rangle} \le p$ be a condition such that $p_{t^\frown\langle i\rangle} \Vdash i \in \mathrm{ran}(\dot{x})$ and $p_{t^\frown\langle i\rangle} \in D_{|t|+1}$. Now each such $p_{t^\frown\langle i\rangle}$ also satisfies condition (3) (at stage $t^\frown \langle i \rangle$), completing the induction step.

Thus we have constructed the tree S, and now we claim that it is an (A, B)-tree. For every $t \in S$, c_t is the set of immediate successors of t in S. By condition (4), $p_t \Vdash \mathrm{ran}(\dot{x}) \subseteq^* \check{c}_t$, and since $p_t \Vdash \Phi(\dot{x})$ and obviously $p_t \Vdash \check{c}_t \in V[s]$, it follows that some $q \le p_t$ forces the consequent of $\Phi(\dot{x})$, i.e., the statement "$\exists b \ (V[s][b] \models \tilde{\psi}(s, b) \wedge |b \cap c_t| = \aleph_0)$". Then for some H' Q-generic over $V[s]$ containing q, this statement holds in $V[s][H']$, and therefore there exists a b such that $V[s][b] \models \tilde{\psi}(s, b)$ and $|b \cap c_t| = \aleph_0$, i.e., there exists $b \in B$ such that $|b \cap c_t| = \aleph_0$. Since this holds for every c_t, one part of the definition of an (A, B)-tree is fulfilled.

It remains to prove that every branch through S is contained in an element of A. Let $z \in [S]$, and let H_z be the filter over Q generated by $\{p_t \mid t \subseteq z\}$. By construction, H_z is Q-generic over $V[S]$. Since all p_t force the statement "$V[s][\dot{x}] \models \tilde{\varphi}(s, \dot{x})$", we get that $V[s][\dot{x}_{H_z}] \models \tilde{\varphi}(s, \dot{x}_{H_z})$, and therefore $\dot{x}_{H_z} \in A$ (here we have identified $\dot{x}_{H_z}$ with its range, but it should be clear that this is fine). Moreover, since by condition (3) we have, for every $t \subseteq z$, that $p_t \Vdash \mathrm{ran}(t) \subseteq \mathrm{ran}(\dot{x})$, it follows that $\mathrm{ran}(t) \subseteq \mathrm{ran}(\dot{x}_{H_z})$ holds for every $t \subseteq z$, and therefore $\mathrm{ran}(z) \subseteq \mathrm{ran}(\dot{x}_{H_z}) \in A$. This is what we wanted to show. $\square$

Corollary 4.4.2. *Let V be a model with an inaccessible cardinal κ and $V[G]$ the Lévy collapse. If (A, B) is a pre-gap in $V[G]$ such that A and B are definable from a countable sequence of ordinals, and moreover A and B are σ-directed, then (A, B) is not a gap.*

Proof. As before, if B is σ-directed then there cannot be an (A, B)-tree by Lemma 4.2.2, and if A is also σ-directed then alternative 1 from Theorem 4.4.1 implies that A and B are separated. $\square$

Corollary 4.4.3. Con(ZFC+ *"there are no projective Hausdorff gaps"*) *and* Con(ZF + DC+ *"there are no Hausdorff gaps"*).

4.5 Axiom of real determinacy

The determinacy of infinite games is often used as a tool to prove regularity properties. The strongest result we could hope to prove in this setting is that AD implies that there are no Hausdorff gaps. We were not able to prove this, but, as already discussed in Question 2.6.4 and Question 2.6.5, $\mathsf{AD}_{\mathbb{R}}$ may be a more appropriate axiom in this case. So, we will take $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ as the ambient theory in this section, and construct a game with real moves whose determinacy proves the non-existence of Hausdorff gaps.

Definition 4.5.1. *Let (A, B) be a pre-gap. The game $G_{\mathrm{H}}(A, B)$ is played as follows:*

$$\begin{array}{c|cccccccc} \mathrm{I}: & c_0 & & (s_1, c_1) & & (s_2, c_2) & & \dots & \\ \hline \mathrm{II}: & & i_0 & & i_1 & & i_2 & & \dots \end{array}$$

where $s_n \in \omega^{<\omega}$, $c_n \in [\omega]^\omega$ and $i_n \in \omega$. The conditions for player I are that

1. $\min(s_n) > \max(s_{n-1})$ *for all* $n \geq 1$,
2. $\min(c_n) > \max(s_n)$,
3. $c_n \not\perp B$ *for all* n, *and*
4. $i_n \in \mathrm{ran}(s_{n+1})$ *for all* n.

Conditions for player II are that

5. $i_n \in c_n$ *for all* n.

If all five conditions are satisfied, let $s^* := s_1{}^\frown s_2{}^\frown \dots$ be an infinite increasing sequence formed by the play of the game. Player I wins iff $\mathrm{ran}(s^*) \in A$.

Theorem 4.5.2.

1. *If player I has a winning strategy in $G_{\mathrm{H}}(A, B)$ then there exists an (A, B)-tree.*
2. *If player II has a winning strategy in $G_{\mathrm{H}}(A, B)$ then A and B are σ-separated.*

Proof. 1. Let σ be a winning strategy for player I and let T_σ be the tree of partial positions according to σ. If $p \in T_\sigma$ is a position of the form $p = \langle c_0, i_0, (s_1, c_1), i_1, \dots, (s_n, c_n)\rangle$, we use the notation $p^* := s_1{}^\frown \dots {}^\frown s_n$.

Now we use T_σ to inductively construct the tree S. To each $s \in S$ we associate a $p_s \in T_\sigma$ (of odd length), such that

1. $s \subseteq t$ iff $p_s \subseteq p_t$, and

2. $\operatorname{ran}(s) \subseteq \operatorname{ran}(p_s^*)$.

First $\varnothing \in S$ and $p_\varnothing = \varnothing$. Suppose $s \in S$ and p_s are already defined and $\operatorname{ran}(s) \subseteq \operatorname{ran}(p_s^*)$ holds. Assume $p_s = \langle \dots, (s_n, c_n)\rangle$. For every $i_n \in c_n$, let (s_{n+1}, c_{n+1}) be the response of the strategy σ to $p_s{}^\frown \langle i_n\rangle$. Let $s^\frown \langle i_n\rangle$ be in S and associate to it $p_{s^\frown\langle i_n\rangle} := p_s{}^\frown \langle i_n\rangle {}^\frown \langle (s_{n+1}, c_{n+1})\rangle$. Since for each $i_n \in c_n$ we know that $i_n \in \operatorname{ran}(s_{n+1})$, it follows that $\operatorname{ran}(s^\frown \langle i_n\rangle) \subseteq \operatorname{ran}(p^*_{s^\frown\langle i_n\rangle})$, completing the induction step.

Now it is clear that the tree S has exactly the c_n's as the branching-points, which all have infinite intersection with some $b \in B$ by assumption. Moreover, if x is a branch through S, then by construction $z := \bigcup\{p_s \mid s \subseteq x\}$ forms a branch through T_σ satisfying $\operatorname{ran}(x) \subseteq \operatorname{ran}(z^*)$. Since z is an infinite play of the game according to the winning strategy σ, it follows that $\operatorname{ran}(z^*) \in A$, so S is an (A, B)-tree.

2. Now let τ be a winning strategy for player II, and let T_τ be the tree of partial plays according to τ. Our method will be similar to the proof of the standard Banach-Mazur theorem, but the problem is that the tree T_τ has uncountable branching. Therefore we first thin it out to another tree $\tilde{T}_\tau$, as follows: for every node of even length $p = \langle \dots, (s_n, c_n), i_n\rangle \in T_\tau$, fix s and i and consider the collection $\operatorname{Succ}_{T_\tau}(p, s, i) := \{(s, c) \mid p^\frown \langle (s, c)\rangle {}^\frown \langle i\rangle \in T_\tau\}$. In other words, this is the collection of all valid moves by player I following position p, such that the first component of the move is s, and such that II's next move according to τ is i. If this collection is non-empty, throw away all members of $\operatorname{Succ}_{T_\tau}(p, s, i)$, and their generated subtrees, except for one, so that $\operatorname{Succ}_{T_\tau}(p, s, i)$ becomes a singleton. Notice that this construction is justified because, since we are working under $\mathsf{AD}_\mathbb{R}$, we have to our disposal the fragment of the Axiom of Choice allowing us to choose from collections indexed by real numbers. Therefore, we can perform this "pruning" operation for every $s \in \omega^{<\omega}$ and every $i \in \omega$, and inductively form the new tree $\tilde{T}_\tau$—this is also going to be a tree of positions according to τ, but it will be a countably branching tree. Now we can use a Banach-Mazur-style argument on $\tilde{T}_\tau$.

For every $p \in \tilde{T}_\tau$ and $x \in \omega^{\uparrow\omega}$, where $p = \langle \dots (s_n, c_n), i_n\rangle$, we say that p *is compatible with* x if $p^* \subseteq x$ and $i_n \in \operatorname{ran}(x)$. We say that p *rejects* x if it is compatible with x and maximally so with respect to $\tilde{T}_\tau$, i.e., if for every (s, c) such that $p^\frown \langle (s, c)\rangle \in \tilde{T}_\tau$ and $p^{*\frown} s \subseteq x$, $i := \tau(p^\frown \langle (s, c)\rangle) \notin \operatorname{ran}(x)$.

It is clear that for every x with $\operatorname{ran}(x) \in A$ there is a $p \in \tilde{T}_\tau$ which rejects x—otherwise we could inductively find an infinite branch z through $\tilde{T}_\tau$ such that $z^* = x$, implying that $\operatorname{ran}(x) \notin A$ since z is a play according to a strategy that was winning for player II. For each $p \in \tilde{T}_\tau$ let $K_p := \{x \mid p \text{ rejects } x\}$. Also, write $K_p^* := \{\operatorname{ran}(x) \mid p \text{ rejects } x\}$. Since $A \subseteq \bigcup_{p \in \tilde{T}_\tau} K_p^*$ and $\tilde{T}_\tau$ is countable, the result will follow if we can prove that each K_p^* is σ-separated from B.

For this, fix some $p = \langle \dots (s_n, c_n), i_n \rangle$, and for every $s \in \omega^{<\omega}$ such that $i_n \in \mathrm{ran}(s)$ and $\min(s) > \max(p^*)$, consider the set

$$a_s := \bigcup \{\mathrm{ran}(x) \mid x \in K_p \text{ and } p^{*\frown} s \subseteq x\}.$$

We claim that the collection $\{a_s \mid i_n \in \mathrm{ran}(s) \text{ and } \min(s) > \max(p^*)\}$ σ-separates K_p^* from B. First, clearly if $x \in K_p$ then there exists some s, satisfying the conditions, such that $p^{*\frown} s \subseteq x$, so that $\mathrm{ran}(x) \subseteq a_s$. Secondly, suppose that there is some s, with $i_n \in \mathrm{ran}(s)$ and $\min(s) > \max(p^*)$, such that a_s has infinite intersection with some $b \in B$. Let $a'_s := a_s \setminus \max(s)$. According to the rules of the game, player I is then allowed to play the move "(s, a'_s)" after position p. The only problem is that $p^\frown \langle (s, a'_s) \rangle$ might not be in $\tilde{T}_\tau$. However, by construction there is some c such that $i := \tau(p^\frown \langle (s, c) \rangle) = \tau(p^\frown \langle (s, a'_s) \rangle)$ and $p^\frown \langle (s, c) \rangle \in \tilde{T}_\tau$. But then we must have $i \in a'_s$, so by definition there is some $x \in K_p$ such that $p^{*\frown} s \subseteq x$ and $i \in \mathrm{ran}(x)$. But then $p^\frown \langle (s, c) \rangle ^\frown \langle i \rangle$ is still compatible with x and hence p does not reject x, contradicting $x \in K_p$.

So we must have $a_s \perp B$ for all s, and this completes the proof. □

Corollary 4.5.3. $\mathsf{AD}_{\mathbb{R}}$ *implies that every pre-gap (A, B) is either σ-separated or there exists an (A, B)-tree.*

Corollary 4.5.4. $\mathsf{AD}_{\mathbb{R}}$ *implies that there are no Hausdorff gaps.*

4.6 Other gaps

In the last section, we briefly consider non-Hausdorff gaps, i.e., gaps (A, B) in which A and B are not necessarily σ-directed, and extend the second main theorem of [Tod96], by combining its proof with results from [Fen93].

As we know, such gaps can be quite explicitly defined. Let us recall the example we mentioned in the introduction: $A := \{a_x \mid x \in 2^\omega\}$ and $B := \{b_x \mid x \in 2^\omega\}$, where $a_x := \{x \restriction n \mid x(n) = 0\}$ and $b_x := \{x \restriction n \mid x(n) = 1\}$ for every $x \in 2^\omega$. In [Tod96, p 57], Todorčević isolated the main ingredient of this construction and defined a concept that he called a *perfect Luzin gap.*

Definition 4.6.1. (A, B) *is called a* perfect Luzin gap *if A can be written as $\{a_x \mid x \in 2^\omega\}$ and B can be written as $\{b_x \mid x \in 2^\omega\}$, such that the functions $x \mapsto a_x$ and $x \mapsto b_x$ are continuous, and so that the following condition* (Luzin's condition) *is satisfied: there exists some $n \in \omega$ such that*

1. *for every $x \in 2^\omega$, $a_x \cap b_x \subseteq n$, and*
2. *for every $x \neq y$, either $a_x \cap b_y \not\subseteq n$ or $a_y \cap b_x \not\subseteq n$.*

Any pre-gap (A, B) satisfying Luzin's condition can be shown to be a gap (see, e.g., [Sch93] for details), and moreover, A and B are perfect subsets of $[\omega]^\omega$. The second main result of Todorčević [Tod96, Theorem 2] shows that a perfect Luzin gap is essentially the only type of analytic gap. First we need a weaker notion of separation.

Definition 4.6.2. *A pre-gap (A, B) is* weakly σ-separated *if there is a countable set C such that for every $a \in A$ and $b \in B$, there is a $c \in C$ such that $a \subseteq^* c$ and $c \cap b$ is finite.*

If (A, B) are σ-separated then they are also weakly σ-separated, though the converse need not be true. Of course, in the case of a Hausdorff gap, the two notions are the same, and are equivalent to (A, B) being separated, but in general we should be more careful.

Definition 4.6.3. *We say that a pre-gap (A, B) satisfies the* perfect Luzin dichotomy *if either*

1. *(A, B) is weakly σ-separated, or*
2. *there is a perfect Luzin sub-gap (A', B') of (A, B) (i.e., $A' \subseteq A$ and $B' \subseteq B$).*

Theorem 4.6.4 (Todorčević)**.** *Every $(\mathbf{\Sigma}^1_1, \mathbf{\Sigma}^1_1)$-pre-gap satisfies the perfect Luzin dichotomy.*

Proof. See [Tod96, Theorem 2] □

Can we extend this theorem, and prove results similar to the ones we proved about Hausdorff gaps? We will show that this is indeed the case, and, in fact, it follows by putting together several existing results. First, we note that the main ingredient of the proof of [Tod96, Theorem 2] is a perfect set version of the Open Colouring Axiom studied by Qi Feng in [Fen93], itself being a variant of the original Open Colouring Axiom (OCA) introduced by Todorčević in [Tod89].

Definition 4.6.5 (Feng)**.** *A set A satisfies OCA_P if for every partition $[A]^2 = K_0 \cup K_1$, where K_0 is open in the relative topology of A, one of the following holds:*

1. *there exists a perfect set $P \subseteq A$ such that $[P]^2 \subseteq K_0$, or*
2. *$A = \bigcup_n A_n$, for some A_n satisfying $[A_n]^2 \subseteq K_1$.*

We write $\mathbf{\Gamma}(\mathsf{OCA}_P)$ to mean that every set in $\mathbf{\Gamma}$ satisfies OCA_P.

Theorem 4.6.6 (Feng)**.**

1. *$\mathbf{\Sigma}^1_1(\mathsf{OCA}_P)$,*

2. *the following are equivalent:*
   (a) $\mathbf{\Sigma}^1_2(\mathsf{OCA}_P)$,
   (b) $\mathbf{\Pi}^1_1(\mathsf{OCA}_P)$,
   (c) $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$.
3. *in the Solovay model, all sets satisfy* OCA_P, *and*
4. $\mathsf{AD} \implies \mathsf{OCA}_P$.

Proof. See Theorem 1.1, Corollary 2.2, Theorem 4.1 and Theorem 3.3 of [Fen93], respectively. □

The proof of [Tod96, Theorem 2] in fact shows the following stronger result:

Theorem 4.6.7 (Todorčević)**.** *For any pointclass* $\mathbf{\Gamma}$, *if* $\mathbf{\Gamma}(\mathsf{OCA}_P)$ *holds then every* $(\mathbf{\Gamma}, \mathbf{\Gamma})$*-pre-gap satisfies the perfect Luzin dichotomy.*

Combining this with Theorem 4.6.6, we immediately get:

Corollary 4.6.8.

1. *The following are equivalent:*
   (a) *Every* $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$*-pre-gap satisfies the perfect Luzin dichotomy,*
   (b) *Every* $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$*-pre-gap satisfies the perfect Luzin dichotomy,*
   (c) $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$.
2. *in the Solovay model, all pre-gaps satisfy the perfect Luzin dichotomy, and*
3. AD *implies that all pre-gaps satisfy the perfect Luzin dichotomy.*

Proof. The only non-trivial direction is $(b) \Rightarrow (c)$ from part (1). For this, simply use the construction from Section 4.3, i.e., the $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap satisfying HC in L. Clearly, it is not weakly σ-separated. On the other hand, if it would contain a perfect Luzin sub-gap (A', B'), then (A', B') would be a Hausdorff gap with a perfect set A', contradicting Theorem 4.1.3. □

Note that here we have an implication from AD rather than just $\mathsf{AD}_{\mathbb{R}}$. Whether the same could be done for Corollary 4.5.4 is still open.

Question 4.6.9. *Does* AD *imply that there are no Hausdorff gaps?*

Chapter 5

Mad families

In this chapter, we turn our attention to another kind of combinatorial object, maximal almost disjoint (mad) families, from the descriptive point of view. Despite the relative simplicity with which a mad family is defined (as opposed to, say, a Hausdorff gap), its properties remain elusive, and many questions about these objects, especially in the definability context, remain unresolved. We will focus on one particular such question, the relationship between $\mathbf{\Sigma}^1_2$ definable mad families and dominating reals. Motivated by the need to preserve mad families while adding dominating reals, we are led to the introduction of a new cardinal invariant $\mathfrak{a}_B$, defined as the least number of Borel almost disjoint sets whose union is a mad family. Our main result is the consistency of $\mathfrak{a}_B < \mathfrak{b}$, which, with only a slight modification, proves the consistency of $\mathbf{\Sigma}^1_2(\mathbb{L})$+ "there is a $\mathbf{\Sigma}^1_2$ mad family" and also the consistency of $\mathfrak{b} > \aleph_1$+ "there is a $\mathbf{\Sigma}^1_2$ mad family". This answers a question recently posed by Friedman and Zdomskyy in [FZ10, Question 16]. The results of this chapter are joint work with Jörg Brendle.

5.1 Introduction

Recall that two sets $a, b \in [\omega]^\omega$ are called *almost disjoint* (*a.d.*) if $|a \cap b| < \omega$, and a set $A \subseteq [\omega]^\omega$ is called *almost disjoint* (*a.d.*) if every two elements $a, b \in A$ are almost disjoint. A set A is called *maximal almost disjoint*, or *mad*, if it is an infinite a.d. family and maximal with respect to that property, i.e., $\forall a \, \exists b \in A(|a \cap b| = \omega)$. It will also be assumed, throughout this chapter, that mad families are *infinite*, since finite ones are of no interest (e.g., any partition of ω into two disjoint sets is mad according to the above definition).

Mad families have been studied for a long time and have applications in various fields of mathematics. They can easily be constructed using a well-ordering of the reals: by induction on $\alpha < 2^{\aleph_0}$, pick the least $a \in [\omega]^\omega$ which is still almost disjoint from all the previously chosen ones. Of course, such objects are in general not definable. Non-maximal almost disjoint families of size $2^{\aleph_0}$, on the other hand, can be explicitly defined: for each $x \in \omega^\omega$, let $a_x := \{\sigma \in \omega^{<\omega} \mid \sigma \subseteq x\}$. If $x \neq y$ then $|a_x \cap a_y| < \omega$, hence $\{a_x \mid x \in \omega^\omega\}$ is an a.d. family in $[\omega^{<\omega}]^\omega$, which can be seen as an a.d. family in $[\omega]^\omega$ upon identification of ω with $\omega^{<\omega}$. Topologically, this is a perfect subset of $[\omega]^\omega$. It is clearly not maximal—for example, $a := \{\sigma \in \omega^{<\omega} \mid |\sigma| = 1\}$ is almost disjoint from it. It can be extended to a mad family using the well-ordering of the reals, but of course at the expense of definability. This is not coincidental, for an early result of Adrian Mathias [Mat77, Corollary 4.7] shows the following:

Theorem 5.1.1 (Mathias)**.** *There is no analytic mad family.*

It is easy to see that if $V = L$, then there is a $\boldsymbol{\Sigma}^1_2$ definable mad family. Also, if A is a $\boldsymbol{\Sigma}^1_2$ mad family then it must be $\boldsymbol{\Delta}^1_2$: if $\phi(a)$ defines A, then $\psi(a) \equiv \exists b\, (\phi(b) \wedge a \neq b \wedge |a \cap b| = \omega)$ defines the complement of A. In fact this shows that any $\boldsymbol{\Sigma}^1_n$ mad family must be $\boldsymbol{\Delta}^1_n$.

A more subtle result is the following theorem of Arnold Miller [Mil89, Theorem 8.23], proved by the same method that we used in the proof of Theorem 4.3.9.

Theorem 5.1.2 (Miller)**.** *If $V = L$ then there is a $\boldsymbol{\Pi}^1_1$ mad family.*

Note that, as in Chapter 4, the *non-existence* of mad families can be considered a regularity hypothesis, and we can look at this on the projective level. In general, there are many open questions here: for example, it is not clear whether the hypotheses "there is no $\boldsymbol{\Sigma}^1_2$ mad family" and "there is no $\boldsymbol{\Pi}^1_1$ mad family" are equivalent; there is no characterization theorem connecting such hypotheses with a transcendence statement over L. Most notably, it is unknown whether the consistency strength of "there are no mad families" is an inaccessible cardinal or higher, i.e., whether collapsing an inaccessible κ is sufficient to obtain a model where there are no mad families (as is the case with most regularity properties). By [Mat77, Metatheorem 5.3], it can be done assuming the existence of a Mahlo cardinal, but the assumption seems stronger than necessary in this context.

But can we say anything about the existence of $\boldsymbol{\Pi}^1_1$ mad families in models larger than L? In the proof of Theorem 5.1.2 above, just as in our proof of Theorem 4.3.9, the $\boldsymbol{\Pi}^1_1$ definition of a mad family $A \subseteq L$ defines the same object A in any larger model V. However, in V, A may fail to be a mad family because V may contain new reals which are almost disjoint from all $a \in A$. So the question is: can we construct a $\boldsymbol{\Pi}^1_1$ mad family in L for which this does not happen? This leads to the following concept:

Definition 5.1.3. *Let $\mathbb{P}$ be a forcing partial order and A a mad family. A is said to be* $\mathbb{P}$-indestructible *if in the generic extension $V[G]$ by $\mathbb{P}$, A is still a mad family.*

Indestructibility of mad families has been widely researched, among others in [Kur01, Hru01, HGF03, BY05, Rag09]. For many standard partial orders

$\mathbb{P}$, combinatorial conditions on mad families have been isolated which ensure $\mathbb{P}$-indestructibility. One particular condition involves the strengthening of the definition of a mad family to a so-called *ω-mad* family (sometimes called *strongly mad* family). By Corollary 37, Corollary 53 and Theorem 65 of [Rag09], such ω-mad families are preserved in iterations of Cohen, Sacks and Miller forcing. On the other hand, by [KSZ08, Theorem 3.1], $\mathbf{\Pi}^1_1$ definable ω-mad families exist in L. Putting this together, we obtain the following result:

Fact 5.1.4. Con($\neg\mathsf{CH}$ + "there is a $\mathbf{\Pi}^1_1$ mad family").

Proof. Construct a $\mathbf{\Pi}^1_1$ ω-mad family A in L. Extend L via an $\aleph_2$-iteration of Cohen, Sacks or Miller forcing. Then in the extension $2^{\aleph_0} = \aleph_2$, A still has a $\mathbf{\Pi}^1_1$ definition (this is just like in the proof of Theorem 4.3.9), and is still maximal. □

If we use Cohen forcing in the above result, we obtain the stronger statement Con(cov($\mathcal{M}$) $> \aleph_1$+ "there is a $\mathbf{\Pi}^1_1$ mad family"), and also Con($\mathbf{\Delta}^1_2$(Baire)+ "there is a $\mathbf{\Pi}^1_1$ mad family").

But what can be said of forcing iterations that have stronger transcendence properties? In particular, what happens if the forcing adds dominating reals? Recall the cardinal invariant $\mathfrak{a}$, defined as the smallest size of an (infinite) mad family, and $\mathfrak{b}$, the smallest size of an unbounded family. It is well-known that $\mathfrak{b} \leq \mathfrak{a}$. In fact, the proof of this inequality tells us that if A is a mad family in V and $\mathbb{P}$ adds a dominating real, then in the generic extension $V[G]$ by $\mathbb{P}$, A is no longer a mad family—in other words, *there are no $\mathbb{P}$-indestructible mad families for forcings $\mathbb{P}$ which add a dominating real.*

This raises the following question: can we iterate a forcing that adds dominating reals, and still have a $\mathbf{\Pi}^1_1$ mad family, or at least a $\mathbf{\Sigma}^1_2$ mad family, in the extension? Another formulation of the same question is: is $\mathfrak{b} > \aleph_1$+ "there is a $\mathbf{\Sigma}^1_2$ mad family" consistent? Initially, one might think that the answer is negative since the method we used so far (i.e., constructing a definable mad family in L and preserving it) cannot possibly work here.

This question was recently asked by Friedman and Zdomskyy in [FZ10, Question 16], who proved the following:

Theorem 5.1.5 (Friedman & Zdomskyy)**.** Con($\mathfrak{b} > \aleph_1$ + "there is a $\mathbf{\Pi}^1_2$ ω-mad family").

This result is optimal for ω-mad families: if A were a $\mathbf{\Sigma}^1_2$ definable ω-mad family, then by Mansfield-Solovay (Theorem 1.3.14) it would either be a subset of L or would contain a perfect set. The former is false because $\aleph_1 < \mathfrak{b} \leq \mathfrak{a}$ and the latter is impossible because, by [Rag09, Corollary 38], an ω-mad family cannot contain a perfect set. However, it was not clear whether this result was also optimal for the more general case of a mad family instead of an ω-mad family.

We answer the question of Friedman and Zdomskyy positively by showing that it is consistent that $\mathfrak{b} > \aleph_1$ and there exists a $\mathbf{\Sigma}^1_2$ mad family.

To avoid the problem with dominating reals we need a somewhat new approach to *preservation*.

Definition 5.1.6.

1. *$A \subseteq [\omega]^\omega$ is called an* $\aleph_1$-Borel mad family *if $A = \bigcup_{\alpha<\aleph_1} A_\alpha$, where each A_α is a Borel a.d. family and A is a mad family.*

2. *Let $\mathbb{P}$ be a forcing partial order. An $\aleph_1$-Borel mad family A is called* $\mathbb{P}$-indestructible *if in the generic extension $V[G]$ by $\mathbb{P}$,*

$$A^{V[G]} := \bigcup_{\alpha<\aleph_1} A_\alpha^{V[G]}$$

is a mad family.

In fact, our families A will be unions of $\aleph_1$ *perfect sets* A_α. Since by $\mathbb{P}$-*preservation* we are now referring to the re-interpreted perfect sets $A_\alpha^{V[G]}$, a dominating real does not necessarily create a problem. Thus, the main focus of our work is the construction of an $\aleph_1$-Borel mad family in L which is preserved by an iteration of a forcing adding dominating reals.

Notice that in Definition 5.1.6, we could replace $\aleph_1$ by an arbitrary cardinal κ and define κ-Borel mad families analogously. A closer look at this allows us to isolate a new cardinal invariant.

Definition 5.1.7. *Let $\mathfrak{a}_B$* (the Borel almost-disjointness number) *be the least infinite cardinal κ such that there exists a sequence $\{A_\alpha \mid \alpha < \kappa\}$ of Borel a.d. sets whose union is a mad family.*

Notice that $\mathfrak{a}_B$ must be uncountable since a countable union of Borel sets is itself Borel, thus cannot be mad by Theorem 5.1.1. It is also clear that $\mathfrak{a}_B \leq \mathfrak{a}$. Moreover, if $\mathfrak{a}_B > \aleph_1$ then there are no $\mathbf{\Sigma}^1_2$ mad families, since a $\mathbf{\Sigma}^1_2$ set is a union of $\aleph_1$ Borel sets. Indeed, the cardinal invariant $\mathfrak{a}_B$ is related to the existence of $\mathbf{\Sigma}^1_2$ mad families in the same way as the covering number of the σ-ideal I was related to I-regularity in Section 2.3 (cf. Corollary 2.3.9 and the discussion after that). Also, the following is an unpublished result of Dilip Raghavan (private communication). For the definition of $\mathfrak{t}$, see Definition 1.2.28.

Theorem 5.1.8 (Raghavan)**.** *$\mathfrak{t} \leq \mathfrak{a}_B$.*

As a consequence, if $\mathfrak{t} > \aleph_1$ then there are no $\mathbf{\Sigma}^1_2$ mad families. In particular, this holds under Martin's Axiom.

Our main result shows that $\mathfrak{a}_B < \mathfrak{b}$ is consistent. To obtain this, we construct a model where $\mathfrak{a}_B = \aleph_1$ and $\mathfrak{b} = \aleph_2$, by using an $\aleph_2$-iteration of Hechler forcing starting with a model of CH. In the proof, we use an essential property of Hechler forcing—preservation of ω-splitting families—first established by Baumgartner and Dordal in [BD85]. This allows us to construct an $\aleph_1$-Borel mad family in the ground model which is preserved by the Hechler iteration.

For simplicity, we will first present the proof of this cardinal inequality. Afterwards, it will be an easy matter to modify the proof so that it yields the consistency of $\mathfrak{b} > \aleph_1$+"there is a $\mathbf{\Sigma}^1_2$ mad family" or of $\mathbf{\Sigma}^1_2(\mathbb{L})$+"there is a $\mathbf{\Sigma}^1_2$ mad family".

5.2 Preparing for the construction

To begin with, we do some preparatory work and lay the foundations necessary for the construction of a Hechler-indestructible mad family.

As the primary component, we will consider partitions of an infinite subset of ω into infinitely many infinite sets, each one indexed by a finite sequence $\sigma \in \omega^{<\omega}$. To be precise, let D be an infinite subset of ω, and let

$$P := \{P_\sigma \mid \sigma \in \omega^{<\omega}\}$$

be a disjoint partition of D into infinite sets. We will call D the *domain of* P, and the elements of each P_σ will be enumerated in order, and we will denote them by $P_\sigma = \{p_\sigma(0), p_\sigma(1), p_\sigma(2), \dots\}$.

The partition P leads to a natural homeomorphism. Consider $\varphi : \omega^{<\omega} \setminus \{\varnothing\} \longrightarrow [D]^{<\omega}$ defined by

$$\varphi(\sigma) := \{p_{\sigma\restriction n}(\sigma(n)) \mid n < |\sigma|\}$$

and its limit function $\Phi : \omega^\omega \longrightarrow [D]^\omega$ defined by

$$\Phi(f) := \{p_{f\restriction n}(f(n)) \mid n < \omega\}.$$

The function Φ is clearly injective and continuous. Also, we define the function $\psi : \omega^{<\omega} \setminus \{\varnothing\} \longrightarrow D$ as follows: if $\sigma \in \omega^{<\omega} \setminus \{\varnothing\}$ and $n := |\sigma| - 1$ then

$$\psi(\sigma) := p_{\sigma\restriction n}(\sigma(n)).$$

Note that ψ is a bijection between $\omega^{<\omega} \setminus \{\varnothing\}$ and D, and that $\varphi(\sigma) = \{\psi(\sigma\restriction n) \mid 1 \leq n \leq |\sigma|\}$ and $\Phi(f) = \{\psi(f\restriction n) \mid n < \omega\}$.

For a fixed partition P of D, define $A_P := \{\Phi(f) \mid f \in \omega^\omega\}$. This is an almost disjoint subfamily of D of size $2^{\aleph_0}$, which forms a perfect set in the natural topology on $[D]^\omega$, with Φ a homeomorphism between ω^ω and A_P. Any model of set theory containing the partition P can interpret the perfect set A_P according to its own reals. Sets of the form A_P will form the basic components of our indestructible mad family, where "indestructibility" will refer to the collection of the re-interpreted perfect sets A_P.

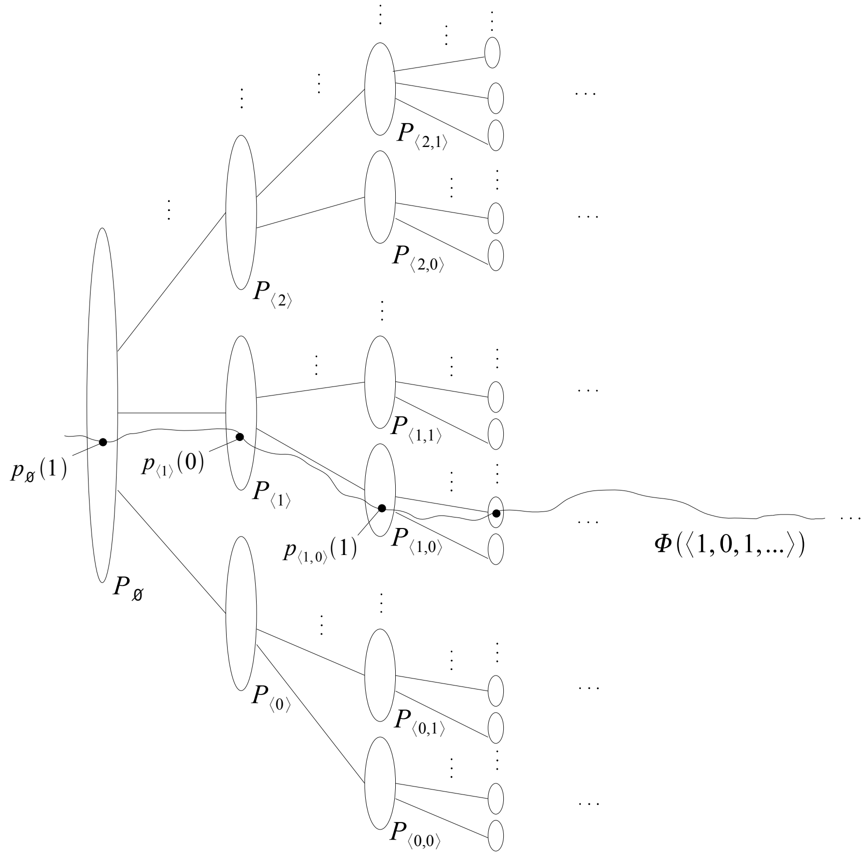


Figure 5.1: A partition P of D.

By Mathias's theorem (Theorem 5.1.1), A_P cannot be a mad family. Hence, there is an infinite set D' which is almost disjoint from every member of A_P. We can then partition D' into a new collection P' and repeat the same construction as above to form $A_{P'}$. Clearly, $A_P \cup A_{P'}$ is still an a.d. family, and we can continue the process. Since any countable union of closed a.d. families is still not maximal, we can even continue this process into the transfinite. In a model of CH, we could thus construct a mad family $\bigcup\{A_{P^\alpha} \mid \alpha < \aleph_1\}$ consisting of $\aleph_1$ perfect sets based on the partitions P^α.

Of itself, however, this construction is of little use because it does not tell us anything about the indestructibility in forcing extensions. Therefore we will consider another method of extending a given a.d. family A_P. This is a brief prelude to the actual construction.

For a partition P and $\sigma \in \omega^{<\omega} \setminus \{\varnothing\}$, define $\tilde{I}(\sigma) := \{\psi(\tau) \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$.

Equivalently, we have $\tilde{I}(\sigma) := \bigcup \Phi``[\sigma]$. For $m \in D$, define $I(m) := \tilde{I}(\psi^{-1}(m))$. Each $I(m)$ is an infinite subset of D, and we let $\mathcal{I}_P$ be the ideal generated by all such $I(m)$, i.e., a set $X \subseteq \omega$ is defined to be in $\mathcal{I}_P$ if there are finitely many $m_0, \ldots, m_k$ such that $X \subseteq^* \bigcup_{i=0}^{k} I(m_i)$. It is clear that $\mathcal{I}_P$ is a proper ideal, i.e., that D (and hence ω) is not in the ideal.

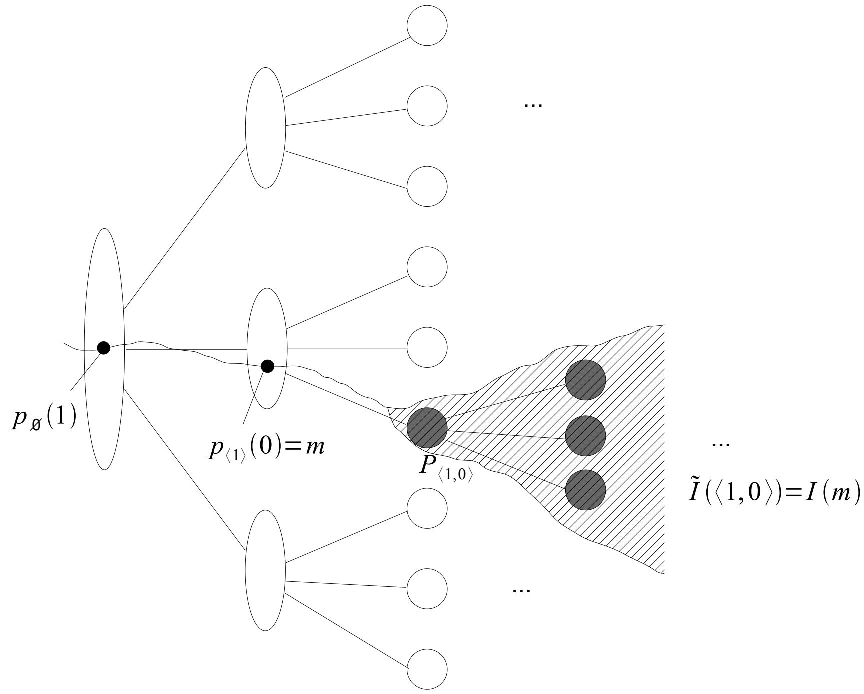


Figure 5.2: The ideal $\tilde{I}(\sigma) = I(m)$

Now, suppose M is a countable model of ZFC, and $P \in M$ is a partition of some D, as defined above. Using finite conditions (i.e., Cohen forcing) in M, we can add a new partition $C = \{C_\sigma \mid \sigma \in \omega^{<\omega}\}$ of some domain D_C, such that for all σ,

$$C_\sigma \cap \bigcup_{m<|\sigma|} I(m) = \varnothing.$$

To be precise, in M, define the partial order of functions $p : \omega^{<\omega} \times \omega \to 2$ with finite domain, ordered by extension, and additionally satisfying the following conditions:

1. for all σ, n, if $p(\sigma, n) = 1$ then $n \notin \bigcup_{m<|\sigma|} I(m)$, and
2. for all $\sigma \neq \tau$, there is no n such that $p(\sigma, n) = p(\tau, n) = 1$.

Once in the extension, define $C_\sigma := \{n \in \omega \mid (\bigcup G)(\sigma, n) = 1\}$, where G is generic over M. Using standard genericity arguments for Cohen forcing, plus the fact that the ideal $\mathcal{I}_P$ is proper, it is easy to see that $C := \{C_\sigma \mid \sigma \in \omega^{<\omega}\}$ forms a partition and satisfies the requirement.

We can now use C to define a new perfect a.d. family A_C. Let the functions $\Phi_C, \varphi_C, \psi_C$, the sets $I_C(m)$ and the ideal $\mathcal{I}_C$ be defined analogously using the new partition C. Moreover, let $\mathcal{I}_{P+C}$ denote the ideal generated by the $I(m)$'s as well as the $I_C(m)$'s, i.e., a set $X \subseteq \omega$ is in the ideal if there are $\ell_0, \dots, \ell_k$ and $m_0, \dots, m_r$ such that $X \subseteq^* I(\ell_0) \cup \dots \cup I(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$.

We claim the following:

Lemma 5.2.1.

1. *$A_P \cup A_C$ is still almost disjoint, and*

2. *$\mathcal{I}_{P+C}$ is still a proper ideal on $D \cup D_C$.*

Proof.

1. Let $f, h \in \omega^\omega$, and we must show that $\Phi(f) \cap \Phi_C(h)$ is finite. Let m be any member of $\Phi(f)$, and let $\sigma \subseteq h$ be sufficiently long so that $m < |\sigma|$. Then for all τ with $\sigma \subseteq \tau \subseteq h$, C_τ is disjoint from $I(m)$, hence $\Phi_C(h)$ is a.d. from $I(m)$. On the other hand, $\Phi(f)$ is almost contained in $I(m)$. Therefore $\Phi(f) \cap \Phi_C(h)$ is finite.

2. Consider a finite union $I(\ell_0) \cup \dots \cup I(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$. By standard genericity arguments, the Cohen real $C_\varnothing$ splits all reals in M. Since the $I(\ell_i)$'s are defined from P, it is clear that $\omega \setminus I(\ell_0) \cup \dots \cup I(\ell_k) \in M$. Therefore, there are infinitely many numbers in $C_\varnothing \setminus I(\ell_0) \cup \dots \cup I(\ell_k)$, and from these, only finitely many can be in $I_C(m_0) \cup \dots \cup I_C(m_r)$. □

It is not hard to see that this way of extending A_P can also be used to extend any countable collection of a.d. families A_{P^i} contained in a countable model M—in that case, C_σ is defined so that it is disjoint from $\bigcup_{m,i<|\sigma|} I_i(m)$. This will be the main idea of our construction, but to guarantee that it is indestructible by an iteration of Hechler forcing, we must adjust and fine-tune this method.

5.3 The Hechler-indestructible mad family

Recall the Hechler forcing partial order $\mathbb{D}$ consisting of conditions of the form $(s, f) \in \omega^{<\omega} \times \omega^\omega$ with $s \subseteq f$, ordered by $(s', f') \leq (s, f)$ iff $s \subseteq s'$ and $f' \geq f$. Hechler forcing satisfies the c.c.c., generically adds a dominating real, and moreover has many useful preservation properties. The one we will rely on in this proof is *preservation of ω-splitting families*. We state it in a slightly stronger form.

Fact 5.3.1. *Let $\dot{a}$ be a $\mathbb{D}$-name for an element of $[\omega]^\omega$. Then there exist $\{a_i \mid i < \omega\}$, explicitly definable from the name $\dot{a}$, such that if c splits all a_i, then $\Vdash_{\mathbb{D}}$ "$\check{c}$ splits $\dot{a}$". This is still true in any iteration of $\mathbb{D}$ with finite support.*

For a proof, see [Bre09, Main Lemma 3.8]. This property of Hechler forcing can be used to show that if $X \subseteq [\omega]^\omega$ is an ω-splitting family, then it will remain so in the Hechler extension, and even in the iterated Hechler extension. We will use it for a slightly different purpose, a kind of "properness"-version of preservation of ω-splitting families.

Fact 5.3.2. *Let M be a countable elementary submodel of some sufficiently large structure $\mathcal{H}_\theta$. Suppose $c \in [\omega]^\omega$ splits all reals in M. Then in the (iterated) Hechler extension $V[G]$, c splits all reals in $M[G]$.*

To see why this follows, note that if $a \in M[G]$ then it has a name $\dot{a} \in M$, so the a_i from Fact 5.3.1 will also be in M because they are definable from $\dot{a}$. Therefore, c will split all a_i, and hence $\Vdash_{\mathbb{D}}$ "$\check{c}$ splits $\dot{a}$" (and the same is still true in an iteration of $\mathbb{D}$ with finite support).

We can now state our main result.

Theorem 5.3.3 (Main Theorem)**.** *In the $\aleph_2$-iteration of Hechler forcing (with finite support) starting from a model of CH, $\mathfrak{b} = \aleph_2$ while $\mathfrak{a}_B = \aleph_1$.*

To prove this theorem, we construct an $\aleph_1$-Borel mad family in the ground model V satisfying CH, i.e., a mad family $A = \bigcup_{\alpha<\aleph_1} A_\alpha$, where each A_α is a Borel (in fact perfect) a.d. set. Then we will show that in $V[G_{\aleph_2}]$, the iteration of Hechler forcing of length $\aleph_2$, $A^{V[G_{\aleph_2}]} := \bigcup_{\alpha<\aleph_1} A_\alpha^{V[G_{\aleph_2}]}$ is still maximal.

We will construct A by induction on $\alpha < \aleph_1$, using ideas described in the previous section. Most of the effort will go into proving the main technical lemma concerned with the induction step, which we now state.

Lemma 5.3.4 (Main Lemma)**.** *Let M be a countable elementary submodel of a sufficiently large structure $\mathcal{H}_\theta$. Let $\{P^i \mid i \in \omega\}$ be a sequence of partitions contained in M. Denote by $\varphi_i, \Phi_i, \psi_i, I_i$ and A_i all the objects derived from the partition P^i, and let $\mathcal{I}_{\overline{P}}$ be the ideal generated by the $I_i(m)$'s, for all $i \in \omega$ (i.e., $\mathcal{I}_{\overline{P}} := \mathcal{I}_{P^0+P^1+P^2+\dots}$). Assume that this ideal is proper, i.e., ω is not in the ideal.*

Then there exists a partition $C := \{C_\sigma \mid \sigma \in \omega^{<\omega}\}$ of some domain D_C, satisfying the following properties:

1. *for every $f, h \in \omega^\omega$ and every $i \in \omega$: $\Phi_i(f)$ and $\Phi_C(h)$ are almost disjoint,*
2. *the ideal $\mathcal{I}_{\overline{P}+C}$ is proper,*
3. *for every $Y \in M$, if Y is almost disjoint from $\Phi_i(f)$ for every f and every $i \in \omega$, then there exists an $h \in \omega^\omega$ such that $\Phi_C(h) \subseteq Y$, and*

3 if G is generic for any iteration of Hechler forcing, then for every $Y \in M[G]$, if Y is almost disjoint from $\Phi_i(f)$ for every $f \in V[G]$ and every $i \in \omega$, then there exists an $h \in V[G]$ such that $\Phi_C(h) \subseteq Y$.*

In the proof of the lemma, conditions 3 and 3* are proved analogously, where the argument for 3* follows from the argument for 3 using Fact 5.3.2. We will prove condition 3 in detail, and then explain how to modify the construction to yield a proof of 3* (the details will then be left out). Most of the work will go into defining a combinatorially involved construction so that condition 3 is satisfied.

Note that if we add C generically using finite conditions, as we did in the previous section, we can easily satisfy conditions 1 and 2. Moreover, recall that Cohen reals are *splitting* over the ground model. To be precise, in the construction of the previous section, each C_σ was a Cohen subset of $\omega \setminus \bigcup_{m,i<|\sigma|} I_i(m)$, and thus split every set $Y \in M$ which had infinite intersection with $\omega \setminus \bigcup_{m,i<|\sigma|} I_i(m)$. This provides an idea for proving condition 3: if we could guarantee that the set $Y \in M$ has infinite intersection with $\omega \setminus \bigcup_{m,i<|\sigma|} I_i(m)$ for arbitrarily large σ, then we could inductively pick $h(n)$ so as to satisfy $\Phi_C(h) \subseteq Y$, using the splitting property of every $C_{h\restriction n}$. But, in general, this is clearly impossible: if $Y \subseteq I_i(m)$ for some m, then eventually σ will be long enough so that $i, m < |\sigma|$, and then Y will not have infinite intersection with the required set and we will not be able to use the splitting property of C_σ.

So we need to modify the construction, but the problem is that we must still guarantee condition 1. This turns out to be more tricky than may seem at first glance. To appreciate the difficulty, suppose $Y \in M$ is an infinite set which is "very close" to some $\Phi_i(f)$ but still almost disjoint from it, e.g., $Y = \{p^i_{f\restriction n}(f(n)+1) \mid n \in \omega\}$. If condition 1 of the lemma is to be satisfied, then $\Phi_C(h)$ must be almost disjoint from $\Phi_i(f)$ for any h. But how can this be achieved without the side-effect that $\Phi_C(h)$ is also almost disjoint from Y, making condition 3 impossible to satisfy? To alleviate this apparent tension between conditions 1 and 3 we will use a careful construction, such that, as σ grows longer, C_σ is disjoint from more and more sets of the form $I_i(m)$, but a special selection of those is excluded from this process. This will allow us, inductively, to select h so that every $C_{h\restriction n}$ splits Y, while making sure that condition 1 is not violated.

Proof of main lemma. The proof will proceed in three steps. First, we will give an inductive definition of C as we wish to have it, then we show that such a construction actually exists, and finally we show that it satisfies conditions 1–3* of the lemma.

Part 1. The sets C_σ are defined by induction on the length of σ. Moreover, to each σ and each $j < |\sigma|$, we associate another sequence $\tau_j(\sigma)$, called a *marker*, satisfying $|\tau_j(\sigma)| = |\sigma|$ and $\sigma \subseteq \sigma' \implies \tau_j(\sigma) \subseteq \tau_j(\sigma')$. This gives rise to the limit function $\bar{\tau}_j : \omega^\omega \to \omega^\omega$ defined by $\bar{\tau}_j(f) := \bigcup_{n>j} \tau_j(f\restriction n)$. The following convention will be useful here: variables $i^0, i^1, \ldots$ denote the digits of σ whereas $i^0_j, i^1_j, \ldots$ denote those of the markers $\tau_j(\sigma)$, and similarly for f versus $\bar{\tau}_j(f)$.

The variable Θ will be used to denote elements of $(\omega^{<\omega})^{<\omega}$, where for all $i < |\Theta|$, $|\Theta(i)| = |\Theta|$.

We describe the first few steps of the inductive construction explicitly.

- $(k = 0)$
  - $C_\varnothing$ is the disjoint union of infinitely many infinite sets $C_\varnothing^{\langle\langle i^0_0\rangle\rangle}$, where i^0_0 is an arbitrary integer. Each $C_\varnothing^{\langle\langle i^0_0\rangle\rangle}$ splits every $Y \in M$.
  - Require that for every i^0_0, $\psi_0(\langle i^0_0\rangle) \notin C_\varnothing^{\langle\langle i^0_0\rangle\rangle}$ (this is called an *excluded point*).
  - For every $i^0 \in \omega$, define

    $$\tau_0(\langle i^0\rangle) := \langle i^0_0\rangle \;:\Longleftrightarrow\; c_\varnothing(i^0) \in C_\varnothing^{\langle\langle i^0_0\rangle\rangle}$$

    (here $c_\varnothing(i^0)$ refers to the enumeration of $C_\varnothing$ in increasing order, as defined in Section 5.2).
- $(k = 1)$ The markers $\tau_0(\langle i^0\rangle) = \langle i^0_0\rangle$ have already been defined in the previous step.
  - Let $\sigma := \langle i^0\rangle$. Call i^1_0 a *forbidden value* if $\psi_0(\langle i^0_0, i^1_0\rangle) = \psi_C(\langle i^0\rangle)$.
  - $C_{\langle i^0\rangle}$ is the disjoint union of infinitely many infinite sets $C_{\langle i^0\rangle}^{\langle\langle i^0_0, i^1_0\rangle, \langle i^0_1, i^1_1\rangle\rangle}$, where $\tau_0(\langle i^0\rangle) = \langle i^0_0\rangle$, i^1_0 is any natural number which is not a forbidden value, and i^0_1, i^1_1 are arbitrary. Define

    $$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < 1 \text{ and } \psi_j^{-1}(\ell) \perp \langle i^0_j\rangle\}.$$

    Each $C_{\langle i^0\rangle}^{\langle\langle i^0_0, i^1_0\rangle, \langle i^0_1, i^1_1\rangle\rangle}$ is disjoint from X_σ and splits every Y which has infinite intersection with $\omega \setminus X_\sigma$.
  - Require that $\psi_0(\langle i^0_0\rangle), \psi_0(\langle i^0_0, i^1_0\rangle), \psi_1(\langle i^0_1\rangle), \psi_1(\langle i^0_1, i^1_1\rangle) \notin C_{\langle i^0\rangle}^{\langle\langle i^0_0, i^1_0\rangle, \langle i^0_1, i^1_1\rangle\rangle}$, or, in other words, $\varphi_j(\langle i^0_j, i^1_j\rangle) \cap C_{\langle i^0\rangle}^{\langle\langle i^0_0, i^1_0\rangle, \langle i^0_1, i^1_1\rangle\rangle} = \varnothing$, for $j = 0, 1$ (*excluded points*).
  - For every $i^1 \in \omega$, define

    $$\left.\begin{array}{c} \tau_0(\langle i^0, i^1\rangle) := \langle i^0_0, i^1_0\rangle \\ \text{and} \\ \tau_1(\langle i^0, i^1\rangle) := \langle i^0_1, i^1_1\rangle \end{array}\right\} \;:\Longleftrightarrow\; c_{\langle i^0\rangle}(i^1) \in C_{\langle i^0\rangle}^{\langle\langle i^0_0, i^1_0\rangle, \langle i^0_1, i^1_1\rangle\rangle}.$$

- ($k = 2$) The markers $\tau_0(\langle i^0, i^1\rangle) = \langle i^0_0, i^1_0\rangle$ and $\tau_1(\langle i^0, i^1\rangle = \langle i^0_1, i^1_1\rangle)$ have already been defined in the first two steps.
  - Let $\sigma := \langle i^0, i^1\rangle$. Call i^2_0 a *forbidden value* if $\psi_0(\langle i^0_0, i^1_0, i^2_0\rangle) \in \{\psi_C(\langle i^0\rangle), \psi_C(\langle i^0, i^1\rangle)\}$ and call i^2_1 a *forbidden value* if $\psi_1(\langle i^0_1, i^1_1, i^2_1\rangle) = \psi_C(\langle i^0, i^1\rangle)$.
  - $C_{\langle i^0,i^1\rangle}$ is the disjoint union of infinitely many infinite sets
  $$C_{\langle i^0,i^1\rangle}^{\langle\langle i^0_0,i^1_0,i^2_0\rangle,\langle i^0_1,i^1_1,i^2_1\rangle,\langle i^0_2,i^1_2,i^2_2\rangle\rangle}$$
  where $\tau_0(\langle i^0, i^1\rangle) = \langle i^0_0, i^1_0\rangle$, $\tau_1(\langle i^0, i^1\rangle) = \langle i^0_1, i^1_1\rangle$, i^2_0 and i^2_1 are any natural number which are not forbidden values, and i^0_2, i^1_2, i^2_2 are arbitrary. Define
  $$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < 2 \text{ and } \psi_j^{-1}(\ell) \perp \langle i^0_j, i^1_j\rangle\}.$$
  Each $C_{\langle i^0,i^1\rangle}^{\langle\langle i^0_0,i^1_0,i^2_0\rangle,\langle i^0_1,i^1_1,i^2_1\rangle,\langle i^0_2,i^1_2,i^2_2\rangle\rangle}$ is disjoint from X_σ and splits every Y which has infinite intersection with $\omega \setminus X_\sigma$.
  - Require that $\varphi_j(\langle i^0_j, i^1_j, i^2_j\rangle) \cap C_{\langle i^0,i^1\rangle}^{\langle\langle i^0_0,i^1_0,i^2_0\rangle,\langle i^0_1,i^1_1,i^2_1\rangle,\langle i^0_2,i^1_2,i^2_2\rangle\rangle} = \varnothing$, for $j = 0, 1, 2$ (*excluded points*).
  - For every $i^2 \in \omega$, define
  $$\left.\begin{array}{c}\tau_0(\langle i^0, i^1, i^2\rangle) := \langle i^0_0, i^1_0, i^2_0\rangle\\ \text{and}\\ \tau_1(\langle i^0, i^1, i^2\rangle) := \langle i^0_1, i^1_1, i^2_1\rangle\\ \text{and}\\ \tau_2(\langle i^0, i^1, i^2\rangle) := \langle i^0_2, i^1_2, i^2_2\rangle\end{array}\right\} :\Longleftrightarrow \begin{array}{l} c_{\langle i^0,i^1\rangle}(i^2) \in \\ \\ C_{\langle i^0,i^1\rangle}^{\langle\langle i^0_0,i^1_0,i^2_0\rangle,\langle i^0_1,i^1_1,i^2_1\rangle,\langle i^0_2,i^1_2,i^2_2\rangle\rangle}.\end{array}$$

This construction is continued in a similar fashion. The general inductive step looks as follows:

- (any k) The markers $\tau_j(\sigma)$ have been defined for all σ with $|\sigma| = k$ and $j < k$.
  - Fix σ such that $|\sigma| = k$. For $j < k$, call i^k_j a *forbidden value* if $\psi_j(\tau_j(\sigma)^\frown \langle i^k_j\rangle) \in \{\psi_C(\sigma\restriction m) \mid j < m \leq k\}$.
  - C_σ is the disjoint union of infinitely many sets C^Θ_σ, where
  $$\Theta := \left\langle \tau_0(\sigma)^\frown \left\langle i^k_0\right\rangle, \ldots, \tau_{k-1}(\sigma)^\frown \left\langle i^k_{k-1}\right\rangle, \left\langle i^0_k, \ldots, i^k_k\right\rangle\right\rangle,$$
  with i^k_j not forbidden and $i^0_k, \ldots, i^k_k$ arbitrary. Define
  $$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < k \text{ and } \psi_j^{-1}(\ell) \perp \tau_j(\sigma)\}.$$
  Each C^Θ_σ is disjoint from X_σ and splits all $Y \in M$ which have infinite intersection with $\omega \setminus X_\sigma$.

– Require that $\varphi_j(\tau_j(\sigma)^\frown \langle i^k_j \rangle) \cap C^\Theta_\sigma = \varnothing$ for all $j < k$ (*excluded points*).
– For every i^k, define

$$\left.\begin{array}{c} \forall j < k \ \left[\tau_j(\sigma^\frown \langle i^k \rangle) := \tau_j(\sigma)^\frown \langle i^k_j \rangle\right] \\ \text{and} \\ \tau_k(\sigma^\frown \langle i^k \rangle) := \langle i^k_0, \dots, i^k_k \rangle \end{array}\right\} :\Longleftrightarrow c_\sigma(i^k) \in C^\Theta_\sigma .$$

Thus we have completed the inductive definition of C.

Part 2. We show that the construction of C, as described above, does indeed exist. Since this time we don't just want every C_σ to be splitting, but every C^Θ_σ considered in the construction, we must add finite conditions in such a way that each C^Θ_σ is essentially a Cohen real over M. At first glance, a potential difficulty seems to arise from the fact that, in order to construct C_σ, we need to know the values of $\tau_j(\sigma')$ for $\sigma' \subseteq \sigma$, and these values are only known when the Cohen real has been added. However, we can avoid this difficulty by first adding C^Θ_σ for *all possible* combinations of σ and Θ, each one being a Cohen subset of the relevant set. Afterwards, we can prune the tree to remove many of the C^Θ_σ's and leave only the ones that correspond to the construction described above.

To be precise, consider partial functions p with $\mathrm{dom}(p)$ being a finite subset of

$$\{(\sigma, \Theta, n) \in \omega^{<\omega} \times (\omega^{<\omega})^{<\omega} \times \omega \mid |\Theta| = |\sigma| + 1\}$$

and $\mathrm{ran}(p) = 2$, ordered by extension, and satisfying the following conditions:

1. For every σ, Θ and n, if $p(\sigma, \Theta, n) = 1$ then $n \notin X_{\sigma,\Theta}$, where

$$X_{\sigma,\Theta} := \bigcup \{I_j(m) \mid j, m < |\sigma| \text{ and } \psi_j^{-1}(\ell) \perp (\mathrm{pr}_j(\Theta) \restriction |\Theta| - 1)\}, \text{ and}$$

2. for all $\sigma, \sigma', \Theta, \Theta'$ such that $(\sigma, \Theta) \neq (\sigma', \Theta')$, there is no n such that $p(\sigma, \Theta, n) = p(\sigma', \Theta', n) = 1$.

Let G be the M-generic filter for this partial order, and in $M[G]$ define $C^\Theta_\sigma := \{n \in \omega \mid (\bigcup G)(\sigma, \Theta, n) = 1\}$. Genericity arguments for Cohen forcing show that all the C^Θ_σ are pairwise disjoint, and that every C^Θ_σ is disjoint from $X_{\sigma,\Theta}$ and splits every $Y \in M$ which has infinite intersection with $\omega \setminus X_{\sigma,\Theta}$.

Now, by induction on the length of σ, we can prune the tree given by the C^Θ_σ's and define the markers $\tau_j(\sigma)$ and the *forbidden values* accordingly. To be more precise, let σ be of length k and suppose that $C_{\sigma \restriction j}$ is already known for $j < k$. Since the values of $\tau_j(\sigma)$ for $j < k$ are then also known, we can compute the *forbidden values* at this step. Then, we throw away all C^Θ_σ except those where Θ is compatible with the already determined sequence of markers $\tau_j(\sigma)$ and the *forbidden values*, i.e., we keep only those C^Θ_σ where $\mathrm{pr}_j(\Theta)$ is of the form $\tau_j(\sigma)^\frown \langle i^k_j \rangle$ and i^k_j is not forbidden. After that, we still need to remove the *excluded points* from each relevant C_σ^Θ. Since this only requires changing finitely many elements, this does not affect the property of C_σ^Θ being a Cohen real.

Now C_σ can be defined as the union of the C_σ^Θ that we left behind and removed *excluded points* from. This allows us to extend τ_j and continue pruning the next levels. It is clear that in this manner we can achieve precisely the construction described above.

Part 3. Finally we show that C satisfies conditions 1–3.

1. Let $f, h \in \omega^\omega$ and $j \in \omega$ be fixed. We must prove that $\Phi_j(f) \cap \Phi_C(h)$ is finite. There are now two methods for proving this. If f does not happen to be $\bar{\tau}_j(h)$, we can use an argument similar to Lemma 5.2.1. Otherwise, we will rely on the *excluded points* and the *forbidden values*.

   Case 1: $f \neq \bar{\tau}_j(h)$. Let $\sigma \subseteq h$ and $\tau \subseteq f$ be long enough so that $\tau_j(\sigma) \perp \tau$. Let $\ell := \psi_j(\tau)$. Then clearly $\Phi_j(f) \subseteq I_j(\ell)$. Moreover, for any σ' such that $\sigma \subseteq \sigma' \subseteq h$ and $|\sigma'| > j, \ell$, we know that $\psi_j^{-1}(\ell) \perp \tau_j(\sigma')$, so, by construction, we know that $C_{\sigma'}^\Theta \cap I_j(\ell) = \varnothing$. This implies that $\Phi_C(h) \cap \Phi_j(f)$ is at most finite.

   Case 2: $f = \bar{\tau}_j(h)$. Ignore the first j values of $\Phi_j(f)$, and let $\sigma := h{\restriction}(k+1)$, for $k > j$. Let $\tau := f{\restriction}(m+1)$ for any m. Clearly, it is sufficient to show that $\psi_C(\sigma) \neq \psi_j(\tau)$.

   Case (a) : $m \leq k$. Then at stage k of the construction, $\psi_j(\tau)$ is an *excluded point* of $C_{\sigma\restriction k}^{\langle \dots, \tau_j(\sigma), \dots \rangle}$. But $\psi_C(\sigma) \in C_{\sigma\restriction k}^{\langle \dots, \tau_j(\sigma), \dots \rangle}$, so indeed $\psi_C(\sigma) \neq \psi_j(\tau)$.

   Case (b) : $k < m$. Let $i_j^m := \tau(m)$. Then at stage m of the construction, i_j^m cannot be a *forbidden value*. Then, by definition, $\psi_j(\tau) \neq \psi_C(h{\restriction}r)$ for any r with $j < r \leq m$, in particular for $r = k+1$. Therefore $\psi_j(\tau) \neq \psi_C(\sigma)$.

2. To show that $\mathcal{I}_{\overline{P}+C}$ is proper, consider any finite union $I_{j_0}(\ell_0) \cup \dots \cup I_{j_k}(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$. Let $Z := I_{j_0}(\ell_0) \cup \dots \cup I_{j_k}(\ell_k)$, and note that Z is in M, since it is constructed from partitions contained in M. Recall that, by construction, every $C_\varnothing^\Theta$ splits every real in M, so in particular, it splits $\omega \setminus Z$. Therefore, there are infinitely many elements in $C_\varnothing^\Theta \setminus Z$. From those, only finitely many can be in $I_C(m_0) \cup \dots \cup I_C(m_r)$. Hence, infinitely many elements are not in $I_{j_0}(\ell_0) \cup \dots \cup I_{j_k}(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$.

3. This is the essence of the proof, and the main reason for setting up the construction as we have done it. Suppose Y is an infinite subset of ω in M, and $Y \cap \Phi_j(g)$ is finite for all j and all $g \in \omega^\omega$. The goal is to construct an h such that $\Phi_C(h) \subseteq Y$.

First, we build functions $g_j \in \omega^\omega$, for every j, and a sequence $Y \supseteq Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots$ of infinite sets in M, making sure that for every j the following condition is satisfied:

$$(*)_j : \quad Y_j \cap \tilde{I}_j(\rho) \text{ is finite for every } \rho \text{ incompatible with } g_j.$$

Start by constructing $g_0 := \langle i_0^0, i_0^1, i_0^2, \dots \rangle$, taking care of the partition P^0. Consider two cases: (a) there exists an i_0^0 such that $Y \cap \tilde{I}_0(\langle i_0^0 \rangle)$ is infinite, and (b) $Y \cap \tilde{I}_0(\langle i_0^0 \rangle)$ is finite for any choice of i_0^0. If case (b) occurs, pick all the $i_0^0, i_0^1, \dots$ arbitrarily (they are irrelevant), and set $K_0 := 0$ (this encodes the fact that case (b) occurred at the 0-th step). If case (a) occurs, fix this i_0^0 and continue. Consider two cases: (a) there exists an i_0^1 such that $Y \cap \tilde{I}_0(\langle i_0^0, i_0^1 \rangle)$ is infinite, and (b) $Y \cap \tilde{I}_0(\langle i_0^0, i_0^1 \rangle)$ is finite for any choice of i_0^1. If case (b) occurs, pick all the remaining $i_0^1, i_0^2, \dots$ arbitrarily, and set $K_0 := 1$; if case (a) occurs, fix this i_0^1 and continue. Go on in a similar fashion: if $g_0 \restriction k$ is defined, there are two cases: (a) there exists an i_0^k such that $Y \cap \tilde{I}_0(g_0 \restriction k ^\frown \langle i_0^k \rangle)$ is infinite, and (b) $Y \cap \tilde{I}_0(g_0 \restriction k ^\frown \langle i_0^k \rangle)$ is finite for any choice of i_0^k. In case (b) pick $i_0^k, i_0^{k+1}, \dots$ arbitrarily and set $K_0 := k$; in case (a) fix i_0^k and continue.

This way we define $g_0 = \langle i_0^0, i_0^1, i_0^2, \dots \rangle$. If case (b) occurred at any finite stage k, we also have $K_0 := k$, otherwise K_0 is undefined. Now, we want to shrink Y to a new infinite set Y_0, in such a way that condition $(*)_0$ is satisfied. There are two possibilities.

(i) If case (b) occurred at some stage, then let $Y_0 := Y \cap \tilde{I}_0(g_0 \restriction K_0)$ (or $Y_0 = Y$ if $K_0 = 0$). By construction, Y_0 is infinite, and it is not hard to check that Y_0 has finite intersection with $\tilde{I}_0(\rho)$ whenever ρ is incompatible with g_0 (in fact, this holds for all ρ except $\rho \subseteq g_0 \restriction K_0$). Therefore, condition $(*)_0$ is satisfied by Y_0.

(ii) If case (a) occurred throughout the construction, then notice the following: for every n, there is a $y \in Y$ such that $g_0 \restriction n \subseteq \psi_0^{-1}(y)$. So, for each n, pick one such y_n, and let $Y_0 := \{y_0, y_1, y_2, \dots\}$. Clearly Y_0 is an infinite subset of Y. Moreover, if ρ is any sequence incompatible with g_0, then, letting n be least such that $\rho(n) \neq g_0(n)$, we see that $\tilde{I}_0(\rho)$ can contain at most n members of Y (because for any y_m for $m > n$ we have $g_0 \restriction m \subseteq \psi_0^{-1}(y_m)$ and hence $y_m \notin \tilde{I}_0(\rho)$). Therefore condition $(*)_0$ is satisfied by Y_0.

Note that, in either case, Y_0 is explicitly constructed using information encoded in the partition $P_0 \in M$, so Y_0 is also in M.

Now we continue with the construction of $g_1 := \langle i_1^0, i_1^1, i_1^2, \dots \rangle$ using Y_0 instead of Y, taking care of the partition P^1 instead of P^0. Consider two cases: (a) there exists an i_1^0 such that $Y_0 \cap \tilde{I}_1(\langle i_0^0 \rangle)$ is infinite, and (b) $Y_0 \cap \tilde{I}_1(\langle i_1^0 \rangle)$ is finite for any choice of i_1^0. If case (b) occurs, pick all the $i_1^0, i_1^1, \dots$ arbitrarily and set $K_1 := 0$; if case (a) occurs, fix this i_1^0 and continue, *etc.* After we have defined g_1, let Y_1 be an infinite subset of Y_0, constructed in the same way as Y_0 was constructed out of Y, i.e., so that condition $(*)_1$ is satisfied, and again $Y_1 \in M$.

It is clear that this method can be continued, so at each step j we deal with the partition P^j, define $g_j = \langle i_j^0, i_j^1, i_k^2, \dots \rangle$ and an infinite set $Y_j \in M$, following the same procedure, and make sure that condition $(*)_j$ is satisfied.

Now, we can define the function $h := \langle i^0, i^1, i^2, \dots \rangle$ so that the following three conditions are satisfied for every k:

1. $c_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) \in C^{\langle \langle i_0^0,\dots,i_0^k \rangle,\dots,\langle i_k^0,\dots,i_k^k \rangle \rangle}_{\langle i^0,\dots,i^{k-1}\rangle}$,
2. $c_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) \in Y_k$, and
3. $c_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) \notin \bigcup_{j\le k} \Phi_j(g_j)$.

The first condition inductively guarantees that for every k and $j < k$, $\tau_j(\langle i^0, \dots, i^k \rangle) = \langle i_j^0, \dots, i_j^k \rangle$. The third condition is crucial: it is to ensure that we will not run into trouble with *forbidden values* i_j^m for $j \le k < m$ in the future. This is the only place in the argument where the assumption that Y is a.d. from all $\Phi_j(g)$ is needed.

To see that numbers i^k satisfying conditions 1–3 can indeed be chosen, proceed inductively. Suppose $\langle i^0, \dots, i^{k-1} \rangle$ has already been defined. Condition 3 inductively implies that i_j^k for $j < k$ are not *forbidden values*, therefore we can consider the set $C^{\langle \langle i_0^0,\dots,i_0^k \rangle,\dots,\langle i_k^0,\dots,i_k^k \rangle \rangle}_{\langle i^0,\dots,i^{k-1}\rangle}$. Recall that this set was defined so that it splits every $Y' \in M$ which has infinite intersection with $\omega \setminus X_{\langle i^0,\dots,i^{k-1}\rangle}$, where

$$X_{\langle i^0,\dots,i^{k-1}\rangle} = \bigcup \{ I_j(\ell) \mid j, \ell < k \text{ and } \psi_j^{-1}(\ell) \perp \langle i_j^0, \dots, i_j^{k-1} \rangle \}.$$

But condition $(*)_k$ implies that Y_k is almost disjoint from any $I_j(\ell)$ with $\psi_j^{-1}(\ell) \perp g_j$. In particular, it is almost disjoint from any $I_j(\ell)$ with $\psi_j^{-1}(\ell) \perp \langle i_j^0, \dots, i_j^{k-1} \rangle$. But then Y_k must be almost disjoint from a finite union of such sets, and therefore, have infinite intersection with $\omega \setminus X_{\langle i^0,\dots,i^{k-1}\rangle}$.

Therefore $C^{\langle \langle i_0^0,\dots,i_0^k \rangle,\dots,\langle i_k^0,\dots,i_k^k \rangle \rangle}_{\langle i^0,\dots,i^{k-1}\rangle}$ splits Y_k, so there are infinitely many numbers n in the set $Y_k \cap C^{\langle \langle i_0^0,\dots,i_0^k \rangle,\dots,\langle i_k^0,\dots,i_k^k \rangle \rangle}_{\langle i^0,\dots,i^{k-1}\rangle}$. Now we recall the fact that, by assumption, $|Y_k \cap \Phi_j(g_j)| < \omega$ for every $j \leq k$. Therefore, it is possible to pick an n even so that condition 3 (at induction step k) is also satisfied. So we pick such an n and let i^k be such that $C_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) = n$. This completes the induction step.

We thus construct the entire function $h = \langle i^0, i^1, \dots \rangle$. The second condition implies that $\Phi_C(h) \subseteq Y$, as had to be shown.

3.* Let G be generic for any iteration of Hechler forcing. It is clear that condition 3* differs from condition 3 only in the sense that it says something about the model $M[G]$ (with respect to $V[G]$) instead of M (with respect to V). For this, we note that the entire proof of 3 above can be repeated, using the preservation property of Hechler forcing, Fact 5.3.2. It is clear that in the statement of Fact 5.3.2 we can replace splitting in the space $[\omega]^\omega$ by splitting in the space $[D]^\omega$ for some infinite subset of ω (provided D is in M). In our case, we have constructed the new partition C out of many reals of the form C^Θ_σ, where each C^Θ_σ was a splitting real over M in the sense of the space $[\omega \setminus X_\sigma]^\omega$, where

$$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < 1 \text{ and } \psi_j^{-1}(\ell) \perp \langle i^0_j \rangle\}.$$

Applying Fact 5.3.2 to the same real C^Θ_σ, we see that it is still splitting over the extended model $M[G]$, in the sense of the space $[\omega \setminus X_\sigma]^\omega$, interpreted in $V[G]$.

Therefore we can repeat the same argument in $V[G]$. If Y is a real in $M[G]$, and if $|Y \cap \Phi_i(g)| < \omega$ for every $g \in \omega^\omega \cap V[G]$, we construct functions g_j the same way as before, except that now $g_j \in V[G]$. However, the sets X_σ have not changed, and the C^Θ_σ split all $Y' \in M[G]$ which have infinite intersection with $\omega \setminus X_\sigma$, so we can apply the same reasoning to produce a real $h \in \omega^\omega \cap V[G]$, such that $V[G] \models \Phi_C(h) \subseteq Y$. This completes the proof. □ (Main Lemma)

Proof of main theorem. Let V be a model of CH. We will construct a $\mathbb{D}_{\aleph_2}$-indestructible $\aleph_1$-Borel mad family in V. If A is such a family, we will denote by $A^{V[G_{\aleph_2}]}$ the re-interpreted $\aleph_1$-Borel mad family, i.e., the $\aleph_1$-union of the re-interpreted Borel sets. Before proceeding with the construction, we show that preservation in iterations of length $\aleph_1$ is sufficient.

Claim. *If an $\aleph_1$-Borel mad family A is $\mathbb{D}_{\aleph_1}$-indestructible, then it is also $\mathbb{D}_{\aleph_2}$-indestructible.*

Proof. For a countable set $S \subseteq \aleph_2$, let $\mathbb{D}_S$ denote the iteration of Hechler forcing with support S. It is known that the $\aleph_2$-iteration of Hechler forcing is the *direct* *limit* of iterations $\mathbb{D}_S$ where S ranges over countable subsets of $\aleph_2$. This is true because Hechler forcing is a Suslin c.c.c. forcing notion, see [Bre10, p 54] for a proof. In particular, any new real added in the iteration $\mathbb{D}_{\aleph_2}$ is already added by some $\mathbb{D}_S$.

Let A be an $\aleph_1$-Borel mad family in V and suppose it is not $\mathbb{D}_{\aleph_2}$-indestructible. Then there is a $Y \in V[G_{\aleph_2}]$ which is almost disjoint from $A^{V[G_{\aleph_2}]}$. By the above, there is a countable $S \subseteq \aleph_2$ such that Y is in $V[G_S]$. Thus in $V[G_S]$, Y is almost disjoint from $A^{V[G_S]}$, and so $V[G_S] \models$ "A is not maximal". Since there is a canonical isomorphism between $\mathbb{D}_S$ and $\mathbb{D}_\gamma$ where $\gamma < \aleph_1$ is the order-type of S, the forcing extensions $V[G_S]$ and $V[G_\gamma]$ satisfy the same sentences, hence $V[G_\gamma] \models$ "A is not maximal". This proves that A is not $\mathbb{D}_{\aleph_1}$-indestructible. $\square$ (Claim)

Because of this claim, it suffices to construct a $\mathbb{D}_{\aleph_1}$-indestructible $\aleph_1$-Borel mad family in V.

Now we can proceed with the construction. First, note that in V there is a set of "canonical $\mathbb{D}_{\aleph_1}$-names for reals" of size $\aleph_1$, such that if $\dot{z}$ is any $\mathbb{D}_{\aleph_1}$-name for a real then there is a name $\dot{x}$ in this set, such that $\Vdash_{\mathbb{D}_{\aleph_1}} \dot{z} = \dot{x}$. This follows by standard arguments for finite support iterations of c.c.c. forcings, such as in the proof of the consistency of Martin's Axiom (see [Jec03, Theorem 16.13]). Therefore we may assume, without loss of generality, that there are only $\aleph_1$ many $\mathbb{D}_{\aleph_1}$-names for reals, and fix some enumeration $\{\dot{x}_\alpha \mid \alpha < \aleph_1\}$ of them in V.

Next, we fix some sufficiently large structure $\mathcal{H}_\theta$. By induction on $\alpha < \aleph_1$, we construct an increasing sequence of countable elementary submodels $M_\alpha \prec \mathcal{H}_\theta$ covering all the $\mathbb{D}_{\aleph_1}$-names for reals, while simultaneously constructing partitions P^α. The corresponding perfect a.d. families will be denoted by A_α, and the ideal generated by it by $\mathcal{I}_\alpha$.

At step α of the construction, the induction hypothesis will guarantee the following four conditions:

(IH1) $\dot{x}_\beta \in M_\alpha$ for all $\beta < \alpha$

(IH2) $P^\beta \in M_\alpha$ for all $\beta < \alpha$,

(IH3) $\bigcup_{\beta<\alpha} A_\beta$ is an a.d. family, and

(IH4) the ideal $\mathcal{I}_{<\alpha}$ generated by all $I_\beta(m)$ for $\beta < \alpha$, is proper (does not cover all of ω).

At successor steps, we will rely on the main lemma to do all the work.

- Basic step: let M_0 be any countable elementary submodel of $\mathcal{H}_\theta$.

- Induction step α: suppose M_α is a countable elementary submodel of $\mathcal{H}_\theta$, and suppose all four inductive conditions hold. Since α is countable and IH2–IH4 are satisfied, we are in the right position to apply the main lemma. From it, we obtain a new partition, P^α, and conditions 1 and 2 of the main lemma make sure that IH3 and IH4 will be satisfied at the next step of the induction, i.e., step $\alpha+1$. Now, define $M_{\alpha+1} \prec \mathcal{H}_\theta$ so that it contains M_α, the new partition P^α, the name $\dot{x}_\alpha$. For instance

$$M_{\alpha+1} := \mathrm{Hull}_{\mathcal{H}_\theta}(M_\alpha \cup \{P^\alpha\} \cup \{\dot{x}_\alpha\}).$$

  This makes sure that IH1 and IH2 are also satisfied at step $\alpha+1$.

- Limit step: at limit stages $\lambda < \aleph_1$, let M_λ contain all the M_α for $\alpha < \lambda$, e.g.

$$M_\lambda := \mathrm{Hull}_{\mathcal{H}_\theta}(\bigcup_{\alpha<\lambda} M_\alpha \cup \{\dot{x}_\lambda\}).$$

  It is clear that conditions IH1–IH4 are satisfied at stage λ.

Note that the construction of P^0 is a trivial application of the main lemma; since the ideal $\mathcal{I}_{<0}$ is empty, P^0 is simply a partition of Cohen reals over M_0, which split all sets $Y \in M_0$.

Let $A := \bigcup_{\alpha<\aleph_1} A_\alpha$. This is our Hechler-indestructible $\aleph_1$-Borel mad family. First, let us see that A is mad in V. Take any $Y \in [\omega]^\omega$, and note that since the sequence of models M_α covers all names for reals (modulo equivalence), in particular it covers ground model reals, so there is an M_α such that $Y \in M_\alpha$. By point 3 of the main lemma, either there is an $f \in \omega^\omega$ and a $\beta < \alpha$ such that Y has infinite intersection with $\Phi_\beta(f)$, or there is an $h \in \omega^\omega$ such that $\Phi_\alpha(h) \subseteq Y$, so in either case Y has infinite intersection with A.

Now, let us check that A is preserved in $V[G_{\aleph_1}]$, the $\aleph_1$-iteration of $\mathbb{D}$. Take any $Y \in [\omega]^\omega \cap V[G_{\aleph_1}]$, and let $\dot{Y}$ be a name for Y. Without loss of generality we may assume that $\dot{Y}$ is a *canonical* $\mathbb{D}_{\aleph_1}$-name, hence there is an M_α such that $\dot{Y} \in M_\alpha$, so $Y \in M_\alpha[G_{\aleph_1}]$. By point 3* of the main lemma, either there is an $f \in \omega^\omega \cap V[G_{\aleph_1}]$ and a $\beta < \alpha$ such that Y has infinite intersection with $\Phi_\beta(f)$, or there is an $h \in \omega^\omega \cap V[G_{\aleph_1}]$ such that $\Phi_\alpha(h) \subseteq Y$. In either case Y has infinite intersection with A.

Thus, the $\aleph_1$-Borel mad family A is preserved by the $\aleph_1$-iteration $V[G_{\aleph_1}]$, and therefore also by the $\aleph_2$-iteration $V[G_{\aleph_2}]$. This witnesses the fact that $\mathfrak{a}_B = \aleph_1$ in $V[G_{\aleph_2}]$. On the other hand, $\mathfrak{b} = \aleph_2$ in $V[G_{\aleph_2}]$, and this completes the proof. □

So we have proved the consistency of $\mathfrak{a}_B < \mathfrak{b}$, and it remains only to verify that the proof of the main theorem can be modified to yield the consistency of $\mathfrak{b} > \aleph_1$+"there is a $\mathbf{\Sigma}^1_2$ mad family". For this, we start with L instead of an arbitrary model of CH, and modify the proof of the main theorem as follows: fix some uniform coding of partitions P^α by reals. In the induction step α of the proof, instead of picking some P^α given to us by the lemma as we did before, pick the P^α with the $<_L$-least code.

Let $\mathcal{P}$ denote the set of all (codes of) $\{P^\alpha \mid \alpha < \aleph_1\}$ produced in this new proof. As always, the absoluteness of $<_L$ and everything else involved in this construction implies that the definition of the set $\mathcal{P}$ is absolute between L and a sufficiently large L_δ. Therefore, we may write $P \in \mathcal{P}$ iff $\exists L_\delta\ (P \in L_\delta \ \wedge\ L_\delta \models P \in \mathcal{P})$, or equivalently: there is $E \subseteq \omega \times \omega$ such that

1. E is well-founded,
2. $(\omega, E) \models \Theta$,
3. $\exists n\,(P = \pi_E(n)$ and $(\omega, E) \models n \in \pi_E^{-1}[\mathcal{P}])$.

This statement is $\boldsymbol{\Sigma}^1_2$.

For $P \in \mathcal{P}$, let A_P denote the Borel a.d. family based on P (i.e., A_α for $P = P_\alpha$). By the main theorem, the $\aleph_1$-Borel mad family, given by $A = \bigcup\{A_P \mid P \in \mathcal{P}\}$, is preserved in the $\aleph_2$-iterated Hechler extension $L[G_{\aleph_2}]$. But in the extension, $A^{L[G_{\aleph_2}]}$ is given by the following definition:

$$a \in A^{L[G_{\aleph_2}]} \iff \exists P\,(P \in \mathcal{P} \wedge a \in A_P),$$

which is a $\boldsymbol{\Sigma}^1_2$ statement. Thus, we have obtained a model of $\mathfrak{b} > \aleph_1$+"there is a $\boldsymbol{\Sigma}^1_2$ mad family", and also of $\boldsymbol{\Sigma}^1_2(\mathbb{L})$+"there is a $\boldsymbol{\Sigma}^1_2$ mad family".

5.4 Open questions

We have succeeded in proving the consistency of $\mathfrak{b} > \aleph_1$ with the existence of a $\boldsymbol{\Sigma}^1_2$ mad family. But recall that in Fact 5.1.4 we proved the consistency of $\neg$CH with the existence of a $\boldsymbol{\Pi}^1_1$ mad family. We conjecture the following:

Conjecture 5.4.1. Con($\mathfrak{b} > \aleph_1$ + "*there is a* $\boldsymbol{\Pi}^1_1$ *mad family*").

The method of Miller (used in Theorem 5.1.2 and in Theorem 4.3.9) can in principle be applied to our construction. Provided that a "coding lemma" can be proved allowing an arbitrary relation E to be encoded into the partition P^α at stage α, Miller's method will give a $\boldsymbol{\Pi}^1_1$ definition of the set $\mathcal{P}$. However, this is not sufficient to prove the conjecture, since in the definition of $A^{L[G_{\aleph_2}]}$ there is an additional existential quantifier, i.e., we would again only obtain a $\boldsymbol{\Sigma}^1_2$ definition of the mad family in $L[G_{\aleph_2}]$. Despite this potential difficulty, we are confident that the conjecture is true. The method of proof would involve the following trick: modify the partitions P^α in such a way that every real from A_α recursively encodes the whole partition P^α. In this way, the additional existential quantifier can be eliminated.

A more fundamental question would be the following:

Question 5.4.2. *Are the statements "there is a $\mathbf{\Sigma}^1_2$ mad family" and "there is a $\mathbf{\Pi}^1_1$ mad family" equivalent?*

Many other questions about projective mad families remain open. The most interesting result would be a characterization theorem.

Question 5.4.3. *Is there some notion of transcendence over L which is equivalent to the statements "there is no $\mathbf{\Sigma}^1_2$ mad family" or "there is no $\mathbf{\Pi}^1_1$ mad family"?*

If characterization is infeasible, one might at least look for upper and lower bounds, i.e., regularity hypotheses that imply, or are implied by, the hypotheses "there is no $\mathbf{\Sigma}^1_2$ mad family" or "there is no $\mathbf{\Pi}^1_1$ mad family". Since the original proof of Mathias (Theorem 5.1.1) involves a Ramsey-style property, the following would be an interesting test-case:

Question 5.4.4. *Does* $\mathbf{\Sigma}^1_2$(Ramsey) *imply that there is no $\mathbf{\Sigma}^1_2$ mad family?*

On the other hand, our own proof relies heavily on splitting reals and the preservation of splitting families by Hechler forcing, so one may ask whether the following holds:

Question 5.4.5. *Does "there is no $\mathbf{\Sigma}^1_2$ mad family" imply that for all r, $L[r]\cap[\omega]^\omega$ is not a splitting family?*

Other questions concern the cardinal invariant $\mathfrak{a}_B$. Although we have established the consistency of $\mathfrak{a}_B < \mathfrak{b}$, many questions remain open. For example, it is not clear whether the converse holds.

Question 5.4.6. *Is $\mathfrak{a}_B \leq \mathfrak{b}$ provable in* ZFC*, or is $\mathfrak{b} < \mathfrak{a}_B$ consistent?*

Concerning lower bounds for $\mathfrak{a}_B$, the following is a conjecture of Dilip Raghavan (for the definition of $\mathfrak{h}$, see Definition 1.2.28):

Conjecture 5.4.7. *$\mathfrak{h} \leq \mathfrak{a}_B$.*

Since the canonical method of increasing $\mathfrak{h}$ is to iterate Mathias forcing, this conjecture seems closely related to Question 5.4.4.

Concerning upper bounds, a question related to Question 5.4.5 would be whether the splitting number $\mathfrak{s}$ (see Definition 1.2.26) is an upper bound for $\mathfrak{a}_B$. Since we have also used the countability of the α's in the construction, it may be more realistic to expect only the weaker result that if $\mathfrak{s} = \aleph_1$ then $\mathfrak{a}_B = \aleph_1$.

Question 5.4.8. *Is $\mathfrak{a}_B \leq \mathfrak{s}$ provable in* ZFC*? Or, at least, is $\mathfrak{s} = \aleph_1 \to \mathfrak{a}_B = \aleph_1$ provable in* ZFC*?*

In a very recent result, Raghavan and Shelah [RS11] showed the weaker statement that if $\mathfrak{d} = \aleph_1$ then $\mathfrak{a}_B = \aleph_1$.

Finally, recall that the a.d. families A_α we constructed were, in fact, closed. We can define the cardinal invariant $\mathfrak{a}_{\text{closed}}$ as the least number of *closed* a.d. sets whose union is a mad family. It is obvious that $\mathfrak{a}_B \leq \mathfrak{a}_{\text{closed}}$, and that our proof actually shows the stronger result $\text{Con}(\mathfrak{a}_{\text{closed}} < \mathfrak{b})$. However, it is not clear whether the two cardinal invariants are different.

Question 5.4.9. *Is* $\mathfrak{a}_B = \mathfrak{a}_{\text{closed}}$ *provable in* ZFC*, or is* $\mathfrak{a}_B < \mathfrak{a}_{\text{closed}}$ *consistent?*

At the moment we do not have an intuition as to which of the above should be true, nor as to how a proof of either would proceed.

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Samenvatting

In dit proefschrift worden vragen bestudeerd die van belang zijn voor de grondslagen van de wiskunde, met name het continuüm der reëele getallen. We bekijken aan de ene kant zogenaamde *regulariteitseigenschappen* van verzamelingen van reëele getallen, en aan de andere kant *definieerbaarheid* van zulke verzamelingen. Met "regulariteitseigenschappen" doelen we op bepaalde wenselijke eigenschappen van verzamelingen, welke garanderen dat deze verzamelingen een goed gedraag vertonen, overeenkomstig zijn met onze intuïtie, of zich eenvoudig laten bestuderen. Klassieke voorbeelden hiervan zijn Lebesgue-meetbaarheid, de eigenschap van Baire en de perfecte verzamleingeigenschap. Met "definieerbaarheid" doelen we op de mogelijkheid een expliciete beschrijving van een verzameling te geven. Klassieke voorbeelden van definieerbare verzamelingen zijn de Borel-, de analytische en de projectieve verzamelingen, en dit leidt tot een maat van complexiteit waarbij een verzameling net zo complex wordt beschouwd als de logische uitdrukking die deze verzameling definieert.

Het verband tussen regulariteit en definieerbaarheid was al vanaf het begin van de 20e eeuw bekend. Zo voldoen bijvoorbeeld alle Borel- en analytische verzamelingen aan de meeste regulariteitseigenschappen. Door gebruik te maken van het Keuzeaxioma kunnen verzamelingen zonder zulke regulariteitseigenschappen worden geconstrueerd, maar deze zijn in het algemeen niet definieerbaar. In het construeerbare universum L van Gödel worden deze tegenvoorbeelden echter ook op het niveau $\boldsymbol{\Sigma}^1_2$ gevonden (het eerstvolgende complexiteitsniveau na het niveau $\boldsymbol{\Sigma}^1_1$ van de analytische verzamelingen).

Doorgaans is de bewering dat alle $\boldsymbol{\Sigma}^1_2$- of $\boldsymbol{\Delta}^1_2$-verzamelingen aan een bepaalde regulariteitseigenschap voldoen, onafhankelijk van ZFC, de gebruikelijke axiomatisering van de verzamelingenleer. Bovendien kunnen zulke beweringen als mogelijke aanvullende hypothesen worden beschouwd, welke onder andere als gevolg hebben dat het huidige wiskundige universum in een bepaalde zin groter is dan L.

Het zwaartepunt van dit proefschrift is de wisselwerking tussen regulariteitseigenschappen en definieerbaarheid, met name het verband tussen hypothesen over regulariteit en meta-mathematische beweringen over het wiskundige universum.

In Hoofdstuk 2 geven we een abstracte behandeling van het bovengenoemde fenomeen in het kader van *Idealized Forcing*, een begrip dat door Jindřich Zapletal werd ingevoerd. We generaliseren een aantal welbekende stellingen in dit gebied, en ook een recent resultaat van Daisuke Ikegami. Daarbij komen veel interessante vragen tevoorschijn. In dit hoofdstuk dient de zogenaamde *forcing*-methode als voornaamste bewijsmiddel.

In Hoofdstuk 3 beschouwen we de *gepolariseerde partitie-eigenschap*, een regulariteitseigenschap dat gemotiveerd is door combinatorische problemen, en aanverwant is aan de klassieke eigenschap van Ramsey. Deze werd onlangs in het werk van, onder andere, Carlos A. Di Prisco and Stevo Todorčević bestudeerd. We bewijzen een aantal resultaten die deze eigenschap met andere bekende eigenschappen op het $\mathbf{\Sigma}^1_2$- en het $\mathbf{\Delta}^1_2$-niveau vergelijken.

In Hoofdstuk 4 richten we onze aandacht op *Hausdorff-gaten*, klassiek objecten die al sinds het vroege 20e eeuw bekend zijn en talrijke toepassingen in verschillende gebieden van de wiskunde kennen, waaronder de topologie en de analyse. Hier bewijzen we een uitbreiding van een stelling van Stevo Todorčević die zegt dat een analytisch Hausdorff-gat niet bestaat.

In Hoofdstuk 5 bekijken we maximaal bijna disjuncte (m.b.d.) families vanuit het definieerbare standpunt. We voeren een nieuw begrip van *onvernietigbaarheid* van m.b.d. families onder forcing in, en gebruiken dit om een behoudsresultaat te bewijzen dat de consistentie van $\mathfrak{b} > \aleph_1$ met het bestaan van een $\mathbf{\Sigma}^1_2$-definieerbare m.b.d familie vaststelt. Dit beantwoordt een vraag van Sy Friedman en Lyubomyr Zdomskyy.

Abstract

In this dissertation we study questions relevant to the foundations of mathematics, particularly the real number continuum. We look at *regularity properties* of sets of real numbers on one hand, and *definability* of such sets on the other. By "regularity properties" we are referring to certain desirable properties of sets of reals, something that makes such sets well-behaved, conforming to our intuition or easy to study. Classical examples include Lebesgue measurability, the property of Baire and the perfect set property. By "definability" we are referring to the possibility of giving an explicit description of a set. Classical examples of definable sets are the Borel, analytic and projective sets, and this leads to a measure of complexity of a set, whereby a set is considered as complex as the logical formula defining it.

The relationship between regularity and definability has been known since the beginning of the 20th century. For example, all Borel and analytic sets satisfy most regularity properties. Using the Axiom of Choice, sets without such regularity properties can easily be constructed, but these are, in general, not definable. On the other hand, working in Gödel's constructible universe L, counterexamples can be found on the $\boldsymbol{\Sigma}^1_2$ level (the next level beyond the analytic).

Typically, the assertion that all $\boldsymbol{\Sigma}^1_2$ or all $\boldsymbol{\Delta}^1_2$ sets of reals satisfy a certain regularity property is independent of ZFC, the standard axiomatization of set theory. Moreover, such an assertion can itself be seen as a possible additional hypothesis, implying among other things that the set-theoretic universe is larger than L in a certain way.

The focus of this dissertation is the interplay between regularity properties and definability, particularly the connection between regularity hypotheses and meta-mathematical statements about the set theoretic universe.

In Chapter 2 we provide an abstract treatment of the above phenomenon, formulated in the framework of *Idealized Forcing* introduced by Jindřich Zapletal. We generalize some well-known theorems in this field and also a recent result of

Daisuke Ikegami, and isolate several interesting questions. All proofs in this chapter rely heavily on the method of *forcing*.

In Chapter 3 we consider the *polarized partition property*, a regularity property motivated by combinatorial questions and a relative of the classical Ramsey property. It has been studied in recent work of Carlos A. Di Prisco and Stevo Todorčević, among others. We prove several results relating this to other well-known regularity properties on the $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ level.

In Chapter 4 we turn our attention to *Hausdroff gaps*, classical objects known since the early 20th century which have numerous applications in various fields of mathematics such as topology and analysis. We extend a theorem of Stevo Todorčević stating that there are no analytic Hausdorff gaps in several directions.

In Chapter 5 we look at maximal almost disjoint (mad) families from the definable point of view. We introduce a new notion of *indestructibility* of a mad family by forcing extensions, and using this notion prove a preservation result establishing the consistency of $\mathfrak{b} > \aleph_1$ together with the existence of a $\mathbf{\Sigma}^1_2$ definable mad family. This answers a question posed by Sy Friedman and Lyubomyr Zdomskyy.

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