Homework 5, due Monday 7 March

1. (a) Show that the following is valid: If $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to (\chi \vee \theta)$, then $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \chi$ or $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \theta$ or $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \varphi$. [3 pts]

   (b)* Give an example such that the first two alernatives of (a) do not apply, but the last one does. [2 pts]

2. (a) Let φ be a propositional formula containing only $\wedge, \to$ and $\bot$.

   Let $\mathfrak{M}$ be a model and w a node in $\mathfrak{M}$ such that w has proper successors (i.e., there is at least one v in $\mathfrak{M}$ with wRv and $w \neq v$). Suppose

   - φ holds in all proper successors of w (i.e., for all v with $wRv, w \neq v$, we have $\mathfrak{M}, v \models \varphi$), and
   - for all propositional variables p, we have that p is true in w iff p is true in all proper successors of w (i.e., $w \in V(p)$ iff $\forall v\ (wRv, w \neq v \Rightarrow v \in V(p))$.

     (In other words, the valuation in w is maximal for propositional variables considering persistency).

   Show that φ is true in w. [4 pts]

   (b) Show on the basis of the above that, if φ is a propositional formula not containing $\vee$ and $\vdash_{\mathbf{IPC}} \varphi \to \psi \vee \chi$, then $\vdash_{\mathbf{IPC}} \varphi \to \psi$ or $\vdash_{\mathbf{IPC}} \varphi \to \chi$. [2 pts]

3. Prove that $\vdash_{\mathbf{IPC}} \varphi$ implies $\vdash_{\mathbf{S4}} \varphi^{\Box}$ in the following manner:

   Assume $\mathfrak{M}$ on $\mathfrak{F}$ is an **S4**-countermodel to $\varphi^{\Box}$. Take the frame $\mathfrak{G}$ that is obtained from $\mathfrak{F}$ by replacing each cluster (collection of nodes that are pairwise accessible from each other) by a single node (try to define this exactly.) There is an obvious function from $\mathfrak{F}$ onto $\mathfrak{G}$. Show that it is a **frame**-p-morphism. Define a valuation on $\mathfrak{G}$ in such a way that the resulting model $\mathfrak{N}$ is an **IPC**-model. Show, by induction on the length of $\psi(p_1, \ldots, p_n)$ that, for each $w \in W$, $\mathfrak{M}, w \models \psi^{\Box}$ (as an **S4**-model) iff $\mathfrak{N}, f(w) \models \psi$ (as an **IPC**-model). Finally conclude that $\mathfrak{N}$ (as an **IPC**-model) falsifies φ. [7 pts]