Canonical tree structure of graphs

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March 7, 2014

Outline

1. motivation
2. decomposing graphs
3. an infinite detour
4. ideas from the proof
5. a new notion: k-profiles
6. k-blocks
7. algorithms

Decompositions of graphs into their 'k-connected pieces'.

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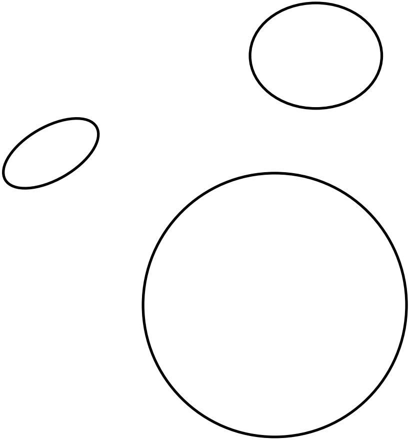

- $k = 1$: components

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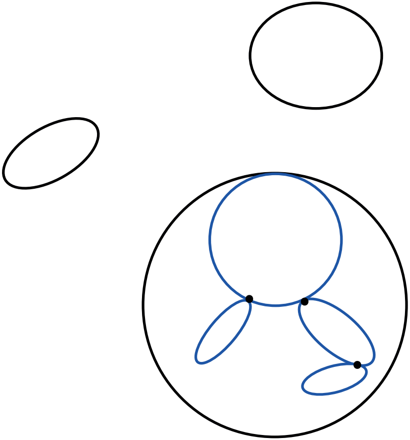

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- $k = 2$: block cut-vertex tree

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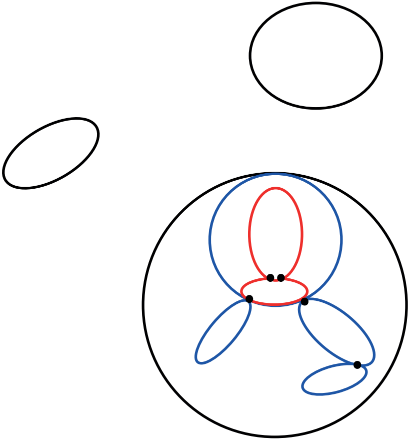

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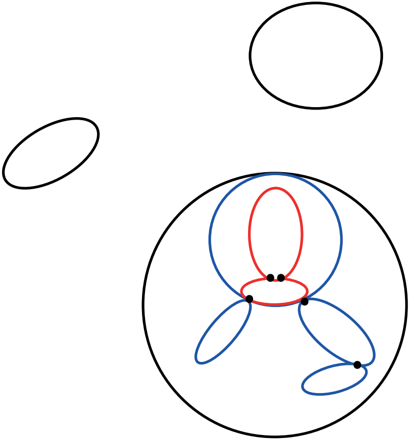

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- $k = 2$: block cut-vertex tree
- $k = 3$: Tutte
- $k \geq 4$: ???

Tutte's result

Theorem (Tutte)

Every 2-connected graphs admits a tree-decomposition (invariant under the automorphisms of the graph) such that each torso is either 3-connected or a cycle.

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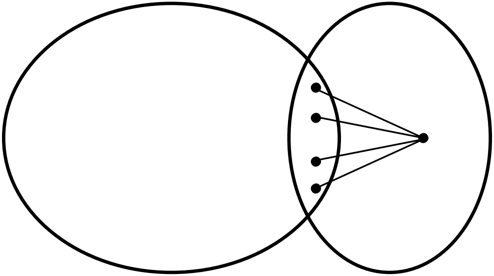
Torsos fail for $k \geq 4$:



Definition

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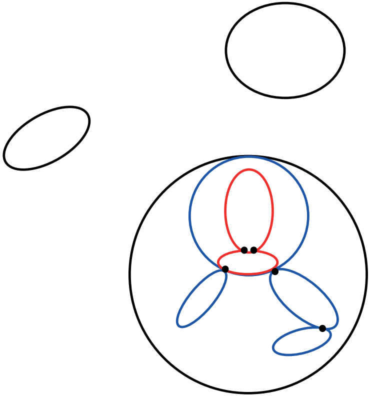
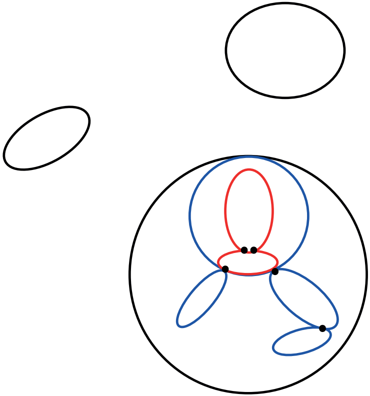
Theorem (Carmesin, Diestel, Hundertmark, Stein)

For every k and every graph G, there exists a canonical tree-decomposition that distinguishes the k-blocks of G.

Iterated decomposition





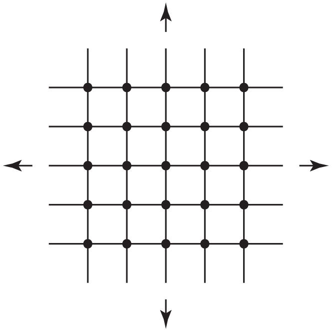
Theorem (Carmesin, Diestel, Hundertmark, Stein)

For every graph G, there exists a canonical tree-decomposition that distinguishes the robust k-blocks of G for all k.

Definition

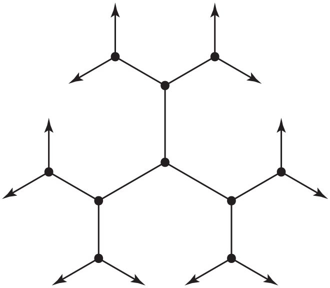
A ray is a one-way infinite path. Two rays in a graph G are equivalent if they lie eventually in the same component of $G - S$ for every finite vertex set S. The equivalence classes of this equivalence relation are the ends of the graph.

Ends of graphs: examples



one end

two ends



infinitely many ends

Definition

A graph is accessible if there is some $n \in \mathbb{N}$ such that each two ends are separated by at most n vertices.

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Theorem (Thomassen & Woess)

A group is accessible if and only if some (and hence any) of its locally finite Cayley graphs is accessible.

THEOREM (H.)

Locally finite vertex-transitive graphs whose cycle spaces are generated by cycles of bounded length are accessible.

Accessibility

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Corollary (H.)

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COROLLARY (H.)

Locally finite hyperbolic vertex-transitive graphs are accessible.

COROLLARY (DUNWOODY)

Finitely presented groups are accessible.

THEOREM (ROBERTSON & SEYMOUR)

For every graph and $k \in \mathbb{N}$, there exists a ~~canonical~~ tree-decomposition that distinguishes all tangles of order k.

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For every graph and $k \in \mathbb{N}$, there exists a ~~canonical~~ tree-decomposition that distinguishes all tangles of order k.

Is it possible, to obtain a *canonical* such tree-decomposition?

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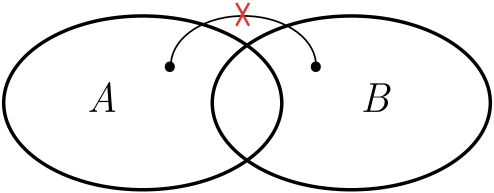
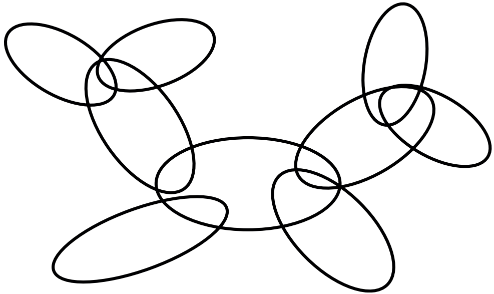
Definition

A separation is a pair (A, B) of vertex sets with $A \cup B = V$ and $G[A] \cup G[B] = G$. Its order is $|A \cap B|$.



Each tree-decomposition belongs to a unique nested set of separations of the graph.



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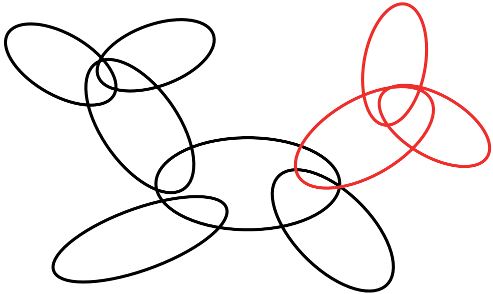

Task: Find a canonical nested set $\mathcal{N}$ of separations of order less than k that distinguishes all k-blocks.

We have a set $\mathcal{S}$ of separations of order less than k and a set $\mathcal{B}$ of k-blocks and we search for a nested set $\mathcal{N}$ distinguishing $\mathcal{B}$.

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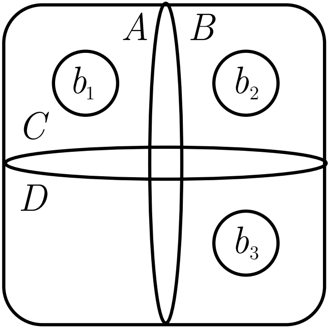

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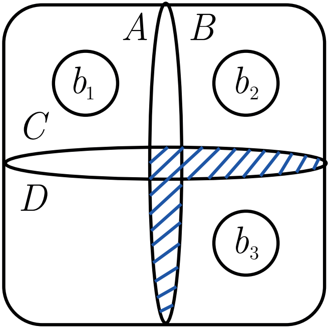

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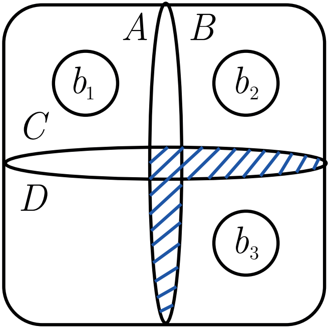
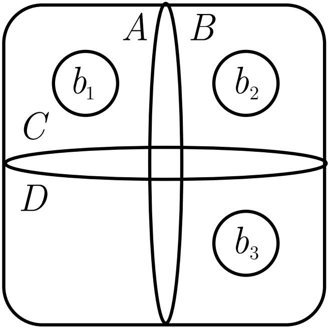


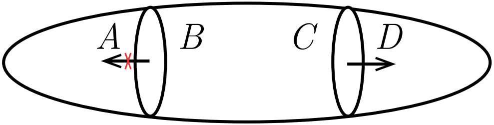
We need a lemma like
"If $\mathcal{S}$ is rich enough in term of $\mathcal{B}$, then we find $\mathcal{N}$."

The only necessary fact about the relation between $\mathcal{S}$ and $\mathcal{B}$ is:
If $(A, B) \in \mathcal{S}$ and $b_2 \in \mathcal{B}$ on which side of (A, B) does b_2 live?



Every k-block induces an orientation of every separation of order less than k which is consistent in that the set P of these orientations satisfies

(P1) $(A, B) \leq (C, D) \in P \Rightarrow (B, A) \notin P$



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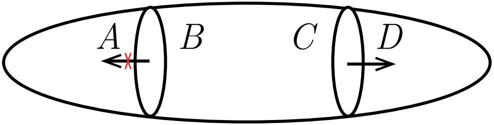
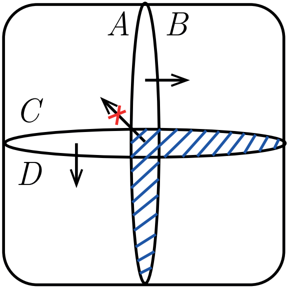


(P2) $(A, B), (C, D) \in P \Rightarrow (B \cap D, A \cup C) \notin P$



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A *k-profile* is an orientation of all separations of order less than k that satisfies (P1) and (P2).

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REMARK

Tangles of order k are k-profiles.

Profiles

Theorem (Carmesin, Diestel, H., Hundertmark)

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COROLLARY

Tangles of order k can be separated canonically.

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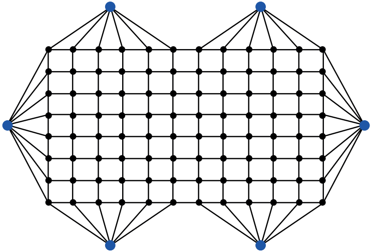
2. connectivity induced by some dense substructure:



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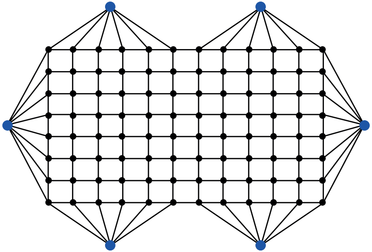

3. connectivity given without any dense substructure:
   Take an independent set on k vertices and add k internally disjoint paths between every two of those vertices.

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Relate other graph parameters to the existence of k-blocks.

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2. *There exists a polynomial time algorithm to find all k-blocks (for fixed and for variable k).*
3. *There exists a polynomial time algorithm to find the decomposition tree (for fixed k).*