# $\mathrm{M}[\mathrm{G}] \vDash$ ZFC 

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## What do we want?

## $\mathrm{M} \vDash \mathbf{Z F C} \Rightarrow \mathrm{M}[\mathrm{G}] \vDash \mathbf{Z F C}$

## Axioms:

Ext o Comprehension

- Foundation
- Replacement
- Infinity
- Dairing
- Union


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## Ext, Foundation

Lemma
If $\mathbf{M}$ is transitive, then $\mathbf{M} \vDash$ Ext.

```
Lemma
If \mathbf{M NWF}\mathrm{ and }\mathbf{M}\mathrm{ is transitive, then }\mathbf{M}\vDash\mathrm{ Foundation}
```


## Ext, Foundation

Lemma
If $\mathbf{M}$ is transitive, then $\mathbf{M} \vDash$ Ext.
Lemma
If $\mathbf{M} \in \mathbf{W F}$ and $\mathbf{M}$ is transitive, then $\mathbf{M} \vDash$ Foundation

## Pairing

Lemma
If $\forall x, y \in \mathbf{M} \exists z \in \mathbf{M}(x \in z \wedge y \in z)$, then $\mathbf{M} \vDash$ Pairing.

Definition

```
up(\sigma,\tau)={(\sigma,1),(\tau,1)}
op}(\sigma,\tau)={\boldsymbol{up}(\sigma,\tau),\mathbf{up}(\sigma,\sigma)
```

Lemma
$\mathrm{M}[\mathrm{G}] \vDash$ Pairing.

## Pairing

Lemma
If $\forall x, y \in \mathbf{M} \exists z \in \mathbf{M}(x \in z \wedge y \in z)$, then $\mathbf{M} \vDash$ Pairing.

## Definition

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\begin{aligned}
\mathbf{u p}(\sigma, \tau) & =\{(\sigma, \mathbf{1}),(\tau, \mathbf{1})\} \\
\mathbf{o p}(\sigma, \tau) & =\{\mathbf{u p}(\sigma, \tau), \mathbf{u p}(\sigma, \sigma)\}
\end{aligned}
$$

## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Pairing.

## Pairing

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## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Pairing.

## Union

## Lemma <br> If $\forall x \in \mathbf{M} \exists z \in \mathbf{M}(\cup x \subseteq z)$, then $\mathbf{M} \vDash$ Union

$\square$
$\mathrm{M}[\mathrm{G}] \vDash$ Union.

## Proof:

## Union

```
Lemma
If }\forallx\in\mathbf{M}\existsz\in\mathbf{M}(\cupx\subseteqz)\mathrm{ , then }\mathbf{M}\vDash\mathrm{ Union
```


## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Union.

## Proof.

Take $\tau \in \mathrm{M}^{\mathrm{P}}$ s.t. $\tau_{G} \in \mathrm{M}[\mathrm{G}] . \cup \operatorname{dom}(\tau) \in \mathrm{M}$ is a name $\pi$ say, s.t. $\bigcup \tau_{G} \subseteq \pi_{G}$.

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## Inf.

## Lemma <br> If $\omega \in \mathrm{M}$, then $\mathrm{M} \vDash$ Inf. .

## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Inf..

## Comprehension and buddies

## Comprehension and buddies



## Comprehension and buddies

```
Definition
Assume M = ZF - P, P}\in\textrm{M}\mathrm{ is a forcing poset, }\psi\in\mathcal{F}\mp@subsup{\mathcal{L}}{\mathbb{P}}{}\cap\textrm{M}\mathrm{ . Then
p}\mp@subsup{|}{\mathbb{P},\textrm{M}}{}\mathrm{ iff M[G]}\vDash\psi\mathrm{ for all filters }G\mathrm{ on }\mathbb{P}\mathrm{ s.t. }p\inG\mathrm{ and }G\mathrm{ is }\mathbb{P}\mathrm{ - generic over M .
```


## Tools <br> Truth Lemma, Definability Lemma.

## Comprehension and buddies

DefinitionAssume $\mathrm{M} \vDash \mathbf{Z F}-\mathbf{P}, \mathbb{P} \in \mathrm{M}$ is a forcing poset, $\psi \in \mathcal{F} \mathcal{L}_{\mathbb{P}} \cap \mathrm{M}$. Then$p \vdash_{\mathbb{P}, \mathrm{M}}$ iff $\mathrm{M}[\mathrm{G}] \vDash \psi$ for all filters $G$ on $\mathbb{P}$ s.t. $p \in G$ and $G$ is $\mathbb{P}$ - genericover M .
Tools
Truth Lemma, Definability Lemma.

## Comp.

## Lemma

If for all formulas $\varphi(x, z, \vec{w})$ :

$$
\forall z, \vec{w} \in \mathrm{M}\left(\left\{x \in z \mid \varphi^{M}(x, z, \vec{w})\right\} \in \mathrm{M}\right)
$$

then $\mathrm{M} \vDash$ Comprehension

## Comp.

## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Comp.

## Proof.

Take $\varphi(x, z, \vec{w})$.

$$
\begin{array}{ll}
z=\pi_{G} \in \mathrm{M}[\mathrm{G}] \\
w_{1}=\sigma_{G}^{0} \in M[G]
\end{array} \quad \begin{aligned}
& S=\left\{x \in \pi_{G} \mid \varphi^{M[G]}\left(x, \pi_{G}, \overrightarrow{\sigma^{i}}\right)\right\} \\
& \Rightarrow
\end{aligned} \quad \begin{aligned}
& \\
& \\
& p \in \mathbb{P} \wedge p \|(v, p) \mid v \in \operatorname{dom}(\pi) \wedge
\end{aligned}
$$

$$
\begin{array}{ll}
w_{n}=\sigma_{G}^{n} \in M[G] & \tau_{\text {exists by Definability Lemma and }} \\
\tau_{G} \subseteq S, S \subseteq \tau_{G}
\end{array}
$$

## Comp.

## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Comp.

## Proof.

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\tau=\{(v, p) \mid v \in \operatorname{dom}(\pi) \wedge \\
\end{gathered} \quad p \in \mathbb{P} \wedge p \Vdash(v \in \pi \wedge \phi(v)\}, ~ \$
$$

$$
w_{n}=\sigma_{G}^{n} \in M[G]
$$

$$
\tau \text { exists by Definability Lemma and }
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w_{n}=\sigma_{G}^{n} \in M[G]
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$\tau$ exists by Definability Lemma and

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\tau_{G} \subseteq S, S \subseteq \tau_{G}
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## Rep.

## Lemma

$\mathrm{M}[\mathrm{G}] \Vdash$ Rep.

## Proof.

Take $\varphi(x, y) \in \mathcal{F} \mathcal{L}_{\mathbb{P}} \cap \mathrm{M}$. Assume $\sigma_{G}=a \in M[G]$ and $\mathrm{M}[\mathrm{G}] \vDash \forall x \in a \exists y \phi(x, y))$.

To show: $b \in \mathrm{M}[\mathrm{G}], \operatorname{rng}(\phi) \subseteq b$.
Using Definability Lemma and Reflection theorem (in M) we can take $Q$ s.t.: $\forall \pi \in \operatorname{dom}(\sigma) \forall p \in \mathbb{P} \exists \mu \in M^{\mathbb{P}}(p \Vdash \phi(\pi, \mu)) \rightarrow \exists \mu \in Q(p \Vdash \phi(\pi, \mu))$. $Q=M^{\mathbb{P}} \cap(R(\alpha))^{M}$

Define $\alpha=Q \times\{\mathbf{1}\}$

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## Power Set

## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Power Set.

## Proof.

To show: $a \in \mathrm{M}[\mathrm{G}] \Rightarrow \exists b \in \mathrm{M}[\mathrm{G}]: \mathcal{P}(a) \cap M[G] \subseteq b$
Take $\tau \in M^{\mathbb{P}}, \tau_{G}=a$.
Define $\pi=Q \times\{\mathbf{1}\}$, where $Q=\mathcal{P}(\operatorname{dom}(\tau) \times \mathbb{P}) \cap \mathrm{M}$.

## Power Set

## Lemma

$\mathrm{M}[\mathrm{G}] \vDash$ Power Set.
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## Choice

## Lemma <br> $\mathrm{M}[\mathrm{G}] \vDash \mathrm{AC}$.

## Proof.

To show: $a \in \mathrm{M}[\mathrm{G}] \Rightarrow a$ can be well-ordered .
Take $\tau_{G}=a \in \mathrm{M}[\mathrm{G}]$ and wellorder $\operatorname{dom}(\tau)$ as $\left\{\sigma^{\eta} \mid \eta<\alpha\right\}$.
Define $f=\left\{o p\left(\hat{\eta}, \sigma^{\eta}\right) \mid \eta<\alpha\right\}$ so that $f_{G}$ is a function with domain $\alpha$ and $a \subseteq \operatorname{ran}(f)$.

Well-order by: $x \triangleleft y$ iff $\min \{\eta<\alpha \mid f(\eta)=x\}<\min \{\eta<\alpha \mid f(\eta)=y\}$

## Choice

## Lemma

$\mathrm{M}[\mathrm{G}] \vDash \mathbf{A C}$.

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