Martin's Axiom - Basics and Two Applications

Anton Chernev and Rodrigo Nicolau Almeida

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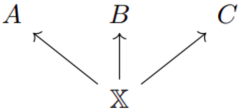

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Definition

We say a forcing poset $\mathbb{P}$ has the (forcing)-ccc if every antichain in $\mathbb{P}$ is countable.

- The key example of a forcing poset with the ccc, for our purposes, is the set of finite partial functions:

Definition

For any I, J, define $Fn(I, J)$ as the set of all finite partial functions ordered by $p \leqslant q$ iff $p \supseteq q$, and with $\mathbf{1} = \emptyset$.

The poset Fn(I,J)

- The proof that the poset $Fn(I, J)$ has the ccc under specific circumstances uses the following lemma:

Lemma

(Δ-System Lemma) Let $(F_\alpha : \alpha < \omega_1)$ be a family of finite subsets of ω_1. Then there is a (stationary) uncountable set S and a finite set R, such that for all $\alpha, \beta \in S$ $A_\alpha \cap A_\beta = R$

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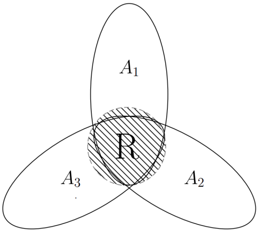

Figure 1: Delta-System Lemma

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- **Proof**: ($\implies$), suppose that J is uncountable, and fix any $i \in I$; then the finite partial functions $\{(i, j)\}$ for $j \in J$ form an uncountable (forcing)-antichain.

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- ($\impliedby$). Suppose that J is countable, and let $(f_\alpha : \alpha < \omega_1)$ be a family of finite partial functions from $\mathbb{P}$. Let $(S_\alpha : \alpha < \omega_1)$ be the domain of these function.

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- ($\impliedby$). Suppose that J is countable, and let $(f_\alpha : \alpha < \omega_1)$ be a family of finite partial functions from $\mathbb{P}$. Let $(S_\alpha : \alpha < \omega_1)$ be the domain of these function.
- By the Δ-System Lemma, let S be uncountable and R finite root such that $S_\alpha \cap S_\beta = R$ for $\alpha, \beta \in S$.

Ccc of Fn(I,J)

- Since R is finite, J^R, the set of functions, is countable. Thus, there must be $\alpha \neq \beta \in S$, such that $f_\alpha \restriction R = f_\beta \restriction R$, since there are uncountably many such f_α. Thus, f_α and f_β agree on R, so they are compatible. Q.E.D

Dense Sets

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- Let I be infinite and J be non-empty.
- $\{q \in Fn(I, J) \mid i \in Dom(q)\}$ is dense for all $i \in I$.
- $\{q \in Fn(I, J) \mid j \in Ran(q)\}$ is dense for all $j \in J$.

Filters

Definition

Let $\mathbb{P}$ be a forcing poset. Then $G \subseteq \mathbb{P}$ is a filter on $\mathbb{P}$ iff

- $\mathbf{1} \in G$
- $\forall p, q \in G \, \exists r \in G (r \leqslant p \wedge r \leqslant q)$
- $\forall p, q \in \mathbb{P} (p \geqslant q \wedge q \in G \to p \in G)$

Martin's Axiom

Definition

- $MA_{\mathbb{P}}(\kappa)$ is the statement that whenever $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leqslant \kappa$, there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.
- $MA(\kappa)$ is the statement that $MA_{\mathbb{P}}(\kappa)$ holds for all ccc $\mathbb{P}$.
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- For $\lambda < \kappa$: $MA_{\mathbb{P}}(\kappa)$ implies $MA_{\mathbb{P}}(\lambda)$ and $MA(\kappa)$ implies $MA(\lambda)$.
- $MA(\kappa)$ implies $\kappa < 2^{\aleph_0}$.
- $MA_{\mathbb{P}}(\aleph_0)$ for all $\mathbb{P}$.
- We cannot drop the ccc requirement – $MA_{\mathbb{P}}(\aleph_1)$ is false for $\mathbb{P} = Fn(\aleph_0, \aleph_1)$.

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Lemma

$MA(\kappa)$ implies $\kappa < 2^{\aleph_0}$.

Proof Sketch:

- If G is a filter on $Fn(I, J)$ (I infinite, J non-empty), then $f_G := \bigcup G$ is a function.

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- We have $\aleph_0$ many D_i and $2^{\aleph_0}$ many E_h.
- MA gives us G that meets these dense sets.
- $f_G : I \to J$ and $f_G \neq h$ for all $h \in J^I$ – contradiction.

Suslin

Figure 2: Mikhail Suslin (1894-1919)

- Context: Trying to axiomatise the properties of the ordered real line.

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- Context: Trying to axiomatise the properties of the ordered real line.
- At the time Suslin was working, it was known by a result of Cantor that the real line was, up to order-isomorphism, characterised as follows:
  - (1) R is complete, dense and unbounded linear order
  - (2) R is separable (having a countable dense subset)

Suslins Hypothesis

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Suslins' Hypothesis

(SH) There are no Suslin lines.

Suslins Hypothesis

- This was shown to be independent of ZFC by several authors: (Jech (1967), Tennenbaum (1968), Jensen (1970)) all proved that there are models where Suslin lines exist.

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- In 1971, Robert Solovay and Stanley Tennenbaum showed that SH was relatively consistent with ZFC, by using a forcing argument. Namely, what they showed was that $MA + \neg CH$ was relatively consistent, by developing a theory of transfinite iterated forcing.
- This was also the first **forcing axiom**: a way of "internalising" forcing to ZFC, and obtaining consequences from it. Other stronger forcing axioms are mentioned by Kunen.

Suslin Lines and Suslin Trees

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- S is complete, dense and unbounded
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Definition

A tree $(T, <)$ is called a (normal) **Suslin tree** if:

- T has height ω_1
- All levels of T are countable
- All (poset)-antichains and all branches of T are countable
- *T has a unique root*
- *Every element $x \in T$ has uncountably many successors*

- We will now brief sketch one implication of a result of Kurepa (1934) that shows that these notions are equivalent:

Suslin line implies Suslin tree

Theorem

If there is a Suslin line, then there is a Suslin tree.

- **Proof Sketch**: Let $(S, <)$ be a Suslin line. We construct the Suslin tree out of the closed intervals for this line, and for $I, J \in \mathbf{C}$, we let $I \preceq J$ if $I \supseteq J$.

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- This is defined by recursion, letting $I_0 = [a_0, b_0]$ for arbitrary $a_0, b_0 \in S$; and for each α, letting C be the set of endpoints of intervals considered so far; since S is not separable, C is not dense, so we let I_α be an interval disjoint from all the endpoints of I_β for $\beta < \alpha$.

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- We let $T = \{I_\alpha : \alpha < \omega_1\}$. Then this will be a tree, as one can prove the predecessors form a well-order.

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- We let $T = \{I_\alpha : \alpha < \omega_1\}$. Then this will be a tree, as one can prove the predecessors form a well-order.
- It has the (poset)-ccc, because S has the ccc; the ccc also yields the non-existence of a branch of size ω_1.

MA+¬CH implies SH

- Assume $MA(\aleph_1)$; let $(T, <)$ be a Suslin tree. We let $(T^*, <^*)$ be the conversely ordered tree (i.e, $x <^* y$ iff $x > y$)

MA+¬CH implies SH

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- Since there is a unique root, this makes T^* into a forcing poset. Note that x, y, we have that if $x \not\perp y$, then there must be some $z <^* x.y$; since T is a tree, this can only happen if x and y are comparable. Thus comparability equals compatibility, so T^* satisfies the (forcing)-ccc.

MA+¬CH implies SH (continued)

- Let $T = \{x_\alpha : \alpha < \omega_1\}$ be an enumeration of the tree. Note that for each α, consider the set:

$$T_{x_\alpha} := \{x_\beta \in T^* : \alpha \leqslant \beta < \omega_1\}$$

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- Each such set is dense. If not, there would be a $p \in T$ such that for no γ, $x_\gamma \in T_{x_\alpha}$ and $x_\gamma \leqslant^* p$. Thus if $y_\gamma \leqslant^* p$, $\gamma < \alpha$. But this is absurd, since then p would only have countably many successors in T.

MA$+\neg$CH implies SH (continued)

- So consider $(T_{x_\alpha} : x_\alpha \in T^*)$ a family of dense sets. By MA, there is a generic filter, D, which intersects each set.

MA+$\neg$CH implies SH (continued)

- So consider $(T_{x_\alpha} : x_\alpha \in T^*)$ a family of dense sets. By MA, there is a generic filter, D, which intersects each set.
- Since all elements of a filter have to be compatible, by our earlier remarks, D has to be totally ordered. So D is a branch in T, and since it intersects all the T_{x_α}, it has to have length ω_1, in contradiction to T being Suslin. Q.E.D

Cardinal Exponentiation

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- *CH* helps us with that.
- *MA* helps too!

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Proof:

- $2^{\aleph_0} \leqslant 2^\kappa$ because $\aleph_0 \leqslant \kappa$.

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MA(κ) *implies* $2^\kappa = 2^{\aleph_0}$.

Proof:

- $2^{\aleph_0} \leqslant 2^\kappa$ because $\aleph_0 \leqslant \kappa$.
- For $2^\kappa \leqslant 2^{\aleph_0}$ we need an injection F that maps subsets of κ to subsets of $\aleph_0$.

Cardinal Exponentiation

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Proof:

- $2^{\aleph_0} \leqslant 2^\kappa$ because $\aleph_0 \leqslant \kappa$.
- For $2^\kappa \leqslant 2^{\aleph_0}$ we need an injection F that maps subsets of κ to subsets of $\aleph_0$.
- What can MA give us?

$(\mathbf{A}, \mathbf{B})$-sets

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Definition

Let $\mathbf{A}$ and $\mathbf{B}$ be collections of subsets of ω. We call C an $(\mathbf{A}, \mathbf{B})$-set if:

- $C \cap A$ is finite for every $A \in \mathbf{A}$.
- $C \cap B$ is infinite for every $B \in \mathbf{B}$.
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- C covers "very little" of A, for all A.
- C covers "a lot" of B, for all B.
- C does not cover "too much" of B, for all B.

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- C does not cover "too much" of B, for all B.
- A does not cover "too much" of B, for all A, B.
- Cannot cover "too much" of B with finitely many As.

A-small sets

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Let $\mathbf{A}$ and $\mathbf{B}$ be collections of subsets of ω. We call C an $(\mathbf{A}, \mathbf{B})$-set if:

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Definition

A set D is $\mathbf{A}$-small if there are sets $A_1, \ldots, A_n \in \mathbf{A}$ such that $D \backslash (A_1 \cup \ldots \cup A_n)$ is finite.

$\mathbf{A}$-small sets

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Definition

A set D is $\mathbf{A}$-small if there are sets $A_1, \ldots, A_n \in \mathbf{A}$ such that $D \backslash (A_1 \cup \ldots \cup A_n)$ is finite.

- Can cover "too much/almost everything" of D with finitely many As.
- If there is an $(\mathbf{A}, \mathbf{B})$-set, then B is not $\mathbf{A}$-small for any B.

The Existence Lemma

Lemma

Let $MA(\kappa)$ hold. Let $\mathbf{A}$ *and* $\mathbf{B}$ *be collections of subsets of* ω *of cardinality* $\leqslant \kappa$. *If no set* $B \in \mathbf{B}$ *is* $\mathbf{A}$*-small, then there is an* $(\mathbf{A}, \mathbf{B})$*-set.*

The Sequence Lemma

Lemma

Let $MA(\kappa)$ hold. Then there is a κ-long sequence $\mathbf{S}$ of subsets of ω such that:

- *If $D \in \mathbf{S}$, then D is not $\mathbf{S} \backslash \{D\}$-small.*

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- *If $D \in \mathbf{S}$, then D is not $\mathbf{S} \setminus \{D\}$-small.*

- **Corollary:** If we split $\mathbf{S}$ into $\mathbf{A}$ and $\mathbf{B}$, then there is an $(\mathbf{A}, \mathbf{B})$-set.

Putting everything together

Theorem

MA(κ) *implies* $2^\kappa = 2^{\aleph_0}$.

Proof:

- For $2^\kappa \leqslant 2^{\aleph_0}$ we need an injection F that maps subsets of κ to subsets of $\aleph_0$.

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- The Sequence Lemma gives us a sequence **S**.

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Proof:

- For $2^\kappa \leqslant 2^{\aleph_0}$ we need an injection F that maps subsets of κ to subsets of $\aleph_0$.
- The Sequence Lemma gives us a sequence $\mathbf{S}$.
- Every subset $X \subseteq \kappa$ determines a partition $\mathbf{A}_X \cup \mathbf{B}_X = \mathbf{S}$.

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Proof:

- For $2^\kappa \leqslant 2^{\aleph_0}$ we need an injection F that maps subsets of κ to subsets of $\aleph_0$.
- The Sequence Lemma gives us a sequence $\mathbf{S}$.
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- $F(X) = C_X$ is injective. Why?
- Let $X \neq Y$. Then $D \in \mathbf{A}_X$ and $D \in \mathbf{B}_Y$ for some D. Now $D \cap C_X$ is finite, $D \cap C_Y$ is infinite. So $C_X \neq C_Y$.

Proof of the Existence Lemma

Lemma

Let $MA(\kappa)$ hold. Let $\mathbf{A}$ and $\mathbf{B}$ be collections of subsets of ω of cardinality $\leqslant \kappa$. If no set $B \in \mathbf{B}$ is $\mathbf{A}$-small, then there is an $(\mathbf{A}, \mathbf{B})$-set.

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- For all $A \in \mathbf{A}$: $D_A = \{g \mid A \subseteq Dom(g)\}$ is dense.
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- If $g \in \mathbb{P}$, then $Dom(g)$ is $\mathbf{A}$-small and B is not $\mathbf{A}$-small so $B \backslash Dom(g)$ is infinite.

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Lemma

Let $MA(\kappa)$ hold. Then there is a κ-long sequence $\mathbf{S}$ of subsets of ω such that:

- *If $D \in \mathbf{S}$, then D is not $\mathbf{S} \backslash \{D\}$-small.*

Proof Idea:

- Let $\mathbf{S}$ of cardinality $\leqslant \kappa$ satisfy the above condition.

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- If $\mathbf{A} \cup \mathbf{B} = \mathbf{S}$ is a partition, then there is an $(\mathbf{A}, \mathbf{B})$-set C such that $S \cup \{C\}$ preserves all conditions.

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- Thank you!