Forcing Relations

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Outline

- Motivation: why forcing relations?
  - Objective of the day: $M[G] \vDash ZFC$
  - Tricky axioms
  - Solution: forcing relations
- The semantic forcing relation $\Vdash$
  - Definition and examples
  - Truth and Definability Lemmas (without proof)
- The syntactic forcing relation $\Vdash^*$
  - Definition
  - Towards Truth and Definability

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- We want $M[G] \models ZFC + \neg CH$, given $M \models ZFC$
- So, in particular, we need: $M[G] \models ZFC$, given $M \models ZFC$
- This (incl. afternoon lecture) is the goal of today
- Turns out some axioms are tricky (*comprehension*, power, replacement)
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Why is comprehension tricky?

Example (motivational)

- Consider

$$S := \{n \in \omega : (\varphi(n, \sigma_G))^{M[G]}\},$$

where $\varphi(x, y)$ formula, $\sigma \in M^{\mathbb{P}}$.

- Recall that $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$, so in order for S to be in $M[G]$, we need $\tau \in M^{\mathbb{P}}$ s.t. $\tau_G = S$.
- Now, how do we know such a name exists?
- Yesterday, we saw instances: $\mathring{G}_G = G, \check{x} = x, op(\sigma, \pi)_G = (\sigma_G, \pi_G)$.
- But this is not enough: we need a general procedure for constructing names for sets such as S given **any** formula φ.
- This seems problematic ...

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Some preliminary definitions

Definition (The forcing language $\mathcal{FL}_{\mathbb{P}}$)

For $\mathbb{P}$ poset, the forcing language $\mathcal{FL}_{\mathbb{P}}$ is the class of formulas build using "$\in$" and the names in $V^{\mathbb{P}}$ as constant symbols.

When dealing with $M[G]$, we restrict ourselves to $\mathcal{FL}_{\mathbb{P}} \cap M$ (which amounts to "$\in$" and the names in $M^{\mathbb{P}}$ as constant symbols).

Definition

Given ψ sentence in $\mathcal{FL}_{\mathbb{P}} \cap M$, $M[G] \vDash \psi$ is defined as usual, however interpreting each $\tau \in M^{\mathbb{P}}$ as τ_G.

- Note: This relation does *not* only depend on $M[G]$ and ψ, but also on G ("$\tau \in M^{\mathbb{P}}$ as $\tau_{\mathbf{G}}$"). That is: there are cases where $M[G] = M[H]$, yet $M[G] \vDash \psi$ while $M[H] \nvDash \psi$.

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The semantic forcing relation $\Vdash$

Definition

Assume countable $M \vDash ZF - P$, $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$, and ψ sentence in $\mathcal{FL}_{\mathbb{P}} \cap M$.
Then $p \Vdash_{\mathbb{P},M} \psi$ *iff* $\forall \mathbb{P}$*-generic* G *over* M *s.t.* $p \in G$*:* $M[G] \vDash \psi$.

- M countable $\Rightarrow \exists \mathbb{P}$-generic G over M (cf. GFEL).

Example (and lemma)

- If $p \leq q$ then $p \Vdash \check{q} \in \overset{\circ}{G}$.
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Seems promising, but two (major) problems remain ...

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presumably naming

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Truth and Definability Lemmas

The Truth Lemma

Assume $M \vDash ZF - P$ countable, $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$, ψ sentence in $\mathcal{FL}_{\mathbb{P}} \cap M$, and G $\mathbb{P}$-generic over M.

Then $M[G] \vDash \psi$ *iff* $\exists p \in G$ *s.t.* $p \Vdash \psi$.

The Definability Lemma

Assume $M \vDash ZF - P$ countable, and let $\psi(x_1, ..., x_n)$ be a formula in the language of set theory. *Then*

$$A := \{(p, \mathbb{P}, \leq, \mathbf{1}, \theta_1, ..., \theta_n) : p \in \mathbb{P} \wedge M \ni \langle \mathbb{P}, \leq, \mathbf{1} \rangle \text{ is a forcing poset} \wedge \theta_1, ..., \theta_n \in M^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P},M} \psi(\theta_1, ..., \theta_n)\},$$

is definable over M without parameters ($A \in \mathcal{D}^-(M)$). I.e., there is $\xi(x)$ s.t. $A = \{x \in M : \xi(x)^M\}$.

Truth and Definability Lemmas

The Truth Lemma

Assume $M \vDash ZF - P$ countable, $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$, ψ sentence in $\mathcal{FL}_{\mathbb{P}} \cap M$, and G $\mathbb{P}$-generic over M.

Then $M[G] \vDash \psi$ *iff* $\exists p \in G$ *s.t.* $p \Vdash \psi$.

Intuition: "Anything true (in the generic extension) is forced (by a condition in the generic filter)"

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Intuition: "Forcing is definable *within* M"

Truth and Definability Lemmas

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Truth and Definability Lemmas to the rescue!

Recall our two problems regarding our comprehension-instance, where

$$\tau = \{(\check{n}, p) : n \in \omega \wedge p \in \mathbb{P} \wedge p \Vdash \varphi(\check{n}, \sigma)\}$$

and

$$S = \{n \in \omega : (\varphi(n, \sigma_G))^{M[G]}\}.$$

We now have that

(1) $\tau \in M^{\mathbb{P}}$ by The Definability Lemma, and

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Propositional lemma

Lemma

Given a forcing poset $\mathbb{P} \in M$, sentences $\varphi, \psi \in \mathcal{FL}_{\mathbb{P}} \cap M$, the following hold:

4. $p \Vdash \varphi \wedge \psi$ *iff* $p \Vdash \varphi$ *and* $p \Vdash \psi$
5. $p \Vdash \neg\varphi$ *iff* $\neg\exists q \leq p(q \Vdash \varphi)$
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7. $p \Vdash \varphi \vee \psi$ *iff* $\{q \leq p : (q \Vdash \varphi) \vee (q \Vdash \psi)\}$ *is dense below p. [As. 3]*

Proof.

4. By definition of the forcing relation.
5. "$\Rightarrow$" follows by previously shown lemma, and that $(q \Vdash \neg\varphi) \wedge (q \Vdash \varphi)$ is a contradiction. "$\Leftarrow$" by contraposition. For generic $G \ni p$ s.t. $M[G] \vDash \varphi$, we get $r \in G$ s.t. $r \Vdash \varphi$ by Truth Lemma. Now, since G filter, we have $q \leq p, q \leq r$. $q \Vdash \varphi$ since $q \leq r$, but also $q \leq p$.
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Given a forcing poset $\mathbb{P} \in M$, and $\varphi(x) \in \mathcal{FL}_{\mathbb{P}} \cap M$, the following hold:

($\forall$) $p \Vdash \forall x \varphi(x)$ *iff* $p \Vdash \varphi(\tau)$ *for all* $\tau \in M^{\mathbb{P}}$

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($\forall$) – $M[G] \models \forall x \varphi(x)$ **iff** $M[G] \models \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$, and
– $p \Vdash \forall x \varphi(x)$ **iff** $\forall \mathbb{P}$−gen. $G \ni p$: $M[G] \models \forall x \varphi(x)$, so in total
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Quantifier lemma

Lemma

Given a forcing poset $\mathbb{P} \in M$, and $\varphi(x) \in \mathcal{FL}_{\mathbb{P}} \cap M$, the following hold:

($\forall$) $p \Vdash \forall x \varphi(x)$ *iff* $p \Vdash \varphi(\tau)$ *for all* $\tau \in M^{\mathbb{P}}$

($\exists$) $p \Vdash \exists x \varphi(x)$ *iff* $\{q \leq p : \exists \tau \in M^{\mathbb{P}}(q \Vdash \varphi(\tau))\}$ *is dense below* p.

Proof.

($\forall$) – $M[G] \vDash \forall x \varphi(x)$ **iff** $M[G] \vDash \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$, and
– $p \Vdash \forall x \varphi(x)$ **iff** $\forall \mathbb{P}-$gen. $G \ni p$: $M[G] \vDash \forall x \varphi(x)$, so in total
– $p \Vdash \forall x \varphi(x)$ **iff** $\forall \mathbb{P}-$gen. $G \ni p$: $M[G] \vDash \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$.
– But this exactly amounts to: $p \Vdash \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$.

($\exists$) Follows from ($\forall$) and 5. from last slide, using $\exists \equiv \neg\forall\neg$.

□

Atomic lemma

The previous lemmas look surprisingly much like propositional and quantifier-steps in a recursive definition of "$\Vdash$". Moreover, one can even prove a lemma for the atomic cases.

Lemma

Given a forcing poset $\mathbb{P} \in M$, and $\tau, \rho, \pi \in M^{\mathbb{P}}$, the following holds:

- $p \Vdash \tau = \rho$ *iff* $\forall \sigma \in [dom(\tau) \cup dom(\rho)] \forall q \leq p (q \Vdash \sigma \in \tau \leftrightarrow q \Vdash \sigma \in \rho)$
- $p \Vdash \pi \in \tau$ *iff* $\{q \leq p : \exists (\sigma, s) \in \tau (q \leq s \wedge q \Vdash \pi = \sigma)\}$ *is dense below* p.

Idea

By proving these lemmas, we have seen that–*given* the Truth and Definability Lemmas–the *semantic* forcing relation "$\Vdash$" behaves recursively. What if we instead define a forcing relation *syntactically* using this recursive definition?

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By proving these lemmas, we have seen that–*given* the Truth and Definability Lemmas–the *semantic* forcing relation "$\Vdash$" behaves recursively. What if we instead define a forcing relation *syntactically* using this recursive definition?

Towards Truth and Definability Lemmas

- Define a forcing relation $\Vdash^*$ which is internal to any $M \vDash ZFC^*$
- Prove the Truth Lemma for $\Vdash^*$ directly.
- Prove the equivalence of $\Vdash$ and $\Vdash^*$.This automatically establishes the Definability Lemma and the Truth Lemma for $\Vdash$.

Syntactic Forcing Relation

Definition

Let $\mathbb{P}$ be a forcing poset and $\tau, \theta, \sigma \in V^{\mathbb{P}}$. Define:

- $p \Vdash^* \tau = \theta \iff \forall \sigma \in \text{dom}(\tau) \cup \text{dom}(\theta) \forall q \leq p \, (q \Vdash^* \sigma \in \tau \leftrightarrow q \Vdash^* \sigma \in \theta)$.
- $p \Vdash^* \pi \in \tau \iff \{q \leq p : \exists (\sigma, r) \in \tau \, (q \leq r \wedge q \Vdash^* \pi = \sigma)\}$ is dense below p.

We denote the set containing these sentences in the forcing language with $\mathcal{AL}_{\mathbb{P}}$.

Note

For our purposes, we might as well define this relation only for names in $M^{\mathbb{P}}$ and prove our results for the fixed model M. Also note that the definition of the syntactic forcing relation for atomic sentences is absolute for transitive models.

Caution

Does this mean $p \Vdash \varphi \iff p \Vdash^* \varphi$?

These clauses are the same as those characterising the semantic forcing relation with atomic sentences. Those clauses were proved using the Truth and Definability Lemma though, meaning that we cannot assume that $p \Vdash \varphi \iff p \Vdash^* \varphi$ for $\varphi \in \mathcal{AL}_{\mathbb{P}}$ until we prove those lemmas independently. On the other hand, given our road map, we have no choice other than defining $p \Vdash^* \varphi$ (for atomic sentence φ) in this way.

Recursion in the definition

This definition is clearly recursive, meaning that to establish $p \Vdash^* \varphi$ for some $\varphi \in \mathcal{AL}_{\mathbb{P}}$, one needs to know a number of $q \Vdash^* \psi$. This is not problematic, as it can be settled using the recursion theorem.

The relation R

Definition

Let $p_1, p_2 \in \mathbb{P}$ and $\sigma_1, \sigma_2, \tau_1, \tau_2 \in V^{\mathbb{P}}$ and define

- $(p_1, \sigma_1 \in \tau_1) R (p_2, \sigma_2 = \tau_2)$ iff ($\sigma_1 \in TC(\sigma_2)$ or $\sigma_1 \in TC(\tau_2)$ and $(\tau_1 = \sigma_2$ or $\tau_1 = \tau_2)$.
- $(p_1, \sigma_1 = \tau_1) R (p_2, \sigma_2 \in \tau_2)$ iff $\sigma_1 = \sigma_2$ and $\tau_1 \in TC(\tau_2)$

R is set like: as an example, there's an obvious surjection between from the set $\mathbb{P} \times TC(\tau)$ to the predecessors of $(p, \sigma \in \tau)$.
R is well-founded: Kunen defines a function from pairs (p, φ) to ordinals which is strictly R-increasing, showing that there cannot be infinite descending R-sequences.

Technical preliminaries

Lemma

Let $p \in \mathbb{P}$ and $\varphi \in \mathcal{AL}_{\mathbb{P}}$, then

- $p \Vdash^* \varphi$ and $q \leq p$ implies $q \Vdash^* \varphi$
- $p \Vdash^* \varphi$ iff $\{q \leq p : q \Vdash^* \varphi\}$ is dense below p

Definition

For $\varphi \in \mathcal{AL}_{\mathbb{P}}$ and $p \in \mathbb{P}$ define $p \Vdash^* \neg\varphi$ iff $\neg\exists q \leq p(q \Vdash^* \varphi)$

Lemma

For $\varphi \in \mathcal{AL}_{\mathbb{P}}$ and $p \in \mathbb{P}$ it holds that $p \Vdash^* \varphi$ iff $\neg\exists q \leq p(q \Vdash^* \neg\varphi)$

The Baby Truth Lemma (for $\Vdash^*$)

Lemma

Let M be a transitive set model for ZF-P and $\mathbb{P} \in M$ a forcing poset, $\varphi \in \mathcal{AL}_{\mathbb{P}} \cap M$ and let G be a $\mathbb{P}$-generic filter over M. Then

$$M[G] \vDash \varphi \iff \text{there is } p \in G \text{ such that } (p \Vdash^* \varphi)^M$$

Proof sketch

By induction on R:

- ($\Longleftarrow$) We show the case $\varphi = \pi \in \tau$. Assume $p \in G$ is such that $p \Vdash^* \pi \in \tau$ and assume that the implication holds for all pairs (q, ψ) such that $(q, \psi) R (p, \pi \in \tau)$, so $D = \{q \leq p : \exists (\sigma, r) \in \tau (q \leq r \wedge q \Vdash^* \pi = \sigma)\}$ is dense below p, hence it meets G at some point q. Fix $(\sigma, r) \in \tau$ such that $q \leq r$ and $q \Vdash^* \pi = \sigma$. Now $(q, \pi = \sigma) R (p, \pi \in \tau)$, therefore by inductive assumption $M[G] \vDash \pi = \sigma$, i.e. $\pi_G = \sigma_G$. Moreover, since $q \leq r$, $r \in G$ and $\sigma_G \in \tau_G$, so $M[G] \vDash \pi \in \tau$.

Proof sketch (cont.)

- ($\Longrightarrow$) Again we show the case $\varphi = \pi \in \tau$. Assume $M[G] \vDash \pi \in \tau$ and assume the implication holds for all pairs(q, ψ) and all $p \in \mathbb{P}$ such that $(q, \psi)R(p, \pi \in \tau)$ (note that the relation R is independent of the first coordinate). Need to show that there exists $p \in G$ such that $\{q \leq p : \exists(\sigma, r) \in \tau \, (q \leq r \wedge q \Vdash^* \pi = \sigma\}$ is dense below p. $M[G] \vDash \pi \in \tau$ means that $\pi_G \in \tau_G$, hence there is $(\sigma, r) \in \tau$ such that $\sigma_G = \pi_G$ and $r \in G$. By inductive assumption, fix $p' \in G$ such that $p' \Vdash^* \sigma = \pi$: by the first technical lemma, all extensions of p' syntactically force $\sigma = \pi$ and since G is a filter, r and p' have a common extension p. p satisfies the requirement above, hence $p \Vdash^* \pi \in \tau$.

$p \Vdash^* \varphi \iff p \Vdash \varphi$ (atomic φ)

Lemma

Let M be a countable transitive set model for $ZF\text{-}P$ and $\mathbb{P} \in M$ a forcing poset, $\varphi \in \mathcal{AL}_{\mathbb{P}} \cap M$ and $p \in \mathbb{P}$. Then

$$p \Vdash \varphi \iff p \Vdash^* \varphi$$

Proof

The previous lemma gives us $\Longleftarrow$ immediately.
For the converse implication, assume $p \Vdash \varphi$ and $p \nVdash^* \varphi$ and, using the second technical lemma let $q \leq p$ be such that $q \Vdash^* \neg\varphi$, so $\neg\exists r \leq q(r \Vdash^* \varphi)$. Now fix a $\mathbb{P}$-generic filter G with $q \in G$ (here we use the assumption that M is countable). $q \leq p$ implies that $p \in G$ so $M[G] \vDash \varphi$. By the baby truth lemma, fix $s \in G$ such that $s \Vdash^* \varphi$ and consider a common extension of q and s, say r. $r \leq s$ means that $r \Vdash^* \varphi$, contradicting $\neg\exists r \leq q(r \Vdash^* \varphi)$.

Extension of $\Vdash^*$ to $\mathcal{FL}_{\mathbb{P}}$

Definition

For a forcing poset $\mathbb{P}$ and $\varphi, \psi \in \mathcal{FL}_{\mathbb{P}}$ define:

- $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$
- $p \Vdash^* \neg\varphi$ iff $\neg\exists q \leq p(q \Vdash^* \varphi)$
- $p \Vdash^* \exists x \varphi(x)$ iff $\{q \leq p : \exists \tau \in V^{\mathbb{P}}(q \Vdash^* \varphi(\tau))\}$ is dense below p.

Note

There are subtleties in this definition that we will address later.

Truth Lemma for $\Vdash^*$ and equivalence of the relations

Lemma

Let M be a transitive set model for ZF-P and $\mathbb{P} \in M$ a forcing poset, $\varphi \in \mathcal{FL}_{\mathbb{P}} \cap M$ and let G be a $\mathbb{P}$-generic filter over M. Then

$$M[G] \vDash \varphi \iff \text{there is } p \in G \text{ such that } (p \Vdash^* \varphi)^M$$

Lemma

Let M be a countable transitive set model for ZF-P and $\mathbb{P} \in M$ a forcing poset, $\varphi \in \mathcal{FL}_{\mathbb{P}} \cap M$ and $p \in \mathbb{P}$. Then

$$p \Vdash \varphi \iff (p \Vdash^* \varphi)^M$$

.

Making sense of the Definability Lemma

The Definability Lemma

Assume $M \vDash ZF - P$ countable, and let $\psi(x_1, ..., x_n)$ be a formula in the language of set theory. *Then*

$$A := \big\{(p, \mathbb{P}, \leq, \mathbf{1}, \theta_1, ..., \theta_n) : p \in \mathbb{P} \wedge M \ni \langle \mathbb{P}, \leq, \mathbf{1}\rangle \textit{ is a forcing poset} \wedge \theta_1, ..., \theta_n \in M^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P},M} \psi(\theta_1, ..., \theta_n)\big\},$$

is definable over M without parameters ($A \in \mathcal{D}^-(M)$).

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is definable over M without parameters ($A \in \mathcal{D}^-(M)$). I.e., there is $\xi(x)$ s.t. $A = \{x \in M : \xi(x)^M\}$.

The extension of $\Vdash^*$ to $\mathcal{FL}_{\mathbb{P}}$ is done by recursion on the complexity of formulas. In the case of quantifiers, the relation one recurses on is not set like because $\{\varphi(\tau) : \tau \in V^{\mathbb{P}}\}$ is class-sized. In fact defining the forcing relation for $\mathcal{FL}_{\mathbb{P}} \cap M$ entirely inside of M would contradict Tarski's Theorem. Luckily the Definability Lemma doesn't need that.