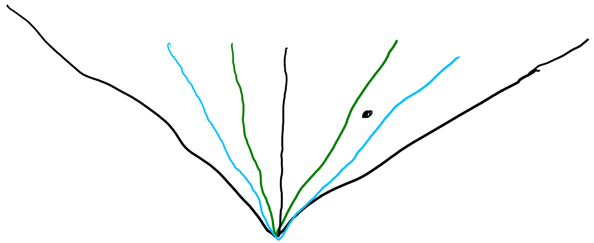
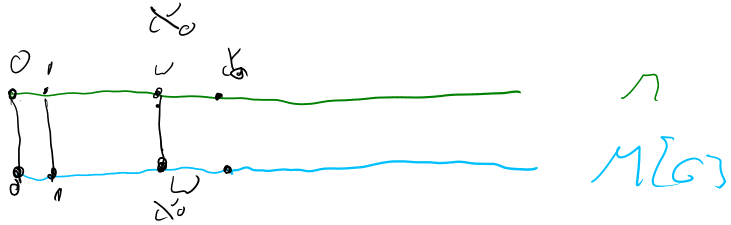
Recap:

- ctm M
- $\mathbb{P} = Fn(\kappa \times \omega, 2)$
- $\kappa = (\aleph_\alpha)^M$
- $(2^{\aleph_0} \geq \kappa)^{M[G]}$

What we want: $\kappa = (\aleph_\alpha)^{M[G]}$

Preservation of cardinalities





$ZFC \vdash n$ is card. $\quad n \in \omega$

$ZFC \vdash \aleph_0 = \omega$

- Approximation lemma $\quad (\mathbb{P}$ has ccc$)^M$
- Thm: regularity preserved

Lemma If $\mathbb{P} \in M$ and $(\mathbb{P}$ has ccc$)^M$ and G generic filter and $f: A \to B$ in $M[G]$ with $A, B \in M$ then there exists a $F: A \to \mathcal{P}(B)$ in M such that

(1) $f(a) \in F(a)$

(2) $(|F(a)| \leq \aleph_0)^M$

Pf Let $\dot{f}$ be a name for f. $M[G] \models$ "$\dot{f}: \check{A} \to \check{B}$" so by truth lemma there is $p \in G$ with $p \Vdash$ "$\dot{f}: \check{A} \to \check{B}$".

Define $F(a) := \{b \in B : (\exists q \leq p)\ q \Vdash \dot{f}(\check{a}) = \check{b}\}$. (Definability lemma)

(1) Let $b = f(a)$. By truth lemma there is a $q \leq p$ such that $q \Vdash \dot{f}(\check{a}) = \check{b}$. So $b \in F(a)$

(2) Let $y, z \in F(a)$ distinct. Then there are $q_y, q_z \leq p$ such that $q_y \Vdash \dot{f}(\check{a}) = \check{y}$ and $q_z \Vdash \dot{f}(\check{a}) = \check{z}$.

If $r \leq q_y, q_z$ then $r \Vdash \dot{f}: \check{A} \to \check{B} \wedge \dot{f}(\check{a}) = \check{y} \wedge \dot{f}(\check{a}) = \check{z} \wedge \check{y} \neq \check{z}$. ↯

$M[H] \models$ ... H gen. filter containing r ↯

So $q_y \perp q_z$. so if we have $(\aleph_1)^M$ many distinct elts of $F(a)$ then we have $(\aleph_1)^M$ many incompatible q's. ↯

□

Thm If $\mathbb{P} \in M$ and $(\mathbb{P}$ has ccc$)^M$ then for any $\beta < o(M)$ we have

G gen

$$(\beta \text{ regular})^M \leftrightarrow (\beta \text{ regular})^{M[G]}.$$

Pf Let $\omega < \beta < o(M)$ be regular in M.

Assume it is singular in $M[G]$ and hope to find a contradiction.

There is a $f: \alpha \to \beta$ increasing cofinal $(\alpha < \beta)$ $\in M[G]$.

So by lemma there is $F: \alpha \to \mathcal{P}(\beta)$ in M such that (1) and (2) hold.

$Y := \bigcup_{a \in \alpha} F(a)$. By (1) we have that Y is cofinal in β. (in M!)

Reason from M: $\beta = cf(\beta) = cf(type(Y))$

So $\beta = |\beta| = |cf(type(Y))| \leq |type(Y)| = |Y| \leq |\alpha| \cdot \aleph_0 = \max\{|\alpha|, \aleph_0\} < \beta$? ↯

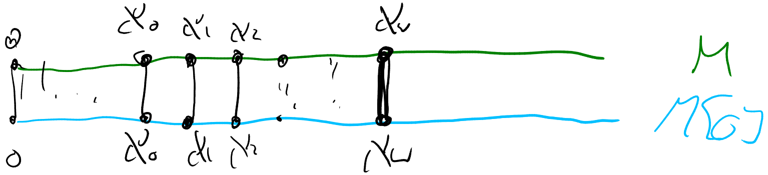
So $(\beta$ regular$)^{M[G]}$

□

Cor If $\mathbb{P} \in M$ and $(\mathbb{P}\ \mathrm{ccc})^M$

then $(\aleph_\alpha)^M = (\aleph_\alpha)^{M[G]}$ for all $\alpha < o(M)$

Pf



$ZFC \vdash \aleph_\omega = \bigcup_{n<\omega} \aleph_n$

$\square$

Strongly: $cf^M(\beta) = cf^{M[G]}(\beta)$

Pf idea: cofinalities are regular

$\alpha = cf^M(\beta)$ then

$cf^{M[G]}(\beta) = cf^{M[G]}(\alpha) = \alpha$

$\square$

$Fn(\kappa \times \omega, 2) \in M$

Yay!