Regularity Properties and Determinacy

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written by

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0. Introduction

One of the most intriguing developments of modern set theory is the investigation of *two-player infinite games of perfect information*. Of course, it is clear that applied game theory, as any other branch of mathematics, can be modeled in set theory. But we are talking about the converse: the use of infinite games as a tool to study fundamental set theoretic questions. When such infinite games are played using integers as moves, a surprisingly rich theory appears, with connections and consequences in all fields of pure set theory, particularly the study of the continuum (the real numbers) and Descriptive Set Theory (the study of "definable" sets of reals).

The concept of *determinacy* of games—a game is determined if one of the players has a *winning strategy*—plays a key role in this field. In the 1960s, the Polish mathematicians Jan Mycielski and Hugo Steinhaus [MySt62, My64, My66] proposed the famous Axiom of Determinacy (AD), which implies that all sets of reals are Lebesgue measurable, have the Baire property, the Perfect Set Property, and in general all the "regularity properties". This contradicts the Axiom of Choice (AC) which allows us to construct irregular sets by using an enumeration of the continuum. A lot of work on determinacy is therefore done in ZF, i.e., Zermelo-Fraenkel set theory without the Axiom of Choice. In such a mathematical universe with AC replaced by AD, the pathological, non-constructive sets that form counter-examples to the regularity properties are altogether banished.

But how should we understand determinacy in the context of ZFC, i.e., standard Zermelo-Fraenkel set theory *with* Choice? The easiest way is to look at determinacy as another kind of regularity property, D, where a set of reals A is determined if its corresponding game is determined. Since in the AD context infinite games are used to prove regularities, one would expect determinacy to be a kind of "mother regularity property", one which subsumes and implies all the others. This is indeed true, but only in the "classwise" sense: assuming for some large collection $\boldsymbol{\Gamma}$ of sets that each of them is determined, we may conclude that each set in $\boldsymbol{\Gamma}$ has the regularity properties. Does determinacy actually have "pointwise" consequences, i.e., if we know of a set A that it is determined, does that imply that A is regular? In general, the answer is no. The real "mother regularity property" is the much stronger property of being *homogeneously Suslin*, which does imply all the regularity properties pointwise.[1] Although there are close similarities between determinacy and being homogeneously Suslin, the crucial difference lies in the fact that the former has only classwise consequences whereas the latter has pointwise consequences. In this sense determinacy is a relatively weak property.

Although, from the beginning, researchers were aware of this fact, a rigorous study of pointwise (non-)implications from determinacy has not been carried

[1] [Ka94, Theorem 32.7]. For a definition of *homogeneously Suslin*, see e.g. [Ka94, p. 450 ff]; other relevant papers include [Ko98, Ko02, KoSc06].

out until [Lö05]. In this thesis, we will continue the research started in that paper and generalize some of its results.

Another focus of this thesis are the regularity properties themselves. We take the view that most regularity properties are naturally connected with special combinatorial objects called *forcing partial orders*. The motivation comes from the theory of *forcing*, a mainstream area dealing with the independence of certain propositions (like the Continuum Hypothesis) from the axioms of set theory. These combinatorial objects are also interesting in their own right, and can be put in connection with classical regularity properties (e.g., the Baire property and the Perfect Set Property) as well as other regularity properties. There are still a number of open questions regarding these connections.

This thesis will combine the study of pointwise consequences of determinacy with the study of these general open questions.

Concretely, we denote a particular forcing partial order by $\mathbb{P}$. Some $\mathbb{P}$ generate a topology, whereas others don't, and this distinction into topological versus non-topological forcing notions will be central to our work. The most important regularity property connected to $\mathbb{P}$ is the Marczewski-Burstin algebra denoted by $\mathrm{MB}(\mathbb{P})$, which can easily be defined for any $\mathbb{P}$. However, when $\mathbb{P}$ is topological, this algebra tends to be a "bad" regularity property and is replaced by the Baire property in the topology generated by $\mathbb{P}$, denoted by $\mathrm{BP}(\mathbb{P})$. But this is only a heuristic distinction, and no research has yet been done on what the precise reason for the dichotomy is. This leads us to formulate our first research question:

Main Question 1: *Why is there a dichotomy between topological and non-topological forcings* $\mathbb{P}$*, i.e., why is it that for non-topological forcings* $\mathbb{P}$ *the right regularity property is* $\mathrm{MB}(\mathbb{P})$ *whereas for topological ones it is* $\mathrm{BP}(\mathbb{P})$*? When is* $\mathrm{MB}(\mathbb{P})$ *a "good" property, and what is the relationship between the two regularity properties?*

Moving on toward pointwise consequences of determinacy, we wish to study the connections between determinacy and the regularity properties introduced above. In [Lö05], the case of non-topological forcings $\mathbb{P}$ and the corresponding algebras $\mathrm{MB}(\mathbb{P})$ is covered, where it is proved that in all interesting cases determinacy does not imply $\mathrm{MB}(\mathbb{P})$ pointwise. Also, a weak version of the Marczewski-Burstin algebra, denoted by $\mathrm{wMB}(\mathbb{P})$, is introduced and studied (where the connections with determinacy are more interesting). We will do an analogous analysis for the topological case.

Main Question 2: *Can we do an analysis of the pointwise connection between determinacy and the Baire property* $\mathrm{BP}(\mathbb{P})$ *(for topological* $\mathbb{P}$*), similar to the one in [Lö05]? Can we also introduce a weak version of the Baire property* $\mathrm{wBP}(\mathbb{P})$*, and if so, what is the pointwise connection between determinacy and* $\mathrm{wBP}(\mathbb{P})$*?*

If BP($\mathbb{P}$) was a generalization of the standard Baire property, then there are also several generalizations of the Perfect Set Property. These so-called *asymmetric regularity properties* can also be connected to forcing partial orders $\mathbb{P}$, in which case we denote them by Asym($\mathbb{P}$). In current research, there are four particular examples but as of yet no general definition. We would like to find that general definition, and also to study the pointwise connections with determinacy, analogously to Question 2. This leads us to the last research question:

Main Question 3: *Can a general definition for the asymmetric property* Asym($\mathbb{P}$) *be given? If so, can we do a similar analysis for the pointwise connections between determinacy and* Asym($\mathbb{P}$) *as we did in Question 2?*

This thesis is structured as follows: in Chapter 1, we introduce the basic definitions and ideas related to the study of the real numbers and the forcing notions. Chapter 2 is still introductory, developing in detail the key ideas: determinacy, regularity properties, pointwise and classwise implications. In Chapter 3 we deal with Main Question 1. The main result there is Theorem 3.4 which provides the connection between MB and BP. In the rest of the chapter we study other aspects of Question 1 (when is MB($\mathbb{P}$) a σ-algebra) and provide a partial answer in Theorems 3.6 and Theorem 3.13.

In Chapter 4 we deal with Main Question 2. Analogously to [Lö05] we prove that determinacy does not imply BP($\mathbb{P}$) pointwise (Theorem 4.8) and characterize the $\mathbb{P}$ for which determinacy does, or does not, imply the weak Baire property pointwise (Theorems 4.13 and 4.18).

Finally, in Chapter 5 we deal with Main Question 3. Although we do not find a clear definition for Asym($\mathbb{P}$), we do give a necessary condition which such a property must satisfy, in terms of a game characterization. This characterization is sufficient to solve the second part of the question: in Theorem 5.12 we do prove that determinacy does not imply Asym($\mathbb{P}$) pointwise in all non-trivial cases.

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1. Preliminaries

1.1 Notation.

For the most part, we use standard set-theoretic notation for notions related to the natural and real numbers. The natural numbers are denoted by ω, ω^ω is the set of all functions from ω to ω and n^ω is the set of functions from ω to n. In general, we use the fixed symbol η to stand for an unspecified ordinal with $2 \leq \eta \leq \omega$ and thus write η^ω to refer to the general case.

The set of all strictly increasing functions is denoted by $\eta^{\uparrow\omega}$, and $[X]^\omega$ is the set of all infinite subsets of X, as usual. Similarly, $[X]^n$ is the set of all subsets of X of size n and $[X]^{<\omega}$ is the set of all finite subsets of X.

The set of all finite sequences with elements from η is denoted by $\eta^{<\omega}$, and for $s \in \eta^{<\omega}$, $s(i)$ is the i-th element of s. A finite sequence can also be specified by writing $s = \langle s_0, s_1, \dots, s_k \rangle$, and in a few cases this notation will be extended to infinite sequences as well, i.e., we can write $x = \langle x_0, x_1, x_2, \dots \rangle$ to refer to some $x \in \eta^\omega$ with $x(i) = x_i$.

For $s, t \in \eta^{<\omega}$, the *concatenation of s with t* is denoted by $s^\frown t$, i.e., if $s = \langle s_0, \dots, s_k \rangle$ and $t = \langle t_0, \dots, t_l \rangle$ then $s^\frown t := \langle s_0, \dots, s_k, t_0, \dots, t_l \rangle$. For $x \in \eta^\omega$, the *restriction of x to the first n values* is denoted by $x \restriction n := \langle x(0), \dots, x(n-1) \rangle$.

Since in set theory functions are represented as sets of pairs, one can (using a notation which might look confusing to non-set-theorists) write $s \subseteq t$ if $s, t \in \eta^{<\omega}$ and s is an initial segment of t. Similarly for $s \in \eta^{<\omega}$ and $x \in \eta^\omega$. In most of this text, therefore, if for a given $x \in \eta^\omega$ we write $s \subseteq x$ it is assumed that s is not just *some* subset of x but an initial segment.

Slightly non-standard is our notation of complements of sets: if the general space X in which we are working is clear from the context (it will always be some space of real numbers) and $A \subseteq X$, we shall write A^c for $X \setminus A$.

Whenever necessary, we use standard notation of first order logic. We also use the symbols $\forall^\infty$ and $\exists^\infty$ to stand for "for all except finitely many" and "there are infinitely many", e.g.

"$\forall^\infty n \; (x(n) = 0)$" stands for "$\exists N$ s.t. $\forall n \geq N \; (x(n) = 0)$"
"$\exists^\infty n \; (x(n) \neq 0)$" stands for "$\forall N \; \exists n \geq N$ s.t. $x(n) \neq 0$".

All results of this thesis are theorems of ZFC.

1.2 The Real Numbers.

Although in ordinary mathematics the term *real numbers* refers to the traditional "$\mathbb{R}$", (defined, e.g., as Dedekind cuts of sets of rationals, limits of Cauchy-sequences etc.), this space is rarely used in pure set theory because it tends to burden us with irrelevant complications. For studying the structural, topological and set-theoretic properties of the continuum, it is most convenient to work with ω^ω or 2^ω, or in general η^ω for $2 \leq \eta \leq \omega$. Hence, following standard set-theoretic practice, these are the spaces that form the standard notions of real numbers. Consequently, an x from η^ω is called a *real number*, or simply *real*.

We will very briefly summarize the main topological and set-theoretic background of these real numbers. A detailed account can be found e.g. in [Ke95].

1.1. Definition. Given $s \in \eta^{<\omega}$, we define the set of all reals extending s:

$$[s] := \{x \in \eta^\omega \mid s \subseteq x\}$$

Then, the *standard topology* on η^ω is the topology which has $\{[s] \mid s \in \eta^{<\omega}\}$ as its basis. $\lhd$

It can easily be checked that this is indeed a topology, and that it is equivalent to the product topology on η^ω provided that each η is endowed with the discrete topology. It is also equivalent to the topology generated by the *metric* d defined by:

$$d(x,y) := \begin{cases} \frac{1}{2^n} & \text{where } n \text{ is least s.t. } x(n) \neq y(n) \text{ or} \\ 0 & \text{if } x = y \end{cases}$$

The space ω^ω with this topology is called *Baire space* and it is homeomorphic to the set of *irrationals* $\mathbb{R} \setminus \mathbb{Q}$. The space 2^ω with this topology is called *Cantor space* and it is homeomorphic to Cantor's standard "$\frac{1}{3}$ set".

Convergence in the Baire and Cantor spaces can conveniently be formulated as follows:

$$x_n \longrightarrow x \quad \text{iff} \quad \forall s \subseteq x \; (\forall^\infty n \; (s \subseteq x_n)) \tag{1}$$

In this context of real numbers, the *rational numbers* (usually just denoted by $\mathbb{Q}$) can be defined as those reals which are eventually 0, i.e.

$$\mathbb{Q} := \{x \in \eta^\omega \mid \forall^\infty n \; (x(n) = 0)\}$$

For the rest, we assume familiarity with basic topological notions, in particular the notions *dense*, *meager*, F_σ, G_δ, *Borel* and so on. We will use the notation A° for the *interior of* A and $\overline{A}$ for the *topological closure of* A.

1.3 Trees.

Since a real x can be approximated by its finite initial segments $s \subseteq x$, sets of reals can similarly be approximated by (countable) sets of finite sequences. This leads to the crucial notion of (descriptive theoretic) trees.

1.2. Definition. A subset $T \subseteq \eta^{<\omega}$ is called a *tree* (*on* η) if it is closed under taking initial segments, i.e., if $t \in T$ and $s \subseteq t$ then $s \in T$. A *branch of* T is an $x \in \eta^\omega$ such that for every n, $x \upharpoonright n \in T$. The set of all branches of T is denoted by $[T]$. $\lhd$

Using the formulation of limits (1) from above, it is not hard to see that for every tree T, the set $[T]$ is closed (contains all its limit points). Furthermore, given any set $A \subseteq \eta^\omega$ we can define *the tree of* A, $T_A := \{x \upharpoonright n \mid x \in A,\ n \in \omega\}$. Obviously, $A \subseteq [T_A]$. Moreover, since $[T_A]$ contains precisely the limit points of A, we can view the operation $A \mapsto T_A \mapsto [T_A]$ as the topological closure, which means that for closed sets C we have $C = [T_C]$. Therefore, via a one-one-correspondence we can identify trees with closed sets.

For the study of trees, the following concepts are of importance:

1.3. Definition.

1. A tree T is called *pruned* if every $t \in T$ has an extension $s \in T$ with $t \subset s$. All the trees which we will consider will be pruned.
2. For a node $t \in T$, $\mathrm{Succ}_T(t) := \{s \in T \mid \exists n(s = t^\frown \langle n \rangle)\}$ is the set of immediate successors of t.
3. A node $t \in T$ is called
   - *splitting* if $|\mathrm{Succ}_T(t)| > 1$ and *non-splitting* otherwise.
   - *ω-splitting* if $|\mathrm{Succ}_T(t)| = \omega$ and *n-splitting* if $|Succ_T(t)| = n < \omega$.
   - *totally splitting* if $\forall n\ (t^\frown \langle n \rangle \in T)$.
   - The *n-th splitting* (*ω-splitting, totally splitting* etc.) *node* if it is a splitting (ω-splitting etc.) node and there are exactly n nodes $s \subsetneq t$ which are also splitting (ω-splitting etc.)
4. The *stem of* T, notation $\mathrm{stem}(T)$, is the largest $s \in T$ such that all $t \subseteq s$ are non-splitting. If such an s does not exist, then $\mathrm{stem}(T) := \{y\}$ for the unique y s.t. $[T] = \{y\}$. Equivalently, $\mathrm{stem}(T) = \bigcup\{s \in T \mid \forall t \in T\ (s \subseteq t \lor t \subseteq s)\}$
5. Clearly, if $S \subseteq T$ then $\mathrm{stem}(T) \subseteq \mathrm{stem}(S)$. We can strengthen the $\subseteq$-relation on trees to a new relation, denoted by $\ll$, by setting $S \ll T$ if and only if $S \subseteq T$ and $\mathrm{stem}(T) \neq \mathrm{stem}(S)$, i.e., a "sub-tree with strictly longer stem"-relation.

6. For a tree T and $s \in T$ we use the notations
   - $T{\uparrow}s := \{t \in T \mid t \subseteq s \vee s \subseteq t\}$ and
   - $T_s := \{t \in \eta^{<\omega} \mid s^\frown t \in T\}$

   for the *subtree of T through s* and the tree "T after s", respectively. $\lhd$

1.4 The Forcing Notions.

The technique of *forcing* is central to higher set theory, where it is used as the primary means of showing independence results, like the independence of the Continuum Hypothesis and other propositions from the axioms of set theory. In each case, a specific *forcing partial order* or *forcing notion* is devised. Although primarily, these are interesting from the point of view of independence proofs, it has been noticed in recent years that the partial orders are interesting objects of study in their own right.

As mentioned in the introduction, this thesis is concerned with special kinds of these partial orders. The theory of forcing as such will not be relevant, and consequently the reader is not required to have any knowledge of forcing. It is, however, nice to remember, in the back of the mind, that the partial orders we are studying are not arbitrary mathematical objects but form a part of a larger and more fundamental theory.

1.4. Definition.

- By a *forcing notion* we mean any partial order $(\mathbb{P}, \leq)$. In standard forcing terminology, elements of $\mathbb{P}$ are called *conditions* and if for $P, Q \in \mathbb{P}$ we have $P \leq Q$, then we say "P is stronger than Q" or "P extends Q".
- A forcing notion $(\mathbb{P}, \leq)$ is called *arboreal* if it is order-isomorphic to a collection $\mathfrak{T}$ of trees ordered by inclusion, with the extra condition that for every $T \in \mathfrak{T}$ there is an $S \in \mathfrak{T}$ with $S \ll T$.
- A forcing notion $\mathbb{P}$ is called *topological* if the collection $\{[P] \mid P \in \mathbb{P}\}$ forms a topology base on η^ω (this is the case if and only if for every $P, Q \in \mathbb{P}$ we have that $[P] \cap [Q]$ is either empty or a union of $[R]$'s for $R \in \mathbb{P}$), and *non-topological* otherwise. When $\mathbb{P}$ is topological, we shall use a slight abuse of notation and write $(\eta^\omega, \mathbb{P})$ to refer to the topological space η^ω endowed with the topology generated by $\mathbb{P}$. $\lhd$

Our central objects of study will be the arboreal forcing notions. In the list below, we present the classical examples of non-topological and topological forcing notions—the key players of this thesis.

1.5. Definition. Non-topological arboreal forcings:

1. A tree T is called *perfect* if $\forall t \in T\ \exists s \in T$ s.t. $t \subseteq s$ and s is a splitting node. *Sacks forcing*, denoted by $\mathbb{S}$, is the partial order of perfect trees on 2^ω.

2. A tree T is called *super-perfect* if all its splitting nodes are ω-splitting and $\forall t \in T\ \exists s \in T$ s.t. $t \subseteq s$ and s is ω-splitting. *Miller forcing*, $\mathbb{M}$, consists of super-perfect trees on ω^ω.

3. A tree T is called a *Spinas tree*[2] if it is super-perfect with the additional requirement that for every node $t \in T$ we have

$$\forall s_1, s_2\ (t^\frown s_1 \text{ and } t^\frown s_2 \text{ are } \omega\text{-splitting nodes of } T \to |s_1| = |s_2|)$$

i.e., the next splitting node is a fixed distance away from t. *Spinas forcing*, denoted by $\mathbb{L}^*$, consists of Spinas trees on ω^ω.

4. A tree T is called *Laver* if $\forall t \in T$ with $\text{stem}(T) \subseteq t$, t is ω-splitting (and $\text{stem}(T)$ is finite, i.e. $[T]$ is not a singleton). *Laver forcing*, $\mathbb{L}$, consists of Laver trees on ω^ω.

5. A tree T is called *uniform* if

$$\forall s, t \in T\ \ (|s| = |t| \to \{n \mid s^\frown \langle n \rangle \in T\} = \{n \mid t^\frown \langle n \rangle \in T\})$$

i.e. at every node the branching is dependent only on the height of the node and not on the previous branchings. *Silver forcing*, $\mathbb{V}$, consists of all uniform perfect trees on 2^ω. $\lhd$

Note that $\mathbb{M}$, $\mathbb{L}^*$ and $\mathbb{L}$ must be defined on ω^ω rather than an arbitrary η^ω. Sacks and Silver forcing $\mathbb{S}$ and $\mathbb{V}$ are usually defined on 2^ω, but for our purposes we will extend the definition onto an arbitrary η^ω.

Topological forcing notions are usually not formulated in terms of trees but in some other way. It is always straightforward to give the isomorphism to the partial order of trees, but often the original formulation is easier to use and understand. Moreover, just as for trees there is a notational difference between the trees themselves ("T") and the sets of their branches ("$[T]$"), we will analogously specify the conditions themselves by "P" and the set of reals it specifies by "$[P]$".[3]

Note that, except $\mathbb{C}$, all the topological forcing notions are defined on ω^ω rather than an arbitrary η^ω.

1.6. Definition. Topological arboreal forcings:

[2] Spinas [Sp93] originally called these "uniform" trees, but we shall need that term for another notion. Also, although Spinas did not originally relate these trees to forcing, it can clearly be used as such, and in [BrHjSp95, Theorem 5.1] it is shown that in forcing-theoretic terms Spinas forcing is equivalent to Laver forcing.

[3] This is essentially a matter of convenience. Since there is a bijection $P \longleftrightarrow [P]$, we could always also consider $\mathbb{P}$ to be a collection of the sets of reals $[P]$ themselves.

1. *Cohen forcing*, $\mathbb{C}$:[4] the conditions are finite sequences ordered by: $s \leq t$ iff $t \subseteq s$. To each s we associate $[s] := \{x \in \eta^\omega \mid s \subseteq x\}$.

   Equivalently, $\mathbb{C}$ is the collection of all trees T whose stem is finite and every node extending the stem is totally splitting.

2. *Hechler forcing*, $\mathbb{D}$: the conditions are $(s, f) \in \omega^{<\omega} \times \omega^\omega$, such that $s \subseteq f$, ordered by:

$$(t, g) \leq (s, f) \iff s \subseteq t \text{ and } \forall n \geq |s| \; (g(n) \geq f(n))$$

   To each condition (s, f) we associate $[s, f] := \{x \in \omega^\omega \mid s \subseteq x \wedge \forall n \geq |s| \; (x(n) \geq f(n))\}$.

   Equivalently, $\mathbb{D}$ is the collection of all trees T such that for some $f \in \omega^\omega$, $\mathrm{stem}(T)$ is finite and $\mathrm{stem}(T) \subseteq f$ and for all t extending (and including) the stem, $\mathrm{Succ}_T(t) = \{t^\frown \langle n \rangle \mid n \geq f(|t|)\}$.

3. *Eventually different forcing*, $\mathbb{E}$: the conditions are $(s, F) \in \omega^\omega \times [\omega^\omega]^{<\omega}$, s.t. $\forall f \in F \; \forall n < |s| \; (f(n) \neq s(n))$, ordered by

$$(t, G) \leq (s, F) \iff s \subseteq t, F \subseteq G \text{ and}$$
$$\forall f \in F \; \forall n \geq |s| \; (t(n) \neq f(n))$$

   To each condition (s, F) we associate $[s, F] := \{x \in \omega^\omega \mid s \subseteq x \wedge \forall f \in F \; \forall n \geq |s| \; (x(n) \neq f(n))\}$.

   The isomorphism to a partial orders of trees is similar as for Hechler forcing.

4. *Mathias forcing*, $\mathbb{R}$: the conditions are $(s, S) \in \omega^{\uparrow<\omega} \times [\omega]^\omega$, with $\max(\mathrm{ran}(s)) < \min(S)$, ordered by:

$$(t, T) \leq (s, S) \iff s \subseteq t, \; T \subseteq S \text{ and } \mathrm{ran}(t \setminus s) \subseteq S$$

   To each condition (s, S) we associate $[s, S] := \{x \in \omega^{\uparrow\omega} \mid s \subseteq x \wedge \forall n \geq |s| \; (x(n) \in S)\}$. $\lhd$

It is easy to check that in each case the collections $\{[P] \mid P \in \mathbb{P}\}$ form a topology base. The topologies generated by $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ are the standard topology (as in Definition 1.1), the *dominating topology* and the *eventually different topology*, respectively. Concerning Mathias forcing, one should note that the sets $[s, S]$ only contain increasing functions, so ω^ω is not open in the corresponding topology. One way to deal with this is to postulate independently that ω^ω is open, and another approach is to treat $\mathbb{R}$ as generating a topology on the space

[4] This is the *original* forcing notion, due to Paul Cohen [Co63, Co64], used in his original 1963 and 1964 proofs of the independence of the CH.

$\omega^{\uparrow\omega}$, or equivalently $[\omega]^\omega$, rather than ω^ω. In this more standard approach, this is called the *Ellentuck topology* (due to [El74]).

We conclude this section with a few general definitions related to the theory of forcing, which will be relevant on an occasional basis.

1.7. Definition. Let $(\mathbb{P}, \leq)$ be a forcing partial order.

1. $P, Q \in \mathbb{P}$ are *compatible* if there is an $R \in \mathbb{P}$ s.t. $R \leq P$ and $R \leq Q$. Otherwise, P and Q are called *incompatible* and this is denoted by $P \perp Q$. item An *antichain* is a collection $S \subseteq \mathbb{P}$ s.t. $\forall P, Q \in S : P \perp Q$.

2. $\mathbb{P}$ has the *countable chain condition*, shorthand *c.c.c.*, if every antichain is countable.

3. $\mathbb{P}$ is *non-atomic* if for every $P \in \mathbb{P}$ there are Q and R with $Q \leq P$ and $R \leq P$ and $Q \perp R$.

4. (For arboreal forcings only:) $\mathbb{P}$ is *strongly non-atomic* if for every $P \in \mathbb{P}$ there are Q and R with $Q \leq P$ and $R \leq P$ and $[Q] \cap [R] = \varnothing$.

2. Classwise Consequences of Determinacy

2.1 Regularity Properties.

In the study of the continuum, *regularity properties* of sets of reals play a crucial role. The classical examples are *Lebesgue measurability*, the *Baire property* (BP) and the *Perfect Set Property* (PSP), although many more examples are around. If a set $A \subseteq \eta^\omega$ has a certain regularity property, we can think of it as being natural in some way, exhibiting no counter-intuitive behavior. For instance, for Lebesgue-measurable sets we have an intuitive concept of "size" or "volume"; sets with the Perfect Set Property form a collection for which a restricted version of the Continuum Hypothesis holds. In ZFC we can prove that for each regularity property there are examples of sets that do not possess them, but these proofs are non-constructive and rely on the full power of the Axiom of Choice. It was even proved by Solovay [So70] that under the assumption that an inaccessible cardinal exists, it is consistent with ZF that *all* sets of reals are Lebesgue measurable, have the Baire property and the Perfect Set Property.[5] So AC is fundamentally necessary to produce irregular sets.

In this section we will give definitions for the regularity properties and discuss their relation to forcing notions.

2.1. Definition. Lebesgue-measurability is defined as follows:

- For each basic open $[s]$, define the *Borel measure* by

$$\mu_{\mathcal{B}}([s]) := \begin{cases} \frac{1}{n^{|s|}} & \text{if } \eta = n < \omega \\ \prod_{i<|s|} \frac{1}{2^{s(i)+1}} & \text{if } \eta = \omega \end{cases}$$

  This can be uniquely extended to a measure on the Borel sets, which is also denoted by $\mu_{\mathcal{B}}$. Then

- We call a set $A \subseteq \eta^\omega$ a *null set* if there is a Borel set B such that $A \subseteq B$ and $\mu_{\mathcal{B}}(B) = 0$.

- We call a set $A \subseteq \eta^\omega$ *Lebesgue measurable* if the symmetric difference $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is null for some Borel set B. In this case, the Lebesgue measure $\mu_{\mathcal{L}}$ is defined by $\mu_{\mathcal{L}}(A) := \mu_{\mathcal{B}}(B)$. $\triangleleft$

[5] The assumption of the inaccessible cardinal is not necessary for the Baire property and the Perfect Set Property but *is* necessary for Lebesgue measurability, as [Sh84] shows.

Lebesgue-measurability is a *symmetric* property in the sense that if A is Lebesgue-measurable then so is its complement A^{c}. Moreover, it can be shown that the collection of Lebesgue-measurable sets forms a σ-algebra.

2.2. Definition. We define the Baire property in the general setting of an arbitrary topological space X.

1. A set $A \subseteq X$ is *nowhere dense* if for every open O there is an open $U \subseteq O$ such that $U \cap A = \varnothing$. Equivalently: if $(\overline{A})^\circ = \varnothing$, i.e., the interior of the closure of A is empty.
2. A set $A \subseteq X$ is *meager* if it is a countable union of nowhere dense sets.
3. A set $A \subseteq X$ has the *Baire property* (BP) if for some open set O, the symmetric difference $A \triangle O$ is meager. $\lhd$

The Baire property is also a symmetric property, and a straightforward computation shows that the collection of sets with the Baire property forms a σ-algebra.

2.3. Definition. A set $A \subseteq \eta^\omega$ has the *Perfect Set Property* (PSP) if it is either countable or else contains the branches of a perfect tree, i.e. $[T] \subseteq A$ for a perfect tree T. $\lhd$

Contrary to Lebesgue-measurability and the Baire property, the Perfect Set Property does not have to be symmetric, i.e., if A has the Perfect Set Property, it is not necessary that A^{c} has it as well.[6] In fact, it shall be taken as the prime example of an *asymmetric* regularity property.[7]

Now, since the focus of this thesis are arboreal forcings, we will need to tie this notion together with the regularity properties just discussed. The only property where this connection is not naturally available is Lebesgue-measurability, and for this reason we will treat it only marginally.

The Baire property is naturally connected to *topological* forcings: if $\mathbb{P}$ is a topological forcing notion then $(\eta^\omega, \mathbb{P})$ is a topological space and consequently we can talk of the Baire property in the $\mathbb{P}$-topology. Formally, we will write

$$\mathrm{BP}(\mathbb{P}) := \{A \subseteq \eta^\omega \mid A \text{ has the Baire property in } (\eta^\omega, \mathbb{P})\}$$

The Perfect Set Property talks about perfect sets, and therefore it is naturally connected to Sacks forcing.

[6]In ZF it cannot be proved that PSP is not symmetric in this sense, since it is consistent with ZF that all sets of reals have the PSP. In ZFC, however, our Theorem 5.12 provides a set which does not have the PSP but its complement contains a perfect tree.

[7]The term "asymmetric" here refers both to the fact that the property is not necessarily closed under complements, and that it talks about being "very large or very small".

Except for these classical regularity properties, there is one more property that is especially important when dealing with arboreal forcing notions. This is the so-called "Marczewski-Burstin algebra of measurability", the investigation of which was started in [Bu14, Ma35].

2.4. Definition. Let $\mathbb{P}$ be an arboreal forcing notion. A set $A \subseteq \eta^\omega$ is *Marczewski-Burstin-*$\mathbb{P}$*-measurable*, or shortly MB($\mathbb{P}$)*-measurable* if

$$\forall P \in \mathbb{P}\ \exists Q \in \mathbb{P},\ Q \leq P \text{ s.t. } [Q] \subseteq A \text{ or } [Q] \subseteq A^{\mathsf{c}}$$

The *Marczewski-Burstin algebra of* $\mathbb{P}$ is:

$$\mathrm{MB}(\mathbb{P}) := \{A \subseteq \eta^\omega \mid A \text{ is MB}(\mathbb{P})\text{-measurable}\} \qquad \triangleleft$$

The MB-algebra is also a *symmetric* regularity property. Together with the Baire property for topological forcings, it is the main regularity property, or *algebra of measurability*, that is naturally attached to $\mathbb{P}$. There is a certain interrelation between the two properties and this will be discussed in detail in Chapter 3.

Note also that both Lebesgue measurability and the standard Baire property (BP($\mathbb{C}$)) can naturally be expressed using a Marczewski-Burstin-like characterization.

2.5. Proposition. *A set A is Lebesgue-measurable if and only if for every perfect, Lebesgue-measurable set with non-zero measure P there is a perfect, Lebesgue-measurable set with non-zero measure $Q \subseteq P$ such that $Q \subseteq A$ or $Q \subseteq A^{\mathsf{c}}$.*[8]

Proof. This was first proved by Burstin in [Bu14] and generalized by Reardon in [Re96, Lemma 3.6]. □

2.6. Proposition. *A set A has the (standard) Baire property if and only if for every open U there is an open $V \subseteq U$ such that either $V \cap A$ is meager or $V \setminus A$ is meager.*

Proof. Suppose A has the Baire property, and let O be open such that $M := A \bigtriangleup O$ is meager. Let U be an arbitrary open set. If $O \cap U = \varnothing$ then we let $V := U$ so that clearly $V \cap A \subseteq M$ is meager. Otherwise, let $V := U \cap O$ so that $V \setminus A \subseteq M$ is meager.

Conversely, suppose the right-hand-side condition holds. Let

$$O := \bigcup\{[s] \mid [s] \setminus A \text{ is meager}\}$$

$$O^* := \bigcup\{[s] \mid [s] \cap A \text{ is meager}\}$$

Then

[8] A perfect set is a set P such that $P = [T]$ for a perfect tree T.

1. $O \cup O^*$ is open dense. To see this, let U be open, so by assumption there is an open $V \subseteq U$, and consequently some basic open $[s] \subseteq V$, such that $[s] \cap A$ or $[s] \setminus A$ is meager. Thus, either $[s] \subseteq O^*$ or $[s] \subseteq O$.

   Consequently, the complement $(O \cup O^*)^{\mathsf{c}}$ is nowhere dense.

2. $O \cap O^* = \varnothing$. If not, we would have some $[s]$ and $[t]$ with $[s] \cap [t] \neq \varnothing$, such that $[s] \cap A$ is meager and $[t] \setminus A$ is meager. But then $[s] \cap [t]$ is meager, which is impossible by the Baire Category Theorem.

3. $O \setminus A$ is meager, because, since there are countably many $[s]$'s, it is a union of countably many meager sets. Similarly, $O^* \cap A$ is meager.

Now we have $A \triangle O \subseteq (O \cup O^*)^{\mathsf{c}} \cup (O \setminus A) \cup (O^* \cap A)$, so it is meager. $\square$

2.2 Infinite Games.

The study of infinite games is a very distinct and intriguing field with consequences throughout set theory. The roots of this subject go as far back as Zermelo in 1913 [Ze13], but the crucial step was the introduction of the Axiom of Determinacy in the 1960s by Jan Mycielski and Hugo Steinhaus [MySt62, My64, My66]. Although it took some time for set theorists to notice these developments, it is fair to say that nowadays infinite games have become *the* central tool in descriptive set theory and the study of the continuum.

Informally, an *infinite two-player perfect information game* can be described as follows:

- There are two players: I and II.
- Player I starts by playing an integer $x_0 \in \eta$, followed by player II who plays a $y_0 \in \eta$, and then they take turns in playing $x_n, y_n \in \eta$.
- In the limit, a real is produced: $x := \langle x_0, y_0, x_1, y_1, x_2, y_2, \dots \rangle$.
- For a fixed set $A \subseteq \eta^\omega$, the game $G(A)$ is defined by stating that I wins if and only if $x \in A$ (and II wins iff $x \notin A$.)

Both players I and II may have *strategies*: these are function $\sigma : \eta^{<\omega} \longrightarrow \eta$ which, given as input the play of the game so far, output the next move of the player. A strategy for I (resp. II) is called a *winning strategy in the game $G(A)$* if, no matter what the opponent plays, the real x resulting from following that strategy is in A (resp. not in A).

Consequently, a game $G(A)$ is called *determined* if either I or II has a winning strategy, and a set $A \subseteq \eta^\omega$ is called *determined* if the game $G(A)$ is determined.

For a better intuitive understanding, note that determinacy can be rendered as a sequence of alternating quantifiers. For example, for games of a fixed length n, I having a winning strategy is equivalent to:

$$\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \ (x \in A)$$

and II having a winning strategy is equivalent to:

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \dots \forall x_n \exists y_n \ (x \notin A)$$

so that *determinacy* becomes a tautology "$\phi \vee \neg\phi$". In the scenario of infinitary logic, one might even think of the sentences:

$$\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots \ (x \in A)$$

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \dots \ (x \notin A)$$

so that, via an infinitary switch of quantifiers, determinacy of infinite games would also become a logical tautology.[9] But in the case of ordinary logic and set theory, determinacy is not self-evident.

Although the informal definition is useful, we will present another, more rigorous definition, which is easily seen to be equivalent but has the advantage of being more "mathematical" and perspicuous, especially when proving pointwise non-implications with which the rest of the thesis is concerned. Interestingly, this definition allows us to skip the actual games, moving straight on to strategies.

2.7. Definition.

- A tree on η is a *strategy for player I* if every node of even length is non-splitting and every node of odd length is totally splitting.
- A tree on η is a *strategy for player II* if every node of odd length is non-splitting and every node of even length is totally splitting.
- A strategy σ for I is *winning for I in the game* $G(A)$ if $[\sigma] \subseteq A$, and a strategy τ for II is *winning for II in the game* $G(A)$ if $[\tau] \subseteq A^{\mathsf{c}}$.
- A set $A \subseteq \eta^\omega$ is *determined* if either I or II has a winning strategy in $G(A)$, i.e.

$$\exists \sigma \text{ for } I \text{ s.t. } [\sigma] \subseteq A \text{ or}$$

$$\exists \tau \text{ for } II \text{ s.t. } [\tau] \subseteq A^{\mathsf{c}}$$

- The collection of all determined sets will be denoted by D. $\triangleleft$

[9] Although an infinitary switch of quantifiers is not an ordinary rule even in infinitary logic.

The strength of this whole theory is that we do not need to limit ourselves to games of the kind described above. Instead, we can let players choose elements of any arbitrary set X instead of integers, arbitrary rules can be imposed on the players during the course of the game, and an arbitrary "winning condition" can be specified. As long as the set X is countable, any such game can be encoded to form a standard game (as in Definition 2.7). Moreover, since the notions of strategies, winning strategies and determinacy are invariant under bijections, the determinacy of any such game is related to determinacy in the sense of Definition 2.7 via an encoding of the set A in question.

As an example, let's look at the Banach-Mazur game, or $**$-game, denoted by $G^{**}(A)$. The rules are as follows: I and II take turns in playing non-empty finite sequences of integers, i.e., elements of $\eta^{<\omega} \setminus \{\varnothing\}$: s_0, t_0, s_1, t_1 etc. In the limit, they construct the real $x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and I wins iff $x \in A$.

Now fix a bijection $\pi : \eta^{<\omega} \longleftrightarrow \omega$ and, to every set $A \subseteq \eta^\omega$ associate the set $A' \subseteq \omega^\omega$ defined by

$$A' := \{x \in \omega^\omega \mid \pi^{-1}(x(0))^\frown \pi^{-1}(x(1))^\frown \pi^{-1}(x(2))^\frown \dots \in A\}$$

We can then view π as a coding function, and every play of the Banach-Mazur game corresponds, via π, to a play in an ordinary game. This carries through to strategies, winning strategies etc. so that consequently we have:

$$G^{**}(A) \text{ is determined iff } G(A') \text{ is determined.}$$

Finally, let us mention the following:

2.8. Definition. The Axiom of Determinacy AD is the statement:

"Every set of reals is determined" $\lhd$

Although this axiom plays an important role in determinacy theory and descriptive set theory, we shall not adopt it here. AD is inconsistent with the Axiom of Choice, and is usually taken as a natural axiom in the context of ZF alone. Since our point of view throughout this paper is ZFC, determinacy is to be considered as another regularity property D, one which holds for many natural sets but not for all sets.

2.3 Classwise Implications.

As this thesis is largely about the difference between pointwise and classwise implications, let us make that more precise. A set $\mathbf{\Gamma} \subseteq \mathscr{P}(\eta^\omega)$ is called a *boldface pointclass* if it is closed under continuous pre-images, i.e., if f is continuous and

$A \in \mathbf{\Gamma}$ then $f^{-1}[A] \in \mathbf{\Gamma}$. Canonical examples of such pointclasses are, of course, complexity classes of the projective hierarchy $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.[10] When dealing with a regularity property Reg, it is natural to ask the question whether all sets from some $\mathbf{\Gamma}$ are regular, i.e., whether $\mathbf{\Gamma} \subseteq \mathsf{Reg}$. For larger $\mathbf{\Gamma}$, such questions are often very subtle, and in many cases independent from ZFC. Certain propositions of the kind "$\mathbf{\Gamma} \subseteq \mathsf{Reg}$" can even be considered strong hypotheses and assumed as additional axioms to prove results.

2.9. Definition. Let $\mathsf{A} \subseteq \mathscr{P}(\eta^\omega)$ and $\mathsf{B} \subseteq \mathscr{P}(\eta^\omega)$ be two collections/properties of sets of reals. We say:

"A *implies* B *pointwise*"	if	$\mathsf{A} \subseteq \mathsf{B}$
"A *implies* B *classwise*"	if	for all boldface pointclasses $\mathbf{\Gamma}$ $(\mathbf{\Gamma} \subseteq \mathsf{A} \to \mathbf{\Gamma} \subseteq \mathsf{B})$ $\lhd$

Occasionally, we will need to put additional requirements on $\mathbf{\Gamma}$ (for example, closure under intersections with basic open sets). We will then also talk about "classwise implications" although technically speaking this is weaker than the classwise implications according to Definition 2.9.

We will be particularly interested in the statements "D implies Reg pointwise" versus "D implies Reg classwise", where D is the collection of determined sets and Reg some regularity property.

The classical regularity properties, as well as many other properties, follow from the determinacy of specific games, i.e., there is a game $G^\star$ such that $A \in \mathsf{Reg}$ iff $G^\star(A)$ is determined (in this case, we call $G^\star$ a *game representation of* Reg). These games are usually not standard integer-games as in Definition 2.7, though they can be encoded in that form. The encoding process, however, alters the set A in question, so that we have $A \in \mathsf{Reg}$ iff $G^\star(A)$ is determined iff $G(A')$ is determined, where A' is some set derived from A (as in the example directly above Definition 2.8). Fortunately, the encoding is usually such that if $A \in \mathbf{\Gamma}$ then also $A' \in \mathbf{\Gamma}$ (because the functions involved in the encoding are continuous).

That is why determinacy has mostly classwise rather than pointwise consequences. Listing the abundance of these classwise consequences is beyond the scope of this work, so let us just mention the classical examples.

2.10. Theorem (Mycielski-Swierczkowski, 1964). *Determinacy implies Lebesgue measurability classwise.*

2.11. Theorem (Mazur, Banach, 1935). *Determinacy implies the Baire property classwise.*

2.12. Theorem (Davis, 1964). *Determinacy implies the Perfect Set Property classwise.*

[10] For a definition, see e.g. [Ke95, p. 313 ff].

The original proofs are in [MySw64], [Mau81, p. 113 ff] and [Da64], respectively. A modern survey of all three results can be found e.g. in [Ka94, pp. 373–377]. Each of these proofs uses a specific type of game. The proof of Theorem 2.11 uses the Banach-Mazur game $G^{**}(A)$ mentioned earlier, and the game used in Theorem 2.12 will be discussed in detail in Chapter 5.

Note that the Axiom of Determinacy can be viewed as a very strong pointclass inclusion "$\mathscr{P}(\eta^\omega) = \mathsf{D}$", concerning the "pointclass of all sets". According to the three theorems, therefore, AD (in ZF) implies that all sets of reals are Lebesgue measurable, have the Baire property, and the Perfect Set Property—and indeed this is the formulation in which the theorems were originally proved and are still usually presented.

Concerning our other regularity property, the Marczewski-Burstin algebra, there is a slight subtlety. It was proved in [Lö98] that in a certain sense determinacy implies MB($\mathbb{P}$) classwise, but the games used to prove this are based on *real* rather than integer moves, i.e., the players are allowed to choose from sets X of cardinality $2^{\aleph_0}$. Such games are not equivalent to integer games, and determinacy in that sense is not equivalent to determinacy in the sense of Definition 2.7.

Whether or not the determinacy of *integer* games (i.e., determinacy in the sense of Definition 2.7) implies MB($\mathbb{P}$) classwise is in some cases still an open problem, and is related to the well-known open problem whether AD implies the property of being completely Ramsey for all sets of reals.[11]

In contrast to these classwise consequences of determinacy, we can ask ourselves whether determinacy implies any of the regularity properties *pointwise*, that is, whether for instance $\mathsf{D} \subseteq$ BP, $\mathsf{D} \subseteq$ PSP etc. In most cases, the answer will be "no". To prove pointwise non-implication one must, first of all, find counterexamples to regularity properties. As mentioned earlier, this can only be done using the Axiom of Choice, and the generic method for this is the *Bernstein construction.*

Because this construction is central to all our results, let us end this chapter by giving its original application.

2.13. Theorem. *There is a set $A \subseteq \eta^\omega$ such that neither A nor its complement A^c contains a perfect tree.*

Proof. Since each tree is a subset of $\eta^{<\omega}$ there are at most $2^{\aleph_0}$ trees. Using AC, fix an enumeration of all perfect trees: $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$. By transfinite recursion on $2^{\aleph_0}$, we construct A_α and B_α in parallel. At each step, we take care that $|A_\alpha| = |B_\alpha| = |\alpha|$.

[11] *Cf.* [Ka94, Question 27.18]. The property of being *completely Ramsey* is just another term for the Marczewski-Burstin algebra of Mathias forcing, MB($\mathbb{R}$). Because the other MB-algebras are very similar in nature, it is likely that the problem is equally difficult to solve for the other cases as well. Of course, there are forcings $\mathbb{P}$ for which this is not an open problem, namely those for which MB($\mathbb{P}$) is not a σ-algebra and for which one can easily construct counterexamples (in ZF).

- $A_0 = B_0 := \varnothing$.
- Suppose for $\alpha < 2^{\aleph_0}$ that A_α and B_α are already defined. By induction, $|A_\alpha \cup B_\alpha| = |\alpha| + |\alpha| = |\alpha| < 2^{\aleph_0}$. Since $|[T_\alpha]| = 2^{\aleph_0}$, it follows that $|[T_\alpha] \setminus (A_\alpha \cup B_\alpha)| = 2^{\aleph_0}$. In particular, we can choose two distinct $a_\alpha, b_\alpha \in [T_\alpha] \setminus (A_\alpha \cup B_\alpha)$. Then we let $A_{\alpha+1} := A_\alpha \cup \{a_\alpha\}$ and $B_{\alpha+1} := B_\alpha \cup \{b_\alpha\}$.
- If $\lambda < 2^{\aleph_0}$ is a limit ordinal, let $A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha$ and $B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha$.
- Finally, set $A := \bigcup_{\alpha<2^{\aleph_0}} A_\alpha$ and $B := \bigcup_{\alpha<2^{\aleph_0}} B_\alpha$.

By construction, it follows that $A \cap B = \varnothing$. Then, for every perfect tree T_α, $[T_\alpha] \not\subseteq A$ because $b_\alpha \in [T_\alpha] \cap B \subseteq [T_\alpha] \setminus A$, and also $[T_\alpha] \not\subseteq A^{\mathsf{c}}$ because $a_\alpha \in [T_\alpha] \cap A$. This completes the proof. $\square$

The set A constructed in the proof is called a *Bernstein set*, and the sets A and B together are called *Bernstein components* (of $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$).

The theorem has some immediate consequences: because a Bernstein set A is clearly uncountable, it does not have the Perfect Set Property. Also, since strategies are perfect trees, neither A nor its complement can contain any strategy, so A is not determined. Therefore, the Axiom of Choice implies that there are sets without the Perfect Set Property, and not determined sets.

Also, since the proof of Theorem 2.13 did not rely on any specific property of perfect trees except that their cardinality is $2^{\aleph_0}$, the same argument clearly applies to any collection of sets of cardinality $2^{\aleph_0}$ which can be well-ordered in order-type $2^{\aleph_0}$.

2.14. Theorem. *Let $\langle X_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ be a collection with $|X_\alpha| = 2^{\aleph_0}$ for all $\alpha < 2^{\aleph_0}$. Then there exist two Bernstein components A and B of $\langle X_\alpha \mid \alpha < 2^{\aleph_0} \rangle$, i.e.:*

- *$A, B \subseteq \bigcup_{\alpha<2^{\aleph_0}} X_\alpha$*
- *$A \cap B = \varnothing$ and*
- *for every $\alpha < 2^{\aleph_0}$ we have $A \cap X_\alpha \neq \varnothing$ and $B \cap X_\alpha \neq \varnothing$.*

Proof. Analogous to Theorem 2.13. $\square$

3. The Marczewski-Burstin Algebra and the Baire Property

In this chapter we will deal with the first of our three main questions: the difference between topological and non-topological forcings and the relationship between MB and BP.

As already mentioned, every arboreal forcing notion has a corresponding (symmetric) regularity property, or a so-called *algebra of measurability*. In practice, the following definition is always used: if $\mathbb{P}$ is a topological notion, the algebra of measurability is the Baire property $\mathrm{BP}(\mathbb{P})$ and if $\mathbb{P}$ is non-topological it is the Marczewski-Burstin algebra $\mathrm{MB}(\mathbb{P})$. This is, for example, clearly evident from the various Solovay-style characterizations and related results (*cf.* [So69, BrLö99, BrHaLö05, Ik06]). Why is there this strange dichotomy? Of course, the Baire property is only available if there is a topology, but the MB-algebra seems to be available no matter what. However, taking the simplest example of a topological forcing, $\mathbb{C}$, there is an immediate problem: the set of rationals $\mathbb{Q}$ is not in $\mathrm{MB}(\mathbb{C})$, since it is clearly not the case that for any basic open $[s]$ there is a basic open $[t] \subseteq [s]$ with $[t] \subseteq \mathbb{Q}$ or $[t] \subseteq \mathbb{Q}^{\mathsf{c}}$. Therefore $\mathrm{MB}(\mathbb{C})$ is not a σ-algebra, it does not contain all F_σ and G_δ sets, and in fact isn't really a "regularity property" at all. The same holds for $\mathrm{MB}(\mathbb{D})$ and $\mathrm{MB}(\mathbb{E})$ as we will see.

Thus, the following natural questions arise:

1. When is the MB-algebra a good regularity property (a σ-algebra)?
2. Is there an inherent reason why $\mathrm{MB}(\mathbb{P})$ is the "correct" notion of measurability for non-topological forcings, whereas $\mathrm{BP}(\mathbb{P})$ is the "correct" one for topological ones?
3. In the case of topological forcings $\mathbb{P}$, what is the relationship between $\mathrm{BP}(\mathbb{P})$ and $\mathrm{MB}(\mathbb{P})$?

We shall give a complete answer to questions 2 and 3 and a partial one to question 1. Whenever necessary, we will refer to these as Main Questions 1.1, 1.2 and 1.3.

3.1 MB and BP.

We start by proving a general but important result about the topological spaces $(\eta^\omega, \mathbb{P})$, i.e., the sets η^ω endowed with the topology generated by a topological forcing $\mathbb{P}$.

3.1. Definition. A topological space is called *a Baire space*[12] if no non-empty open set is meager, or, equivalently, if the intersection of countably many open dense sets is dense. ◁

The classical Baire Category Theorem showed that the standard real numbers with the usual topology form a Baire space. Modern versions of this theorem say the same thing for completely metrizable spaces and locally compact Hausdorff spaces (see e.g. [Ke95, Theorem 8.4]). Since $(\eta^\omega, \mathbb{P})$ is in general neither metrizable nor locally compact,[13] we will need to prove another version which applies to the present setting. It relies on the fact that all basic open sets are trees.

3.2. Theorem. *Let $\mathbb{P}$ be any topological forcing notion on η^ω. Then $(\eta^\omega, \mathbb{P})$ is a Baire space.*

Proof. Let $\{O_n \mid n \in \omega\}$ be a sequence of open dense sets, and let $A := \bigcap_n O_n$. We must show that A is dense. Let U be an arbitrary open set. Inductively, define a sequence $P_0 \gg P_1 \gg P_2 \gg \dots$ from $\mathbb{P}$ as follows: let $[P_0'] \subseteq U$ be arbitrary, and since O_0 is open dense, let $[P_0] \subseteq [P_0'] \cap O_0$. For a given P_n, first extend it to a P_{n+1}' with strictly longer stem, i.e. $P_{n+1}' \ll P_n$ (this is possible by definition of arboreal forcings). Then, since O_{n+1} is open dense, let $[P_{n+1}] \subseteq [P_{n+1}'] \cap O_{n+1}$.

Since we have a sequence with increasing stems, we can define $x := \bigcup_n \mathrm{stem}(P_n)$, which is a real number. But then, we claim that for all n and k, we have $x \restriction k \in P_n$. Let m be such that $x \restriction k \subseteq \mathrm{stem}(P_m)$. If $n \leq m$ then $P_m \subseteq P_n$, so $x \restriction k \subseteq \mathrm{stem}(P_m) \in P_m \subseteq P_n$. And if $m < n$ then $x \restriction k \subseteq \mathrm{stem}(P_m) \subseteq \mathrm{stem}(P_n) \in P_n$.

Therefore $x \in [P_n]$ for all n. Hence, $x \in [P_0] \subseteq U$ and $x \in [P_n] \subseteq O_n$ for all n. Therefore $x \in U \cap A$, as had to be shown. □

Next, we discuss the connection between the MB-algebra and the Baire property. Suppose $\mathbb{P}$ is a topological forcing. Note that $A \in \mathrm{MB}(\mathbb{P})$ is equivalent to saying:

For all open O there is an open $U \subseteq O$ such that $U \subseteq A$ or $U \subseteq A^{\mathsf{c}}$.

Therefore we easily get the following:

3.3. Proposition. $\mathrm{MB}(\mathbb{P}) \subseteq \mathrm{BP}(\mathbb{P})$.

Proof. Let $A \in \mathrm{MB}(\mathbb{P})$, and let $B := A \setminus A^\circ$. We claim that B is nowhere dense. For any open O there is an open $U \subseteq O$ such that $U \subseteq A$ or $U \subseteq A^{\mathsf{c}}$.

[12]The terminology is slightly confusing because "*the* Baire space" usually refers only to ω^ω, but "*a* Baire space" can be any topological space, e.g., 2^ω.

[13]For example the dominating topology $\mathbb{D}$ is not metrizable since it's not second countable. The space $(\omega^\omega, \mathbb{C})$ is not locally compact because it is a product of infinitely many non-compact spaces. (This is not the case for $(n^\omega, \mathbb{C})$).

The former implies $U \subseteq A^\circ$, so in either case $U \cap B = \varnothing$. Writing $A \triangle A^\circ = B$, we have that A° is open and B is nowhere dense, hence meager. Therefore A has the Baire property. $\square$

The converse direction does not always hold. In the example given above, $\mathrm{MB}(\mathbb{C})$ is not a σ-algebra and is, relatively speaking, a rather small collection. What about topological forcings in general? We prove the following characterization:[14]

3.4. Theorem. *Let $\mathbb{P}$ be a topological forcing notion. Then the following are equivalent:*

1. $\mathrm{MB}(\mathbb{P})$ *is a σ-algebra.*
2. $\mathrm{MB}(\mathbb{P})$ *contains all meager sets.*
3. *All meager sets are nowhere dense.*
4. $\mathrm{MB}(\mathbb{P}) = \mathrm{BP}(\mathbb{P})$.

Proof. $1 \Longrightarrow 2$: Clearly, every nowhere dense set A is in $\mathrm{MB}(\mathbb{P})$ since for every open O there is an open $U \subseteq O$ such that $U \subseteq A^{\mathsf{c}}$. If $\mathrm{MB}(\mathbb{P})$ is closed under countable unions, it follows that every meager set is also in $\mathrm{MB}(\mathbb{P})$.

$2 \Longleftrightarrow 3$: Let A be meager. Since it is in $\mathrm{MB}(\mathbb{P})$, for every open O there is an open $U \subseteq O$ such that either $U \subseteq A$ or $U \subseteq A^{\mathsf{c}}$. But the former is impossible, since then U would be an open meager set, contradicting Theorem 3.2. Therefore the latter holds, which means that A is nowhere dense. The converse direction is obvious since $\mathrm{MB}(\mathbb{P})$ contains all nowhere dense sets.

$2 \Longrightarrow 4$: We know that $\mathrm{MB}(\mathbb{P}) \subseteq \mathrm{BP}(\mathbb{P})$ so it remains to prove the reverse inclusion. Suppose A has the Baire property. Then there is some open set O such that $A \triangle O$ is meager, hence in $\mathrm{MB}(\mathbb{P})$. Now, given any U, there is a $U' \subseteq U$ such that either (a) $U' \subseteq A \triangle O$ or (b) $U' \subseteq (A \triangle O)^{\mathsf{c}}$. Also, since O is open, there is a $U'' \subseteq U'$ such that either (i) $U'' \subseteq O$ or (ii) $U'' \subseteq O^{\mathsf{c}}$. Then

Cases (a) + (i)	imply	$U'' \subseteq A^{\mathsf{c}}$
Cases (a) + (ii)	imply	$U'' \subseteq A$
Cases (b) + (i)	imply	$U'' \subseteq A$
Cases (b) + (ii)	imply	$U'' \subseteq A^{\mathsf{c}}$

This shows that $A \in \mathrm{MB}(\mathbb{P})$.

$4 \Longrightarrow 1$: this is immediate because $\mathrm{BP}(\mathbb{P})$ is always a σ-algebra. $\square$

This characterization at once answers a number of the questions originally posed. First of all, it shows that $\mathrm{MB}(\mathbb{P})$ cannot be a σ-algebra strictly smaller

[14] This result is partly based on Ellentuck's proof [El74] that the completely Ramsey sets are precisely those with the Baire property in the Ellentuck topology.

than BP($\mathbb{P}$): in other words, either MB($\mathbb{P}$) is "bad", in which case it is a useless notion of measurability, or it is "good", in which case it is just BP($\mathbb{P}$) anyway. The practical consequence of this is that we never need to use MB($\mathbb{P}$) as a notion of measurability when $\mathbb{P}$ is topological.

Secondly, it indicates why MB($\mathbb{P}$) tends to behave badly for topological $\mathbb{P}$: condition 3 of the theorem is, of course, rather unnatural, and it is clear why many ordinary topologies would not satisfy it. Note that an equivalent formulation of this condition would be: "a countable intersections of open dense sets is open dense."

This practically settles Questions 1.2 and 1.3 above. It also answers Question 1.1 for topological $\mathbb{P}$, but we would still like to have a more perspicuous explanation, which applies to all $\mathbb{P}$, for the reason that MB($\mathbb{P}$) is a σ-algebra in some cases and not a σ-algebra in other cases. This question turns out to be more difficult than expected, but in the next two sections we can give at least a partial answer.

3.2 Fusion Sequences.

In the literature, there is a heuristic method of proving that a given MB($\mathbb{P}$) is a σ-algebra (the method is more commonly used for other technical purposes in forcing theory). We shall give a general criterion in order for this method to work. On the other hand, we can generalize the method used to prove that MB($\mathbb{C}$), MB($\mathbb{D}$) and MB($\mathbb{E}$) are *not* σ-algebras. Together, these two results give easily verifiable criteria for proving either one case or the other, but unfortunately we will not be able to give an exhaustive characterization. However, one should note that all the standard arboreal forcing notions fall into one of these two categories, i.e., either one or the other of the verifiable criteria is satisfied.

We begin with the method of *fusion sequences*. Although it is used in the existing literature (e.g. [Je86, p. 15 ff]) it has never been formally defined in an abstract setting. This is the definition we propose:

3.5. Definition. Let $(\mathbb{P}, \leq)$ be an arboreal forcing notion.

1. Given a $P \in \mathbb{P}$, we say that a set $A \subseteq \eta^\omega$ is *dense below* P if for all $P' \leq P$ there exists a $P'' \leq P'$ such that $[P''] \subseteq A$.[15]
2. Let $\{\leq_n\}_{n\in\omega}$ be a sequence of sub-relations of the main relation $\leq$. If this sequence has the property that whenever $P_0 \geq_0 P_1 \geq_1 P_2 \geq_2 \dots$ then $P := \bigcap_n P_n \in \mathbb{P}$, then we say that $\{\leq_n\}_{n\in\omega}$ *has the fusion property*. A

[15] Translating this into standard forcing terminology, this would mean that $D_A := \{Q \in \mathbb{P} \mid [Q] \subseteq A\}$ is dense below P.

particular sequence $P_0 \geq_0 P_1 \geq_1 P_2 \geq_1 \ldots$ is then called a $\{\leq_n\}_{n\in\omega}$-*fusion sequence.*

We say that the sequence $\{\leq_n\}_{n\in\omega}$ *has the amalgamation property* if for each $n \in \omega$ and $P \in \mathbb{P}$, if A is dense below P then there is a $Q \leq_n P$ such that $[Q] \subseteq A$.

3. Finally, we say that $(\mathbb{P}, \leq)$ *has the fusion and the amalgamation property* if there is a sequence $\{\leq_n\}_{n\in\omega}$ with the fusion property and the amalgamation property. $\triangleleft$

3.6. Theorem. *Let $\mathbb{P}$ be an arboreal forcing notion. If $\mathbb{P}$ has the fusion and amalgamation property then* $\mathrm{MB}(\mathbb{P})$ *is a σ-algebra.*

Proof. Since $\mathrm{MB}(\mathbb{P})$ is closed under complements by definition, it suffices to show closure under countable intersections. Let $A_0, A_1, \ldots \in \mathrm{MB}(\mathbb{P})$. Fix some $Q \in \mathbb{P}$. The goal is to find a $P \leq Q$ such that $[P] \subseteq \bigcap_n A_n$ or $[P] \subseteq (\bigcap_n A_n)^{\mathsf{c}}$, so that $\bigcap_n A_n \in \mathrm{MB}(\mathbb{P})$.

Note that if we ever find a $P \leq Q$ such that $[P] \subseteq A_n^{\mathsf{c}}$ for any n, then $[P] \subseteq \bigcup_n A_n^{\mathsf{c}} = (\bigcap_n A_n)^{\mathsf{c}}$ so we are done. Therefore, in the following we shall assume that that is not the case. Inductively, we will build a fusion sequence in such a way that $[P_n] \subseteq A_n$ for all n.

Let $P_0 \in \mathbb{P}$ be such that $P_0 \leq Q$ and $[P_0] \subseteq A_0$. For each n, suppose $[P_n] \subseteq A_n$. Since $A_{n+1} \in \mathrm{MB}(\mathbb{P})$, for each $P' \leq P_n$ there exists a $P'' \leq P'$ such that $[P''] \subseteq A_{n+1}$ or $[P''] \subseteq A_{n+1}^{\mathsf{c}}$ — but the latter is impossible since it contradicts our assumption above. Therefore A_{n+1} is dense below P_n. Using the amalgamation property, there is a $P_{n+1} \leq_n P_n$ such that $[P_{n+1}] \subseteq A_{n+1}$. This gives us a fusion sequence

$$Q \geq P_0 \geq_0 P_1 \geq_1 P_2 \geq_2 \ldots$$

so $P := \bigcap_n P_n \in \mathbb{P}$ and $[P] \subseteq [P_n] \subseteq A_n$ for all n, so $[P] \subseteq \bigcap_n A_n$. $\square$

Thus, we have reduced the question of $\mathrm{MB}(\mathbb{P})$ being a σ-algebra to an abstract combinatorial property of the forcing partial order. In concrete applications, one still needs to find a way to make the conditions of the theorem true. Typically, the sub-relations $\leq_n$ are explicitly defined. The idea is that if $P \leq_n Q$ then, though P is a smaller tree than Q, it is not too much smaller. In this way it guarantees that in the limit we still get a tree of the desired type.

Perhaps some explanation for the term "amalgamation" is also required: in our abstract definition we did not state how the Q in question should be constructed. In practice, however, this is usually done by taking all the trees that are produced by the property of being *dense below* P, and somehow "amalgamating" these into one tree $Q \leq_n P$.

All of this is best clarified in the form of a few examples.

3.7. Proposition.

1. *Sacks forcing* $\mathbb{S}$ *has the fusion and amalgamation property.*
2. *Miller forcing* $\mathbb{M}$ *has the fusion and amalgamation property.*

Proof. The proof is essentially taken from [Je86, p. 15 ff].

1. For two perfect trees S and T, define $T \leq_n S$ iff $T \leq S$ and every n-th splitting node of S is in T and is a splitting node of T (and $S \leq_0 T$ iff $S \leq T$). To see that $T_0 \geq_0 T_1 \geq_1 T_2 \geq_2 \ldots$ is a fusion sequence, let $T := \bigcap_n T_n$. Consider any $t \in T$, and let n be the number of splitting nodes before t. Then, since $T_{n+1} \subseteq T$, there are also n splitting nodes before t in T_{n+1} and, since T_{n+1} is perfect, let s be the first splitting node extending t. Then s is actually the $(n+1)$-th splitting node of T_{n+1} and so it is a splitting node of T_{n+2}. Then for all $i > n+2$, s is also a splitting node of T_i and hence s is a splitting node of T. This proves that T is a perfect tree.

   Now, let us check amalgamation: For any n, $A \subseteq \eta^\omega$ and $T \in \mathbb{S}$ s.t. A is dense below T, let $t_1, \ldots, t_k$ be all the successors of all the n-th splitting nodes of T. Then $T \uparrow t_i \leq T$ so there is an $S_i \leq T \uparrow t_i$ with $[S_i] \subseteq A$. Then $S := S_1 \cup \cdots \cup S_k$ is a perfect tree, $S \leq_n T$ and $[S] \subseteq A$. (Here, we have "amalgamated" $S_1, \ldots, S_k$ to form S.)

2. Miller forcing is entirely analogous, except that there are ω-many successors $t_0, t_1, \ldots$ of all the n-th splitting nodes, but that does not affect the rest of the argument. □

Using a similar, but more technically involved method, we can prove that all the other non-topological forcing notions have the fusion and amalgamation property. We will not give the details here, referring instead to [Je86, p. 15 ff]. Let us just note one other case of interest, namely Mathias forcing $\mathbb{R}$. It turns out that Mathias forcing also has the fusion and amalgamation property, and hence that MB($\mathbb{R}$) is a σ-algebra. This is the only case of a *topological* forcing notion where MB($\mathbb{R}$) is indeed a σ-algebra. Consequently, this implies that meager sets in the Ellentuck topology are nowhere dense in the Ellentuck topology.

3.8. Proposition. *Mathias forcing* $\mathbb{R}$ *has the fusion and amalgamation property.*

Proof. See [Ke95, pp. 133–134] and [Je86, pp. 17–19]. □

3.3 Counter-examples.

We have given an easily verifiable criterion which guarantees that MB($\mathbb{P}$) is a σ-algebra. We would like to find a similar, easily verifiable criterion which guarantees that MB($\mathbb{P}$) is *not* a σ-algebra? In the case of Cohen forcing, we have already done that by specifying the counter-example $\mathbb{Q}$. Here we shall do the same for $\mathbb{D}$ and $\mathbb{E}$, and then give an indication how this can be generalized to form an easily verifiable criterion for $\mathbb{P}$ in general.

Let us start with Hechler forcing $\mathbb{D}$. Here, we cannot use a counter-example as simple as with $\mathbb{C}$ because every countable set is nowhere dense in the dominating topology. Nevertheless, we do have a nice counter-example:

3.9. Theorem. MB($\mathbb{D}$) *is not a σ-algebra.*

Proof. For each $N < \omega$, let

$$A_N := \{x \mid \forall n \geq N,\ x(n) \text{ is even}\}$$

Each A_N is nowhere dense since any (s, f) can be extended to a (t, g) such that $t(n)$ is odd for some $n \geq N$, and then $[t, g] \cap A_N = \varnothing$. However

$$A := \bigcup_N A_N = \{x \mid \forall^\infty n\ (x(n) \text{ is even})\}$$

i.e., the set of those x which are eventually even, is clearly not MB-measurable: given any (s, f) we can extend f outside s to some function having only even values, and also to one having only odd values. So $[s, f] \not\subseteq A$ and $[s, f] \not\subseteq A^{\mathsf{c}}$, and since this holds for any (s, f), clearly $A \notin$ MB($\mathbb{D}$). $\square$

The situation with eventually different forcing $\mathbb{E}$ is analogous to $\mathbb{D}$.

3.10. Theorem. MB($\mathbb{E}$) *is not a σ-algebra.*

Proof. Define A_N identically as in Theorem 3.9; then the A_N are nowhere dense, since every (s, F) can be extended to a (t, F) such that $t(n)$ is odd for some $n \geq N$ (this is because F, and hence $\{f(n) \mid f \in F\}$, is finite). On the other hand, $A := \bigcup_N A_N$ is not in MB($\mathbb{E}$) for the same reason as with $\mathbb{D}$. $\square$

This idea can be generalized, even beyond the scope of topological forcings. For this, we introduce the following definitions.

3.11. Definition. Let $(\mathbb{P}, \leq)$ be an arboreal forcing notion and $\{X_n\}_{n\in\omega}$ a collection of subsets of η.

1. We say that $\{X_n\}_{n\in\omega}$ has the *tail property relative to* $\mathbb{P}$ if
   - $\forall P \in \mathbb{P}\ \exists x \in [P]$ such that $\forall^\infty n\ (x(n) \in X_n)$ and
   - $\forall P \in \mathbb{P}\ \exists y \in [P]$ such that $\neg\forall^\infty n\ (y(n) \in X_n)$.
2. We say that $\{X_n\}_{n\in\omega}$ has the *stem property relative to* $\mathbb{P}$ if
   - $\forall P \in \mathbb{P}\ \forall n < \omega$, $\exists Q \leq P$ such that $|\mathrm{stem}(Q)| > n$ and $\mathrm{stem}(Q)(n) \notin X_n$.
3. We say that $(\mathbb{P}, \leq)$ has the *pruning property* if
   - $\forall P \in \mathbb{P}\ \forall t \in P\ \exists Q \leq P$ such that $t \subseteq \mathrm{stem}(Q)$. $\lhd$

3.12. Lemma. *If* $(\mathbb{P}, \leq)$ *has the pruning property and* $\{X_n\}_{n\in\omega}$ *has the tail property relative to* $\mathbb{P}$*, then* $\{X_n\}_{n\in\omega}$ *has the stem property relative to* $\mathbb{P}$*.*

Proof. Fix a $P \in \mathbb{P}$ and an $n < \omega$. By the tail property, there is a $y \in [P]$ such that $\neg\forall^\infty i\ (y(i) \in X_i)$. Hence there is an $m \geq n$ such that $y(m) \notin X_m$. Now let $t := y \restriction (m+1)$. By the pruning property, there is a $Q \leq P$ such that $t \subseteq \mathrm{stem}(Q)$. This is the Q we needed to find. Hence, $\{X_n\}_{n\in\omega}$ has the stem property. $\square$

The method of showing that a certain MB-algebra is not a σ-algebra is then given in the following theorem:

3.13. Theorem. *Let* $\mathbb{P}$ *be an arboreal forcing notion. If there is a collection* $\{X_n \subseteq \eta\}_{n\in\omega}$ *which has both the tail and the stem property relative to* $\mathbb{P}$*, then* $\mathrm{MB}(\mathbb{P})$ *is not a* σ*-algebra.*

Proof. Define
$$A_N := \{x \in \omega^\omega \mid \forall n \geq N\ (x(n) \in X_n)\}$$
$$A := \bigcup_N A_N = \{x \in \omega^\omega \mid \forall^\infty n\ (x(n) \in X_n)\}$$
By the stem property, each $A_N \in \mathrm{MB}(\mathbb{P})$ since for all $P \in \mathbb{P}$ there is a $Q \leq P$ with $|\mathrm{stem}(Q)| > N$ and $\mathrm{stem}(Q)(N) \notin X_N$, which implies that $[Q] \cap A_N = \varnothing$.

However, from the tail property it directly follows that for all $P \in \mathbb{P}$, $[P] \nsubseteq A$ and $[P] \nsubseteq A^\mathsf{c}$. Since this holds for all P, it cannot be that A is in $\mathrm{MB}(\mathbb{P})$. Therefore $\mathrm{MB}(\mathbb{P})$ is not a σ-algebra. $\square$

Note that all standard arboreal forcings have the pruning property, which is a very natural property for forcing notions to have. Therefore, in all standard cases the tail property is sufficient to prove that $\mathrm{MB}(\mathbb{P})$ is not a σ-algebra.

Conclusion, indication of further research.

We have found a clear relationship between $\mathrm{MB}(\mathbb{P})$ and $\mathrm{BP}(\mathbb{P})$ and answered Questions 1.2 and 1.3 posed at the start. We also found a (relatively) easily verifiable condition on $\mathbb{P}$ which guarantees that $\mathrm{MB}(\mathbb{P})$ is a σ-algebra, and a contrary condition which guarantees that it is not, thus partially answering Question 1.1. But it still remains to find an exhaustive characterization, which could be a suggestion for further research.

Moreover, the condition of *having the fusion and amalgamation property* used in Theorem 3.6 is very abstract and not immediately suitable for applications. It would be better to replace this with some condition of which it is immediately clear whether $\mathbb{P}$ satisfies it or not. This is something that future research might be able to provide. On the other hand, the fact that proving the result for Mathias forcing is so involved suggests that perhaps a convenient characterization does not exist and that, ultimately, each particular forcing partial order $\mathbb{P}$ has to be studied individually. In any case, a general theory along these lines should be at least as complex as the proof of Proposition 3.8.

4. Determinacy and the Baire Property.

In Section 2.3. we saw how determinacy implies most regularity properties classwise. We now come to Main Question 2, and thereby to the main topic of this thesis, namely *pointwise (non-)implications*. Of course, as we saw Section 2.3, proving regularity properties from determinacy usually involves a game characterization for some game other than the standard integer-game. Therefore it is by no means surprising that determinacy should have mostly classwise rather than pointwise implications.

Still, the rigorous study of pointwise consequences of determinacy was only begun in [Lö05]. In that paper, Löwe studied the pointwise relationship between determinacy and the Marczewski-Burstin algebra generated by non-topological arboreal forcing notions $\mathbb{P}$, and another, related notion of the *weak Marczewski-Burstin algebra*, $\mathrm{wMB}(\mathbb{P})$. It was proved that determinacy does *not* imply $\mathrm{MB}(\mathbb{P})$ pointwise, i.e., that $\mathsf{D} \nsubseteq \mathrm{MB}(\mathbb{P})$, and a characterization for $\mathbb{P}$ was given for deciding when determinacy does, and when it does not, imply $\mathrm{wMB}(\mathbb{P})$.

In this chapter we carry on the formal investigation of pointwise consequences. The final goal is to extend the result of [Lö05] to *topological* forcing notions $\mathbb{P}$, where the notion of measurability is the Baire property $\mathrm{BP}(\mathbb{P})$, as well as to study a weak form of the Baire property, $\mathrm{wBP}(\mathbb{P})$. We will indeed prove that determinacy does not imply $\mathrm{BP}(\mathbb{P})$ pointwise, and develop a characterization of topological forcings $\mathbb{P}$ similar to that of [Lö05] for studying pointwise implications from D to $\mathrm{wBP}(\mathbb{P})$.

On the way, we shall develop a useful general theory of proving pointwise non-implications, one which simultaneously covers the MB-algebra, the Baire property, Lebesgue measurability and potentially other (symmetric) regularity properties that might show up.

4.1 Generalized MB-algebras.

The primary method of proving pointwise non-implications is via Bernstein constructions. In [Lö05] this was pretty straightforward since $\mathrm{MB}(\mathbb{P})$ lends itself perfectly for an application of the Bernstein construction. In the case of regularity properties in general, it would therefore be useful to have an MB-like characterization. We already noted in Section 2.1 that both Lebesgue measurability and the Baire property have these characterizations, but in order to make this precise we need the following notion of a generalized Marczewski-Burstin algebra.

4.1. Definition. Given any two collections $\mathcal{A}, \mathcal{B} \subseteq \mathscr{P}(\eta^\omega)$, we define the *generalized Marczewski-Burstin algebra* $\mathrm{MB}(\mathcal{A}, \mathcal{B})$ by setting

$$A \in \mathrm{MB}(\mathcal{A}, \mathcal{B}) \iff \forall P \in \mathcal{A}\ \exists Q \in \mathcal{B}, Q \subseteq P \text{ s.t. } Q \subseteq A \vee Q \subseteq A^{\mathsf{c}}$$

If we are dealing with $\mathrm{MB}(\mathcal{A}, \mathcal{A})$ we simply write $\mathrm{MB}(\mathcal{A})$. Clearly, the MB-algebra for arboreal forcings $\mathbb{P}$ is the special case where $\mathcal{A} = \{[P] \mid P \in \mathbb{P}\}$. $\lhd$

Using this notion, we can prove pointwise non-implications analogously to [Lö05]. The only thing that remains to be checked are some cardinality conditions.

4.2. Theorem. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathscr{P}(\eta^\omega)$, and suppose there is some strategy σ (for I or II) and a corresponding $P \in \mathcal{A}$ such that:*

1. *$|\mathcal{B} \cap \mathscr{P}(P)| \leq 2^{\aleph_0}$ and*
2. *For every $Q \in \mathcal{B} \cap \mathscr{P}(P)$ we have $|Q \setminus [\sigma]| = 2^{\aleph_0}$.*

Then determinacy does not imply $\mathrm{MB}(\mathcal{A}, \mathcal{B})$ pointwise.

Proof. Let $\{Q_\alpha \mid \alpha < 2^{\aleph_0}\}$ enumerate $\mathcal{B} \cap \mathscr{P}(P)$, which automatically gives an enumeration of $\{(Q_\alpha \setminus [\sigma]) \mid \alpha < 2^{\aleph_0}\}$. Since this is a collection of $2^{\aleph_0}$ sets of cardinality $2^{\aleph_0}$, we can apply the general Bernstein construction as in Theorem 2.14 and find two disjoint sets A and B such that $\forall \alpha\ (Q_\alpha \setminus [\sigma]) \cap A \neq \varnothing$ and $(Q_\alpha \setminus [\sigma]) \cap B \neq \varnothing$.

Let $A' := A \cup [\sigma]$. Then A' does not contain, and is not disjoint from, any $Q \in \mathcal{B} \cap \mathscr{P}(P)$, so P witnesses the fact that neither A' nor its complement is in $\mathrm{MB}(\mathcal{A}, \mathcal{B})$. But if σ is a strategy for player I then A' is determined, whereas if it's a strategy for player II then $(A')^{\mathsf{c}}$ is determined. This proves $\mathsf{D} \not\subseteq \mathrm{MB}(\mathcal{A}, \mathcal{B})$. $\square$

Some research has been done on Marczewski-Burstin-like characterizations of certain properties, among others in [BaBaCo00/01, BaCi01/02, BrEl99]. Given some collection Reg of interest—for example, a regularity property—the aim was to find a suitable collection $\mathcal{A}$ such that $\mathsf{Reg} = \mathrm{MB}(\mathcal{A})$. For us, a one-way inclusion is in fact sufficient.

4.3. Corollary. *Let $\mathsf{Reg} \subseteq \mathscr{P}(\eta^\omega)$ be any regularity property. Suppose we can find collections $\mathcal{A}$ and $\mathcal{B}$ satisfying the conditions of Theorem 4.2, and such that $\mathsf{Reg} \subseteq \mathrm{MB}(\mathcal{A}, \mathcal{B})$. Then determinacy does not imply Reg pointwise.*

Proof. $\mathsf{D} \subseteq \mathsf{Reg} \subseteq \mathrm{MB}(\mathcal{A}, \mathcal{B})$ would contradict Theorem 4.2. $\square$

In many cases, the rather complicated conditions of Theorem 4.2 will be easy to verify. For example, we might have $|\mathcal{B}| \leq 2^{\aleph_0}$, or we might have $|Q \setminus [\sigma]| = 2^{\aleph_0}$ for all Q, or we might have the conditions go through for all σ, or all P etc. All of these would make life easier, but of particular interest is also the following result from [Lö05].

4.4. Lemma. *Let $\mathcal{A} \subseteq \mathscr{P}(\eta^\omega)$ be a collection such that $|\mathcal{A}| \leq 2^{\aleph_0}$ and for every $P \in \mathcal{A}$, $|P| = 2^{\aleph_0}$. Then the conditions of Theorem 4.2 are satisfied, and consequently $\mathsf{D} \not\subseteq \mathrm{MB}(\mathcal{A})$.*

Proof. Condition 1 is trivially satisfied, so it remains to show 2. If for all $P \in \mathcal{A}$ and all strategies σ, we have $|P \setminus [\sigma]| = 2^{\aleph_0}$ then we are done. Otherwise, fix P, σ such that $|P \setminus [\sigma]| < 2^{\aleph_0}$. Let τ be any strategy such that $[\sigma] \cap [\tau] = \varnothing$ (this is always possible). For any $Q \in \mathcal{A} \cap \mathscr{P}(P)$, we then have $Q \cap [\tau] \subseteq P \setminus [\sigma]$ and so $|Q \cap [\tau]| < 2^{\aleph_0}$. Since $|Q| = 2^{\aleph_0}$, this implies that $|Q \setminus [\tau]| = 2^{\aleph_0}$. So the pair (P, τ) satisfies condition 2. □

Since all standard arboreal forcings $\mathbb{P}$ from Section 1.4 (and, in fact, *any* arboreal forcing notion of interest) satisfy the requirements of Lemma 4.4, determinacy does not imply MB($\mathbb{P}$) pointwise, as we already knew from [Lö05]. But this situation also applies to Lebesgue measurability:

4.5. Theorem. *Determinacy does not imply Lebesgue measurability pointwise.*

Proof. We use the MB-like characterization from Proposition 2.5. Let

$$\mathcal{A}_{\mathcal{L}} := \{P \subseteq \eta^\omega \mid P \text{ is perfect, Lebesgue-measurable with non-zero measure}\}$$

This obviously satisfies the conditions of Lemma 4.4 so $\mathsf{D} \not\subseteq \mathrm{MB}(\mathcal{A}_{\mathcal{L}})$. But from [Bu14] we know that A is Lebesgue measurable iff it is in $\mathrm{MB}(\mathcal{A}_{\mathcal{L}})$. □

Although this proof is a nice illustration of the application of our general theory, we should note that the theorem can also be proved more directly: starting with a non-Lebesgue-measurable set A, define $A^* := \{\langle 0 \rangle^\frown x \mid x \in A\} \cup \{x \mid x(0) = 1\}$. By definition of Lebesgue measurability it follows that A^* is also non-Lebesgue-measurable (otherwise $\mu_{\mathcal{L}}(A) = \mu_{\mathcal{L}}(A^*) - \mu_{\mathcal{L}}(\{x \mid x(0) = 1\})$). But A^* is clearly determined since I has a winning strategy by playing a "1" in the first move.

4.2 Determinacy and BP($\mathbb{P}$).

Having built up the technology of proving pointwise non-implications, let us get back to the main task of proving $\mathsf{D} \not\subseteq \mathrm{BP}(\mathbb{P})$ for topological $\mathbb{P}$. Our first inclination would be to use the MB-like characterization of the Baire property given in Proposition 2.6. This general idea is correct, but there are some unexpected difficulties along the way.

First of all, the characterization from Proposition 2.6 is no good the way it is stated. Brown and Elalaoui-Talabi [BrEl99, Theorem 4] give a precise version of this characterization, namely that $\mathrm{BP}(\mathbb{P}) \subseteq \mathrm{MB}(\mathbb{G}_{\mathbb{P}})$ where $\mathbb{G}_{\mathbb{P}}$ is the collection of G_δ sets which are co-meager in an open set:

$$\mathbb{G}_{\mathbb{P}} := \{P \mid P \text{ is } G_\delta \text{ and } \exists O \text{ open s.t. } P \subseteq O \text{ and } O \setminus P \text{ is meager}\}$$

(where all topological concepts refer to the $\mathbb{P}$-topology). Formulated this way, one is already quite tempted to apply some version of Theorem 4.2. Unfortunately, proving the cardinality conditions for this characterization turns out to be surprisingly difficult. In particular, although in the standard topology there are only $2^{\aleph_0}$ G_δ sets and each G_δ set is either countable or has cardinality $2^{\aleph_0}$ (which makes the proof easy), for $\mathbb{D}$ and $\mathbb{E}$ this does not work: there are $2^{(2^{\aleph_0})}$ G_δ sets and a G_δ set can have any cardinality between 0 and $2^{\aleph_0}$ whatsoever![16]

Therefore, we will adopt a different (though related) approach. Let PT := $\{[T] \mid T$ is a perfect tree$\}$ and let $\mathcal{O}_\mathbb{P}$ be the collection of all open sets in the $\mathbb{P}$-topology. We will prove the characterization

$$\mathrm{BP}(\mathbb{P}) \subseteq \mathrm{MB}(\mathcal{O}_\mathbb{P}, \mathrm{PT})$$

in Proposition 4.7 and, once this is ready, we get the cardinality conditions for free.

Our result will cover all topological $\mathbb{P}$ provided that one very simple condition is satisfied: $(\mathbb{P}, \leq)$ must be non-atomic in the sense of Definition 1.7. Since any partial order interesting from the point of view of forcing satisfies this condition,[17] this is not really an issue.

Most of the work in proving the inclusion is contained in the following technical lemma.

4.6. Lemma. *Let $\mathbb{P}$ be any non-atomic, topological forcing notion. Let O be open and $C \subseteq O$ be G_δ co-meager in O (in the $\mathbb{P}$-topology). Then C contains a perfect tree.*

Proof. By Theorem 3.2 our space is Baire, so C is also dense in O. Let $C = \bigcap_n C_n$ with each C_n open. Since $C \subseteq C_n$, each C_n is actually open dense in O. By induction, we will construct a collection of basic open sets $[P_u]$ indexed by $u \in 2^{<\omega}$, while taking care that $[P_u] \subseteq C_{|u|}$. We proceed as follows:

- Since C_0 is open dense in O, let $P_\varnothing \in \mathbb{P}$ be such that $[P_\varnothing] \subseteq O \cap C_0$.
- Suppose we have $u \in 2^{<\omega}$ and $[P_u] \subseteq O$, with $|u| = n$. Note that by definition of arboreal forcings, there is a $P'_u \subseteq P_u$ with strictly longer stem, i.e. $P'_u \ll P_u$. By non-atomicity, there are $P^0_u \subseteq P'_u$ and $P^1_u \subseteq P'_u$ such that $[P^0_u] \cap [P^1_u] = \varnothing$. Since C_{n+1} is open dense in O, there are $[P_{u^\frown\langle 0\rangle}] \subseteq [P^0_u] \cap C_{n+1}$ and $[P_{u^\frown\langle 1\rangle}] \subseteq [P^1_u] \cap C_{n+1}$

Now, for each $y \in 2^\omega$ we get a sequence

$$P_\varnothing \gg P_{y\restriction 1} \gg P_{y\restriction 2} \gg P_{y\restriction 3} \gg \dots$$

and correspondingly a real $x := \bigcup_n \mathrm{stem}(P_{y\restriction n})$. Thus we define the following mapping:

$$\varphi : \begin{array}{ccc} 2^\omega & \longrightarrow & \eta^\omega \\ y & \longmapsto & \bigcup_n \mathrm{stem}(P_{y\restriction n}) \end{array}$$

[16] We will show that every $H \subseteq 2^\omega$ is a G_δ set in $(\omega^\omega, \mathbb{E})$. For $x \in 2^\omega$, let $\check{x}(n) := 1 - x(n)$. Given any $H \subseteq 2^\omega$, define $H^\bullet := \bigcup_{x \in H}[\varnothing, \check{x}]$ which is open. Clearly $H \subseteq H^\bullet \cap 2^\omega$. But conversely, if $x \in H^\bullet \cap 2^\omega$ then $x \in [\varnothing, \check{y}]$ for some $y \in H$. But for $x, y \in 2^\omega$, this can only happen if $x = y \in H$. Therefore $H = H^\bullet \cap 2^\omega$. Moreover, since we can write $2^\omega = \bigcap_n (\bigcup_{s \in 2^n} [s, \varnothing])$ we see that 2^ω is G_δ, and hence H is also G_δ.

[17] See e.g. [Ku80, Lemma VII 2.4]

Using an argument like the one in the proof of Theorem 3.2, it also follows that $\varphi(y)$ is the unique real in the singleton set $\bigcap_n [P_{y\restriction n}]$.

It is also clear that φ is injective: if $y \neq y'$, let n be least such that $y(n) \neq y'(n)$, say, $y(n) = 0$ and $y'(n) = 1$, and let $u := y \restriction n = y' \restriction n$. Then, by construction, $[P_{u^\frown\langle 0\rangle}] \cap [P_{u^\frown\langle 1\rangle}] = \varnothing$, but $\varphi(y) \in [P_{u^\frown\langle 0\rangle}]$ and $\varphi(y') \in [P_{u^\frown\langle 1\rangle}]$. Therefore $\varphi(y) \neq \varphi(y')$.

Now let T be the tree of $\varphi[2^\omega]$, i.e.

$$T := \{s \in \eta^{<\omega} \mid \exists u \in 2^{<\omega}\ (s \subseteq \mathrm{stem}(P_u))\}$$

It follows that $[T] = \varphi[2^\omega]$. Clearly, T is a perfect tree: let $s \in T$ and u be least s.t. $s \subseteq \mathrm{stem}(P_u)$. Let $y, y' \in 2^\omega$ with $y \neq y'$ extend u. Then $\varphi(y) \neq \varphi(y')$, and both $\varphi(y)$ and $\varphi(y')$ are extensions of s in $[T]$.

It remains only to show that $[T] \subseteq C$. But by construction, for every $x \in [T]$ there is a $y \in 2^\omega$ such that $x = \varphi(y)$, i.e $\bigcap_n [P_{y\restriction n}] = \{x\}$. Hence, for every n, $x \in [P_{y\restriction n}] \subseteq C_n$, so $x \in \bigcap_n C_n = C$.

This completes the proof. $\square$

We can now combine Lemma 4.6 with the MB-representation of the Baire property from Proposition 2.6 to prove the desired inclusion.

4.7. Proposition. *Let $\mathbb{P}$ be a non-atomic, topological forcing notion. Then* $\mathrm{BP}(\mathbb{P}) \subseteq \mathrm{MB}(\mathcal{O}_{\mathbb{P}}, \mathrm{PT})$.

Proof. Let A have the Baire property, and let U be open. By Proposition 2.6 we know that there is an open $V \subseteq U$ such that $V \cap A$ is meager or $V \setminus A$ is meager. Assume the former. Let M be an F_σ meager set such that $V \cap A \subseteq M$. Then $V \setminus M$ is G_δ, and it is comeager in V. By Lemma 4.6, there is a perfect tree T with $[T] \subseteq V \setminus M \subseteq A^{\mathsf{c}}$. Similarly, if $V \setminus A$ is meager then there is a perfect tree T with $[T] \subseteq V$ and $[T] \subseteq A$. This completes the proof. $\square$

Having completed the most difficult task of proving the inclusion $\mathrm{BP}(\mathbb{P}) \subseteq \mathrm{MB}(\mathcal{O}_{\mathbb{P}}, \mathrm{PT})$, the rest is straightforward.

4.8. Theorem. *Let $\mathbb{P}$ be a non-atomic, topological forcing notion. Then determinacy does not imply* $\mathrm{BP}(\mathbb{P})$ *pointwise.*

Proof. Fix any basic open $P \in \mathbb{P}$ with $|\mathrm{stem}(P)| \geq 2$. Then fix any strategy σ such that $[P] \cap [\sigma] = \varnothing$, which is always possible just by letting the beginning of σ be different from the stem of P. Clearly, then, the cardinality conditions of Theorem 4.2 are satisfied, i.e., there are at most $2^{\aleph_0}$ perfect trees, and for every perfect tree $T \subseteq P$, obviously $|[T] \setminus [\sigma]| = |[T]| = 2^{\aleph_0}$. The rest then follows by Theorem 4.2. $\square$

4.3 Determinacy and wBP($\mathbb{P}$).

The weak Marczewski-Burstin algebra is a local version of the full MB-algebra. Essentially, it was introduced in [BrLö99] and studied further in [Lö05].

4.9. Definition. Let $\mathbb{P}$ be a forcing notion. The *weak Marczewski-Burstin algebra*, denoted by wMB($\mathbb{P}$), is defined by

$$A \in \mathrm{wMB}(\mathbb{P}) \;:\Longleftrightarrow\; \exists P \in \mathbb{P}\ ([P] \subseteq A \text{ or } [P] \subseteq A^{\mathsf{c}}) \qquad \lhd$$

4.10. Proposition. MB($\mathbb{P}$) *and* wMB($\mathbb{P}$) *are classwise equivalent. To be precise, if* $\mathbf{\Gamma}$ *is a boldface pointclass closed under intersections with basic open sets (i.e.* $A \in \mathbf{\Gamma} \to A \cap [s] \in \mathbf{\Gamma}$*), then* $\mathbf{\Gamma} \subseteq \mathrm{MB}(\mathbb{P}) \leftrightarrow \mathbf{\Gamma} \subseteq \mathrm{wMB}(\mathbb{P})$.

Proof. See [BrLö99, Lemma 2.1]. $\square$

In [Lö05], the author classified all arboreal forcings $\mathbb{P}$ according to whether D implies wMB($\mathbb{P}$) pointwise or not. He identified three classes of forcings: for the first class the inclusion holds, for the second class it does not and for the third class there are examples either way.

Intuitively, this result can be understood as follows: if we would denote by $\mathbb{S}$trat the collection of all strategies (which is not an arboreal forcing according to Definition 1.4 because we cannot extend the stems arbitrarily), then determinacy is almost wMB($\mathbb{S}$trat) (it is not quite, because determinacy involves a difference between strategies of I and II). Therefore, the question whether determinacy implies wMB($\mathbb{P}$) is inherently related to the question: how *different* is $\mathbb{P}$ from $\mathbb{S}$trat? For example, if $\mathbb{S}\mathrm{trat} \subseteq \mathbb{P}$ then determinacy implies wMB($\mathbb{P}$) pointwise. This is the case with Sacks, Miller and Spinas forcing but not with the other forcing notions.

In the case of the Baire property, there is no "weak Baire property" in the existing literature, so we need to search for the right definition. Recall that in Proposition 2.6 a Marczewski-Burstin-like characterization of the Baire property was given. Such a characterization of course lends itself easily for the definition of a "weak" version.

4.11. Definition. Let $\mathbb{P}$ be a topological forcing notion. The *weak Baire property*, denoted by wBP($\mathbb{P}$), is defined by:

$$A \in \mathrm{wBP}(\mathbb{P}) \;:\Longleftrightarrow\; \exists \text{ open } O\ (O \cap A \text{ is meager or } O \setminus A \text{ is meager.}) \qquad \lhd$$

We must still show that this is an appropriate definition. It definitely looks like a very natural analogue of the situation with the MB-algebra. But a criterion which wMB must satisfy in order to make sure that the definition is appropriate, is the classwise equivalence with the full Baire property, analogously to Proposition 4.10. This is true for the standard topology:

4.12. Proposition. $\mathrm{BP}(\mathbb{C})$ *and* $\mathrm{wBP}(\mathbb{C})$ *are classwise equivalent. Precisely, if* $\boldsymbol{\Gamma}$ *is a boldface pointclass closed under intersection with basic open sets, then* $\boldsymbol{\Gamma} \subseteq \mathrm{wBP}(\mathbb{C}) \leftrightarrow \boldsymbol{\Gamma} \subseteq \mathrm{BP}(\mathbb{C})$.

Proof. The left-to-right direction is obvious so let's prove the reverse. Let $A \in \boldsymbol{\Gamma}$ and let O be an arbitrary open set. Let $[s] \subseteq O$ be basic open. Then define the following function:

$$f: \begin{array}{ccc} \eta^\omega & \longrightarrow & [s] \\ x & \longmapsto & s^\frown x \end{array}$$

It is not difficult to check that f is a bijection between η^ω and $[s]$ and that both f and f^{-1} are continuous, i.e., f is a homeomorphism. Now consider the set $A' := f^{-1}[[s] \cap A]$. By assumption, $A' \in \boldsymbol{\Gamma}$. Therefore $A' \in \mathrm{wBP}(\mathbb{C})$, so there is an open set U' such that $U' \cap A'$ is meager or $U' \setminus A'$ is meager. Now, let $U := f[U']$. By construction, $U \subseteq [s] \subseteq O$, and $f[U' \cap A'] = U \cap A$ and $f[U' \setminus A'] = U \setminus A$. Since f is a homeomorphism, either $U \cap A$ or $U \setminus A$ is meager. But since $U \subseteq O$, it follows that A has the Baire property by Proposition 2.6. $\square$

This confirms that the definition of the "weak Baire property" is indeed the appropriate one. We are now interested in the pointwise relationship between determinacy and $\mathrm{wBP}(\mathbb{P})$. Just as in [Lö05] the feature characterizing $\mathbb{P}$ was a notion called *fatness* (which was based on the interrelation between strategies and members of $\mathbb{P}$) we will present a similar characterization based on a topological descriptions of strategies.

Consider a topological forcing $\mathbb{P}$ and the topological space $(\eta^\omega, \mathbb{P})$. There are three cases:

- **Case 1.** For every strategy σ, the set $[\sigma]$ is somewhere open dense (i.e. $\exists O$ s.t. $\forall U \subseteq O \; \exists V \subseteq U \cap [\sigma]$.
- **Case 2.** There is a strategy σ such that $[\sigma]$ is nowhere dense.
- **Case 3.** Neither of the above: all strategies are somewhere dense but not necessarily somewhere open dense.

Just as in [Lö05], we will be able to give a definitive answer for Case 1 and Case 2 but not for Case 3. Unlike [Lö05], we will not be able to say anything detailed about what goes on in Case 3, which we can only suggest for further research.

Let us begin with the easiest case:

4.13. Theorem. *Let $\mathbb{P}$ be a topological forcing notion, and suppose that Case 1 holds, i.e., every strategy is somewhere open dense. Then determinacy implies* wBP($\mathbb{P}$) *pointwise.*

Proof. Let A be determined. Then there is either a strategy σ for I s.t. $[\sigma] \subseteq A$ or a strategy τ for II s.t. $[\tau] \subseteq A^{\mathsf{c}}$. Suppose the former. Since $[\sigma]$ is somewhere open dense, let O be open such that $\forall U \subseteq O \, \exists V \subseteq U \cap [\sigma]$. But then it is clear that $O \setminus A$ is nowhere dense, since for all $U \subseteq O$ there is a $V \subseteq U \cap [\sigma] \subseteq O \cap A$. Similarly, if $[\tau] \subseteq A^{\mathsf{c}}$ then we find an open O such that $O \cap A$ is nowhere dense. This shows that $A \in \mathrm{wBP}(\mathbb{P})$. □

The situation in Case 2 is noticeably more difficult, but can be resolved using the techniques developed in Section 4.2. First, we need the following notion of a generalized weak Marczewski-Burstin-algebra:

4.14. Definition. For a collection $\mathcal{A} \subseteq \mathscr{P}(\eta^\omega)$, the *generalized weak Marczewski-Burstin algebra*, denoted by $\mathrm{wMB}(\mathcal{A})$, is defined by

$$A \in \mathrm{wMB}(\mathcal{A}) \;:\Longleftrightarrow\; \exists P \in \mathcal{A}\, (P \subseteq A \text{ or } P \subseteq A^{\mathsf{c}}) \qquad \triangleleft$$

The following is an analogue of the general Theorem 4.2. Condition 2 is the formalization of the vague notion "$\mathcal{A}$ is different from $\mathbb{S}$trat".

4.15. Theorem. *Suppose $\mathcal{A}$ is a collection such that*

1. $|\mathcal{A}| \leq 2^{\aleph_0}$ *and*
2. *There is some strategy σ s.t. $\forall P \in \mathcal{A}\, (|P \setminus [\sigma]| = 2^{\aleph_0})$.*

Then determinacy does not imply wMB($\mathbb{P}$) *pointwise.*[18]

Proof. Using an enumeration $\langle P_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ of $\mathcal{A}$, we get an enumeration of $\langle P_\alpha \setminus [\sigma] \mid \alpha < 2^{\aleph_0} \rangle$. Then, using Theorem 2.14 we can find two Bernstein components A and B which are disjoint, intersect every P_α but are both disjoint from $[\sigma]$. Then, let $A' := A \cup [\sigma]$, so that either A' or its complement is determined but neither is in wMB($\mathbb{P}$) by construction. □

4.16. Corollary. *If* Reg *is some regularity property with* $\mathsf{Reg} \subseteq \mathrm{wMB}(\mathcal{A})$ *and $\mathcal{A}$ satisfies the conditions of Theorem 4.15, then determinacy does not imply* Reg *pointwise.*

Let us apply this to wBP($\mathbb{P}$). Using results from the previous section, we can relate wBP($\mathbb{P}$) to the weak MB-algebra of perfect trees, wMB(PT). At first, it is tempting to use the characterization $\mathrm{wBP}(\mathbb{P}) \subseteq \mathrm{wMB}(\mathrm{PT})$ and thus show pointwise non-implication. The problem is that if we diagonalize against all perfect trees, we also diagonalize against all strategies (since strategies *are*

[18] Note that this is a generalization of Theorem 4.4. from [Lö05].

perfect trees). However, since at least one strategy σ is nowhere dense (Case 2), there is a way around that difficulty.

Given a strategy σ, define

$$\mathrm{PT}_{\neg\sigma} := \{[T] \mid T \text{ is a perfect tree and } [T] \cap [\sigma] = \varnothing\}$$

i.e., all perfect trees that lie completely outside of σ.

4.17. Lemma. *Let $\mathbb{P}$ be any non-atomic, topological forcing notion. Let σ be a strategy such that $[\sigma]$ is nowhere dense in the $\mathbb{P}$-topology. Then* $\mathrm{wBP}(\mathbb{P}) \subseteq \mathrm{wMB}(\mathrm{PT}_{\neg\sigma})$.

Proof. Let $A \in \mathrm{wBP}(\mathbb{P})$. Let O be open so that either $O \cap A$ or $O \setminus A$ is meager. Since the situation is clearly symmetric, suppose the latter. Now, since $[\sigma]$ is nowhere dense, there is an open $U \subseteq O \setminus [\sigma]$. Then $U \setminus A \subseteq O \setminus A$, hence it is meager. Using Lemma 4.6. as in the previous section, we then find a G_δ set $C \subseteq U \cap A$ and then a perfect tree T such that $[T] \subseteq C \subseteq A$. But because $U \cap [\sigma] = \varnothing$ and $[T] \subseteq U$, we also have $T \in \mathrm{PT}_{\neg\sigma}$ as required. □

4.18. Theorem. *Let $\mathbb{P}$ be any non-atomic, topological forcing notion and suppose Case 2 holds, i.e., there is some σ with $[\sigma]$ nowhere dense. Then determinacy does not imply* $\mathrm{wBP}(\mathbb{P})$ *pointwise.*

Proof. Choose that σ for which $[\sigma]$ is nowhere dense. Then by Lemma 4.17 we have $\mathrm{wBP}(\mathbb{P}) \subseteq \mathrm{wMB}(\mathrm{PT}_{\neg\sigma})$, and the two conditions of Theorem 4.15 are satisfied (with σ as the witness). Hence, the desired result follows. □

Note that for all standard examples of topological forcings, namely $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$ and $\mathbb{R}$, Case 2 applies, since in those topologies *every* strategy is in fact nowhere dense. This, in any case, settles the question for the standard examples encountered in the literature.

Conclusion, indications of further research.

We have proved that in all cases of interest, determinacy does not imply the Baire property $\mathrm{BP}(\mathbb{P})$ pointwise. In the process we have also developed a general method of proving pointwise non-implications for (symmetric) regularity properties, which include the Baire property, the Marczewski-Burstin algebra, Lebesgue measurability and potential new properties.

Based on a topological characterization of strategies in the space $(\eta^\omega, \mathbb{P})$, we were able to find cases when determinacy does, and when it does not, imply the weak Baire property $\mathrm{wBP}(\mathbb{P})$ pointwise. We have left open the question of what happens in the remaining Case 3, although that does not affect the solution regarding the standard examples of topological forcings. Also, it is interesting that all standard examples belong to Case 2.

A suggestion for further research would be to study what exactly happens in Case 3. It would be especially nice if we could find examples of forcings that belong to Case 3 where pointwise implication holds, and other examples where pointwise implication does not hold, analogously to the results [Lö05, Theorems 5.1, 5.2, 5.3, 6.4].

On the other hand, it might be interesting to find some condition which all natural forcings $\mathbb{P}$ satisfy, and which would imply that $\mathbb{P}$ belongs either to Case 1 or Case 2.

5. Determinacy and Asymmetric Properties

In this chapter we deal with Main Question 3: the asymmetric regularity properties. The classical and paradigmatic example is the Perfect Set Property. In several papers, analogues of this idea have been developed. The aim of this chapter is two-fold: firstly, we would like to find a general definition of the asymmetric property of $\mathbb{P}$, based on a generalization of the existing properties. Secondly, we would like to do the same analysis about pointwise implications from determinacy as we did in the previous chapter. We will not be able to define $\mathrm{Asym}(\mathbb{P})$ as a simple combinatorial property derived from $\mathbb{P}$. Instead, we will give a game-theoretic definition of a very general kind of asymmetric property which subsumes $\mathrm{Asym}(\mathbb{P})$. Then we will use this game characterization to show that, under a certain non-triviality condition, determinacy does not imply the asymmetric properties pointwise.

First we introduce and define the standard asymmetric properties, together with their game characterizations. In the second section we give a general definition for $\mathrm{Asym}(\mathbb{P})$, and finally, in the third section, we will analyze pointwise implications from determinacy.

5.1 The Asymmetric Properties.

The Perfect Set Property says of a set A that it is either countable or contains a perfect tree. With countability being a notion of "smallness" whereas containing a perfect tree a notion of "largeness", this property says of a set A that it is "either large or small". These ideas have been generalized to three other asymmetric properties: K_σ-regularity, u-regularity and Laver-regularity, in [Ke77], [Sp93] and [GoReShSp95], respectively. In each case, the asymmetric property says of a set A that it is "either large or small". By largeness is meant: containing a certain type of tree, whereas by smallness various notions are meant (defined independently for each case.) Connecting all of this to our arboreal forcings, we think of the trees $P \in \mathbb{P}$ as corresponding to a particular notion of largeness. The traditional Perfect Set Property, for example, corresponds to Sacks forcing, since it deals with perfect trees.

Let us start by giving the definitions we will require.

5.1. Definition.

- For two reals $x, y \in \eta^\omega$, we define the pointwise ordering $\leq$ and the dominating ordering $\leq^*$:

$$x \leq y \;:\Longleftrightarrow\; \forall n\, (x(n) \leq y(n))$$

$$x \leq^* y \;:\Longleftrightarrow\; \forall^\infty n\, (x(n) \leq y(n))$$

- A set $A \subseteq \omega^\omega$ is *σ-bounded* if there exists a bound $f \in \omega^\omega$ such that $\forall x \in A\ (x \leq^* f)$.
- A set $A \subseteq \omega^\omega$ is *dominating* if for every $f \in \omega^\omega$ there is an $x \in A$ such that $f \leq^* x$.
- A set $A \subseteq \omega^\omega$ is *strongly dominating* if for every $f \in \omega^\omega$ there is an $x \in A$ such that $\forall^\infty n\ (x(n+1) > f(x(n)))$ $\lhd$

We can now proceed to define the standard asymmetric regularity properties. The Perfect Set Property is classical, having originated from Cantor's attempts to solve the Continuum Hypothesis, and the others are due to [Ke77, p. 195 ff], [Sp93, Definition 1.6] and [GoReShSp95, p. 1576 ff], respectively. The special types of trees are defined in Definition 1.5.

5.2. Definition.

1. Perfect Set Property: $A \subseteq \eta^\omega$ contains a perfect tree or is countable.
2. K_σ-regularity: $A \subseteq \omega^\omega$ contains a super-perfect tree or is σ-bounded.
3. u-regularity: $A \subseteq \omega^\omega$ contains a Spinas tree or is not dominating.
4. Laver-regularity: $A \subseteq \omega^\omega$ contains a Laver tree or is not strongly dominating. $\lhd$

Since each of these asymmetric properties deals with containing a tree from a particular forcing partial order (namely Sacks, Miller, Spinas and Laver) we can also denote them by Asym($\mathbb{S}$), Asym($\mathbb{M}$), Asym($\mathbb{L}^*$) and Asym($\mathbb{L}$), respectively. Note that the Perfect Set Property can be defined on any η^ω but the other properties must be defined on ω^ω to avoid triviality.

The one main feature connecting all these asymmetric properties is that they have game characterizations, i.e., there is a game $G_{\mathbb{P}}(A)$ such that I has a winning strategy in $G_{\mathbb{P}}(A)$ if and only if $\exists P \in \mathbb{P}([P] \subseteq A)$. The "smallness" property then corresponds to II having a winning strategy in that game.

The classical game associated to the Perfect Set Property is the $*$-game $G^*(A)$, and by the proof of Morton Davis [Da64], I has a winning strategy in $G^*(A)$ iff A contains a perfect tree and II has a winning strategy in $G^*(A)$ iff A is countable. Keeping in line with the original papers, the three other games will be denoted by $\tilde{G}(A)$, $G_u(A)$ and $D(A)$, respectively.

We will start by presenting Davis's proof, because it is simple and representative for the whole theory. However, the problem with the $*$-game is that it is played on 2^ω, and the proof depends crucially on the property of that space. We shall therefore present a slightly different game $G^\bullet(A)$ which can be played on any η^ω and proves the same result (and coincides with the $*$-game if played on 2^ω). It has the advantage of showing the underlying mechanism more perspicuously, and moreover the proof falls in line with other games and thus lends itself naturally for generalizations.

5.3. Definition. The game $G^\bullet(A)$ is played on any η^ω as follows:

- I plays non-empty sequences from η, and II plays elements of η.

$$\begin{array}{c|ccccccc} I: & s_0 & & s_1 & & s_2 & & \dots \\ \hline II: & & n_1 & & n_2 & & \dots & \end{array}$$

I wins if

- $\forall i \geq 1$: $s_i(0) \neq n_i$, and
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \dots \in A$.

5.4. Theorem (Davis, 1964).

1. *I has a winning strategy in $G^\bullet(A)$ iff A contains a perfect tree.*
2. *II has a winning strategy in $G^\bullet(A)$ iff A is countable.*

Proof.

1. If A contains a perfect tree, a strategy for I can informally be defined as follows: "at each step, play s_i until the next splitting node. If II plays n_{i+1}, play s_{i+1}, staying in the tree, such that $s_{i+1}(0) \neq n_{i+1}$ and until the next splitting node." Because we have a splitting node, choosing such an s_{i+1} is always possible.

Conversely, if σ is a winning strategy for I, then it is easy to see that $\{x \mid x$ is a play according to $\sigma\}$ is a perfect set.

2. If A is countable, say $A = \{a_0, a_1, a_2, \dots\}$, then II can play each n_i to "lie on a_i" (at the corresponding digit), so that I is forced to avoid it.

The only interesting direction is the converse. Fix a winning strategy τ for II. Let p be a partial play $\langle s_0, n_1, s_1, \dots, s_{i-1}, n_i\rangle$ according to τ, and let $p^* := s_0{}^\frown \dots {}^\frown s_{i-1}$. For such p and $x \in \eta^\omega$ we say:

- p is *compatible* with x if $p^* \subseteq x$ and there exists an s_i such that $s_i(0) \neq n_i$ and $p^{*\frown} s_i \subseteq x$ (this just means that n_i doesn't "lie on x").
- p *rejects* x if it is compatible with x and maximally so, i.e., for all s_i with $s_i(0) \neq n_i$, we have that $p^\frown \langle s_i, n_{i+1} := \tau(p^\frown \langle s_i\rangle)\rangle$ is *not* compatible with x.

Define $K_p := \{x \mid p \text{ rejects } x\}$. Clearly, if $x \in A$ then there is a p which rejects it, as otherwise I could continue playing s_i's against τ and produce an $x \in A$, contrary to τ being a winning strategy for II. Therefore $A \subseteq \bigcup_p K_p$, which is a countable union since there are only countably many p's. But now we claim that each K_p contains just one element. For suppose $x, y \in K_p$ with $x \neq y$. Then p is compatible with x and y, so let s_i be maximal such s.t. $p^{*\frown} s_i \subseteq x$ and $p^{*\frown} s_i \subseteq y$. Then consider $n_{i+1} := \tau(p^\frown \langle s_i \rangle)$. Clearly, n_{i+1} cannot lie on both x and y, so $p^\frown \langle s_i, n_{i+1} \rangle$ is still compatible with either x or y. Therefore, x and y cannot both be in K_p.

Therefore, $A \subseteq \bigcup_p K_p$ is a countable union of singletons, hence A is countable.

□

We now define the three other games and give the corresponding theorems (without proof) stating the relationship with the asymmetric properties.

5.5. Definition. The game $\tilde{G}(A)$ is played as follows:

- I plays non-empty sequences from ω, and II plays natural numbers.

$$\begin{array}{r|ccccccc} I: & s_0 & & s_1 & & s_2 & & \dots \\ \hline II: & & n_1 & & n_2 & & \dots & \end{array}$$

I wins if

- $\forall i \geq 1$: $s_i(0) \geq n_i$, and
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \dots \in A$.

The next game, $G_u(A)$, is as follows:

- I plays pairs (s_i, k_i) where $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$ and $k_i \in \omega \setminus \{0\}$, and II plays elements of ω.

$$\begin{array}{r|ccccccc} I: & (s_0, k_0) & & (s_1, k_1) & & (s_2, k_2) & & \dots \\ \hline II: & & n_1 & & n_2 & & \dots & \end{array}$$

I wins if

- $\forall i \geq 1 : |s_i| = k_{i-1}$
- $\forall i \geq 1$: $s_i(0) \geq n_i$, and
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \dots \in A$.

Finally, $D(A)$ is as follows:

- I first plays a non-empty sequences, then natural numbers, and II always plays natural numbers.

$$\begin{array}{r|ccccccc} I: & s_0 & & k_1 & & k_2 & & \dots \\ \hline II: & & n_1 & & n_2 & & \dots & \end{array}$$

I wins if

- $\forall i \geq 1$: $k_i > n_i$, and
- $x := s_0{}^\frown \langle k_1, k_2, \dots \rangle \in A$. $\triangleleft$

5.6. Theorem.

1. (a) *I wins $\tilde{G}(A)$ if and only if A contains a super-perfect tree.*
   (b) *II wins $\tilde{G}(A)$ if and only if A is σ-bounded.*
2. (a) *I wins $G_u(A)$ if and only if A contains a Spinas tree.*
   (b) *II wins $G_u(A)$ if and only if A is not dominating.*
3. (a) *I wins $D(A)$ if and only if A contains a Laver tree.*
   (b) *II wins $D(A)$ if and only if A is not strongly dominating.*

Proof.

1. [Ke77, Theorem 3.1]. The proof is completely analogous to the proof of Theorem 5.4 except that each K_p is no longer a singleton, but bounded (i.e. $\exists f \in \omega^\omega$ s.t. $\forall x \in K_p$ $(x \leq^* f)$).
2. [Sp93, Theorem 1.4].
3. [GoReShSp95, Lemma 2.3]. $\square$

It is easy to show that all four games can be encoded as standard integer-games, and that the encoding involved is such that the relevant sets stay within a boldface pointclass $\boldsymbol{\Gamma}$. Thus, following the ideas of Section 2.3, we get the following:

5.7. Corollary. *Determinacy implies the Perfect Set Property, K_σ-regularity, u-regularity and Laver-regularity classwise.* $\square$

5.2 The General Definition of Asym($\mathbb{P}$).

We have defined the four regularity properties Asym($\mathbb{S}$), Asym($\mathbb{M}$), Asym($\mathbb{L}^*$) and Asym($\mathbb{L}$). The natural question is: can a general definition of Asym($\mathbb{P}$) be given for arbitrary $\mathbb{P}$? Ideally, Asym($\mathbb{P}$) should be some explicitly defined combinatorial property derived from $\mathbb{P}$, valid for any given $\mathbb{P}$. So far we have not been able to find such a definition.

However, as the previous section illustrates, each asymmetric property has a game characterization, and all those games have one thing in common: they are *asymmetric* in the sense that I and II play different moves, with II playing "requirements" which I must "satisfy". Thus, the idea of the asymmetric game characterization can be understood as follows: II tries to set challenges which I must overcome while staying within a set A. If I can always do that then A is "large". On the other hand, if II has a way of setting challenges which I cannot overcome and stay within A, then A is "small".

We will now make this idea precise by defining a class of *general asymmetric games*, which will be used to defined general asymmetric properties based on those games. In some cases, these general asymmetric properties will correspond to $\mathrm{Asym}(\mathbb{P})$ for particular forcing notions $\mathbb{P}$. Although this does not uniquely define $\mathrm{Asym}(\mathbb{P})$, it does give a necessary condition which all $\mathrm{Asym}(\mathbb{P})$ must certainly satisfy in order to qualify for that name. This condition will be enough to prove pointwise non-implication for all $\mathrm{Asym}(\mathbb{P})$ in Theorem 5.12.

The inspiration for the general asymmetric games comes from [Ke77, p. 203 ff] where Kechris introduces a general class of games, as well as an asymmetric property connected to them, meant to generalize Davis's $*$-game, the Banach-Mazur game and his own $\tilde{G}$-game. Essentially, the games are defined as follows: I plays non-empty sequences of integers and II plays *requirements:* collections of non-empty sequences from which I must pick one in the next move. But the Kechris games are not sufficient for our purpose, because they do not include the two other asymmetric games $G_u(A)$ and $D(A)$ connected with u-regularity and Laver-regularity. We will therefore expand Kechris's definition to the following definition:

5.8. Definition. We define a class of *general asymmetric games*. Each game is based on a set of parameters $\Phi := (R, r^0, \{\Theta_n\}_{n\in\omega}, f)$ where

- $R \subseteq \mathscr{P}(\eta^{<\omega})$ is a countable set of *requirements*.
- $r^0 \subseteq \eta^{<\omega}$ is the *initial requirement*.
- The Θ_i are countable sets of *additional information*, which may, in principle, contain any sort of objects.
- $f : \bigcup_n \Theta_n \longrightarrow \mathscr{P}(R)$.

Then the game $G_\Phi(A)$ is defined as follows:

In each move player I plays a non-empty sequence s_i and, optionally, some $\theta_i \in \Theta_i$ (an additional piece of information). Player II responds by playing a requirement from $R \cap f(\theta_i)$, i.e., he or she may play any requirement from R but a restriction may be imposed upon his or her choice, depending on the information θ_i played by I in the previous move.

$$\begin{array}{l|cccccccc} I: & (s_0,\theta_0) & & (s_1,\theta_1) & & (s_2,\theta_2) & & \dots \\ \hline II: & & r_1 \in R\cap f(\theta_0) & & r_2 \in R\cap f(\theta_1) & & \dots & \end{array}$$

I wins $G_\Phi(A)$ iff

- $s_0 \in r^0$,
- $\forall i \geq 1 : s_i \in r_i$ and
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$. $\triangleleft$

Using this formal definition, the game $G_u(A)$ can be rendered in this form by setting

- $R := \{r_0, r_1, \dots\}$ with $r_i := \{s \in \omega^{<\omega} \mid s(0) \geq i\}$
- $r^0 := \omega^{<\omega} \setminus \{\varnothing\}$
- $\Theta_n := \omega$ for all n
- $f(n) := \{s \in \omega^{<\omega} \mid |s| = n\}$

Similarly, $D(A)$ becomes:

- $R := \{r_0, r_1, \dots\}$ with $r_i := \{s \in \omega^{<\omega} \mid s = \langle n \rangle$ for some $n > i\}$
- $r^0 := \omega^{<\omega} \setminus \{\varnothing\}$
- Θ_n and f are not used.

We will never need to write out such definitions of asymmetric games, since in practice more perspicuous definitions are available. Intuitively, we are allowed to define all kinds of games, where interdependencies between I's and II's moves may take an arbitrary level of complexity (encoded via R, r^0, Θ_n and f), but the important thing is that player I keeps playing non-empty sequences throughout, which will eventually be part of the real x produced in the limit, and that the winning condition depends on whether $x \in A$ or not. In this sense, the progress of a particular play corresponds to a progress deeper into the tree, with no possibilities to "take back moves" or something alike.

The asymmetric games may, or may not, be connected to forcing partial orders $\mathbb{P}$. We use the following definition:

5.9. Definition. Let $\mathbb{P}$ be some arboreal forcing notion, and let G_Φ be a general asymmetric game. We say that G_Φ *represents* $\mathbb{P}$ if

$$\forall A \subseteq \eta^\omega \text{ } (I \text{ has a winning strategy in } G_\Phi(A) \text{ iff } \exists P \in \mathbb{P} \, ([P] \subseteq A)). \quad \triangleleft$$

This leads naturally to the following definition:

5.10. Definition. Let $\mathbb{P}$ be some arboreal forcing notion and let G_Φ be an asymmetric game which represents $\mathbb{P}$. Then we define the *asymmetric property of* $\mathbb{P}$ *generated by* Φ, denoted by $\mathrm{Asym}_\Phi(\mathbb{P})$, by

$$A \in \mathrm{Asym}_\Phi(\mathbb{P}) \;:\Longleftrightarrow\; \exists P \in \mathbb{P} \, ([P] \subseteq A) \;\;\vee\;\; II \text{ wins } G_\Phi(A). \quad \triangleleft$$

Note that this is not an abstract definition of Asym($\mathbb{P}$) derived from $\mathbb{P}$ as we had envisaged it beforehand (in the same style as BP($\mathbb{P}$), MB($\mathbb{P}$) etc.) There may be $\mathbb{P}$ which cannot be represented by any G_Φ, and similarly there are $\mathbb{P}$ which can be represented by various G_Φ, so that there is not one well-defined asymmetric property related to $\mathbb{P}$. For a particular example of the latter case, consider the game $G^\sharp(A)$ defined as follows:

• I plays non-empty sequences and two natural numbers > 0; II plays natural numbers.

$$\begin{array}{c|ccccccc} I: & (s_0,k_0,l_0) & & (s_1,k_1,l_1) & & (s_2,k_2,l_2) & & \dots \\ \hline II: & & n_1 & & n_2 & & \dots & \end{array}$$

I wins iff

- $\forall i \geq 1\ (s_i(0) \neq n_i)$,
- $\forall i \geq 1\ (|s_i| = k_{i-1}$ or $|s_i| = l_{i-1})$ and
- $x := s_0{}^\frown s_1{}^\frown \cdots \in A$.

Clearly, this can be rendered as a general asymmetric game. Also, it is easy to check that this game actually represents Sacks forcing $\mathbb{S}$, just like $G^\bullet(A)$ from Theorem 5.4, although it is clearly different from $G^\bullet(A)$ (in particular, the proof of Davis would not work with this game since we can no longer conclude that $K_p := \{x \mid p \text{ rejects } x\}$ is a singleton). So $G^\bullet(A)$ and $G^\sharp(A)$ are two distinct games both of which represent $\mathbb{S}$.

Regardless of this, our definition still gives a precise condition which every asymmetric property Asym($\mathbb{P}$), whatever its potential definition, must certainly satisfy.

5.3 Determinacy and Asym($\mathbb{P}$).

Using the notion of G_Φ-representation and the corresponding asymmetric property, we can now prove that determinacy does not imply Asym$_\Phi$($\mathbb{P}$) pointwise. Essentially, the main observation is that the asymmetric games G_Φ are *different* from standard games (in the sense of Definition 2.7). This difference will be sufficient to prove pointwise non-implication by diagonalizing directly against strategies.

To deal more easily with the different types of perfect trees, we can define a weaker version of the asymmetric property wAsym$_\Phi$($\mathbb{P}$) by replacing the condition "contains a $P \in \mathbb{P}$" by the weaker condition "contains a perfect tree".

5.11. Definition. Given a forcing notion $\mathbb{P}$ and an asymmetric game G_Φ which represents $\mathbb{P}$, the weak asymmetric property $\mathrm{wAsym}_\Phi(\mathbb{P})$ is defined by:

$$A \in \mathrm{wAsym}_\Phi(\mathbb{P}) \;:\Longleftrightarrow\; \begin{array}{c} A \text{ contains a perfect tree} \\ \text{or} \\ II \text{ has a winning strategy in } G_\Phi(A). \end{array} \qquad \lhd$$

Since all $\mathbb{P}$ we are interested in have perfect trees as conditions, $\mathrm{Asym}_\Phi(\mathbb{P}) \subseteq \mathrm{wAsym}_\Phi(\mathbb{P})$, and hence, if $\mathsf{D} \nsubseteq \mathrm{wAsym}_\Phi(\mathbb{P})$ then also $\mathsf{D} \nsubseteq \mathrm{Asym}_\Phi(\mathbb{P})$. It is thus sufficient to show the former.

Assuming one very simple non-triviality condition on the games, we will prove pointwise non-implication.

5.12. Theorem. *Let $\mathbb{P}$ be an arboreal forcing notion represented by an asymmetric game G_Φ. Suppose G_Φ is non-trivial in the following sense: in all except finitely many positions of the game, player I has a choice of playing at least two sequences s and t which are incompatible (i.e. $s \nsubseteq t$ and $t \nsubseteq s$). Then determinacy does not imply* $\mathrm{wAsym}_\Phi(\mathbb{P})$ *pointwise.*

Proof. First of all, note that by definition of arboreal forcings $\mathbb{P}$, for every n there is a $P \in \mathbb{P}$ with $|\mathrm{stem}(\mathbb{P})| \geq n$. This means that, whatever the other properties of the game G_Φ, for every n we can find an $s \in \eta^{<\omega}$ with $|s| \geq n$ such that player I can make sure that s is eventually played, regardless of the strategy II is currently using (in all our examples, this s can be played in I's first move). Now fix any such s with $|s| \geq 2$. Then it immediately follows that there is a (standard) strategy σ for II such that $s \notin \sigma$. Fix this σ.

Now let $\{T_\alpha \mid \alpha < 2^{\aleph_0}\}$ be an enumeration of all perfect trees, and $\{\tau_\alpha \mid \alpha < 2^{\aleph_0}\}$ an enumeration of all strategies for II in the game G_Φ. For each α, let $K_\alpha := \{x \in \eta^\omega \mid s \subseteq x \ \wedge \ x \text{ is the result of a } G_\Phi\text{-game according to } \tau_\alpha\}$. Now, by the condition on the ability of I to have at least two choices infinitely many times during a run of the game, it follows that $|K_\alpha| = 2^{\aleph_0}$. Also, $K_\alpha \cap [\sigma] = \varnothing$ for all α.

By induction on $2^{\aleph_0}$ we construct two disjoint Bernstein components A and B similarly as in Theorem 2.14 but with a slight extra: At each step α, suppose we have already constructed A_α and B_α with $|A_\alpha| = |B_\alpha| = |\alpha| < 2^{\aleph_0}$. Then $K_\alpha \setminus (A_\alpha \cup B_\alpha)$ still has $2^{\aleph_0}$ elements to choose from, so we choose some $a_{\alpha+1}$ out of it. Then, $[T_\alpha] \setminus (A_\alpha \cup B_\alpha \cup \{a_{\alpha+1}\})$ still has $2^{\aleph_0}$ elements to choose from, so we choose some $b_{\alpha+1}$ out of it. Then we set $A_{\alpha+1} := A \cup \{a_{\alpha+1}\}$ and $B_{\alpha+1} := B \cup \{b_{\alpha+1}\}$ as usual. Finally, we set $A := \bigcup_{\alpha < 2^{\aleph_0}} A_\alpha$ and $B := \bigcup_{\alpha < 2^{\aleph_0}} B_\alpha$.

Then:

- $A \cap [\sigma] = \varnothing$ by construction, so A is determined (in the standard sense).
- A doesn't contain a perfect tree, because B intersects every perfect tree and $A \cap B = \varnothing$.

- II doesn't have a winning strategy in the game G_Φ because for every strategy τ_α, there is a play according to τ_α, namely a_α, which is in A.

Therefore determinacy does not imply the weak asymmetric property for $\mathbb{P}$, $\mathrm{wAsym}_\Phi(\mathbb{P})$. □

This means that, in particular, determinacy does not imply the Perfect Set Property, K_σ-regularity, u-regularity and Laver-regularity pointwise. Also, because of the weakening to wAsym, any other combination of "$\mathbb{P}$-largeness" with "$\mathbb{Q}$-smallness", like e.g. the notion of w-regularity briefly introduced in [BrHjSp95, p. 299], is covered here as well.

Conclusion, indication of further research.

The first part of Main Question 3 proved to be more subtle than expected. We were able to give a definition of general asymmetric games G_Φ which seem to adequately generalize the four standard asymmetric games. Using these games we were able to define $\mathrm{Asym}_\Phi(\mathbb{P})$ for those $\mathbb{P}$ which are represented by an asymmetric game G_Φ.

Using this definition we answered the second part of Question 3, namely, we proved that in all non-trivial cases determinacy does not imply $\mathrm{Asym}_\Phi(\mathbb{P})$ pointwise.

There are several questions still left open. Firstly, it is not clear whether all $\mathbb{P}$ are G_Φ-representable. Because the nature of our asymmetric games is such that the players, as it were, only "look deeper inside the tree", in seems likely that many $\mathbb{P}$ are not G_Φ-representable. In particular, we conjecture that Silver forcing $\mathbb{V}$ is not G_Φ-representable. Perhaps it is also possible to find conditions on $\mathbb{P}$ which imply that it is, or is not, G_Φ-representable, and maybe something can even be said about a canonical way of deriving G_Φ from $\mathbb{P}$.

Secondly, although we did mentioned that there can be distinct Φ_1 and Φ_2 such that both G_{Φ_1} and G_{Φ_2} represent $\mathbb{P}$, it would be nice to find examples for which we can prove that $\mathrm{Asym}_{\Phi_1}(\mathbb{P}) \neq \mathrm{Asym}_{\Phi_2}(\mathbb{P})$.

Another question is, of course, whether $\mathrm{Asym}(\mathbb{P})$ could be defined in some other way, hopefully using a direct derivation from $\mathbb{P}$. Such a definition might, again, have its basis in infinite games, or it might be a purely combinatorial definition. If such a definition exists, it would be interesting to see what the relationship is between that definition and our definition of $\mathrm{Asym}_\Phi(\mathbb{P})$.

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