Accessible groups and graphs

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2 October 2015

Outline

1. Accessibility in groups
2. Reinterpreting Dunwoody's accessibility theorem
3. Accessibility in graphs
4. Outlook

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Stallings's structure theorem

Theorem (Stallings 1971)

Every finitely generated group G with more than one end splits non-trivially over a finite subgroup C, that is, $G = *_C A$ or $G = A *_C B$ for some subgroups $A \neq C \neq B$.

Splitting recursively

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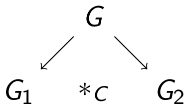

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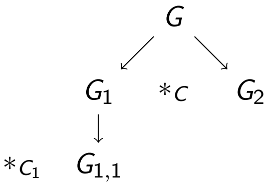

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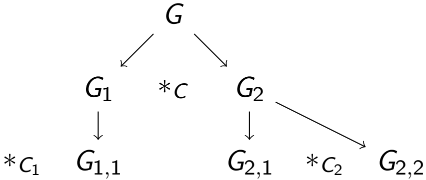

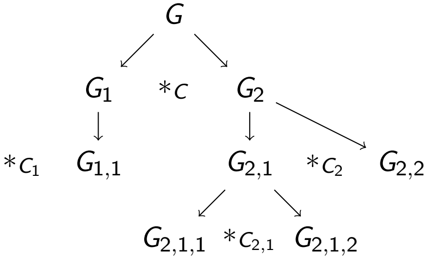
G
G_1
$*C$
G_2
$*C_1$
$G_{1,1}$
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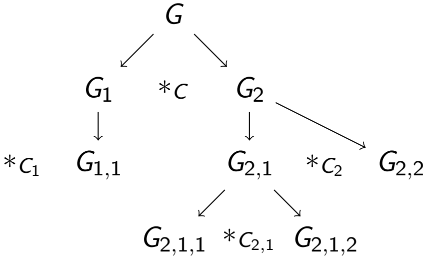

Definition

A finitely generated group is *accessible* if this process of successively decomposing factors with more than one end terminates after finitely many steps.

Wall's conjecture

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Every finitely generated group is accessible.

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- Disproved by Dunwoody 1993.

Remark

Finitely generated free groups are accessible.

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Theorem (Gromov 1987)

Finitely generated hyperbolic groups are finitely presented.

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Theorem (Droms 2006)

Finitely generated planar groups are finitely presented and accessible.

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The idea

groups	$\longleftrightarrow$	Cayley graphs	$\longrightarrow$	other graph classes
$\Downarrow$		$\Downarrow$		$\Downarrow$
accessibility theorem	$\longleftrightarrow$	?	$\longrightarrow$	??

Reformulating Dunwoody's theorem

Theorem (Dunwoody 1985)

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A finitely presented group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ has a locally finite Cayley graph Γ whose first homology group is generated by $\{g(C) \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of closed walks corresponding to the relators in $\mathcal{R}$

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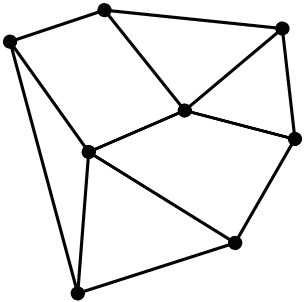

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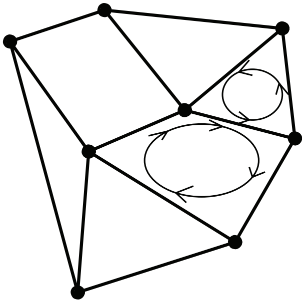

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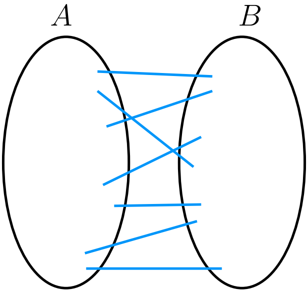

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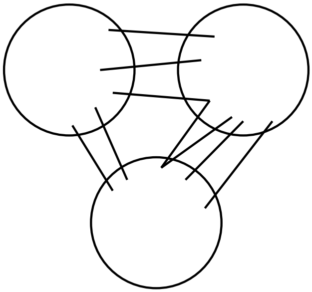

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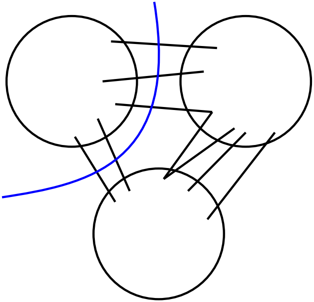

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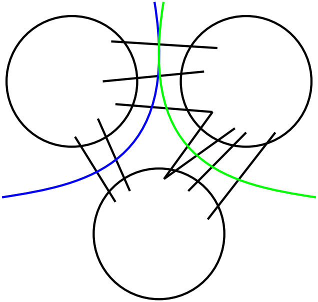

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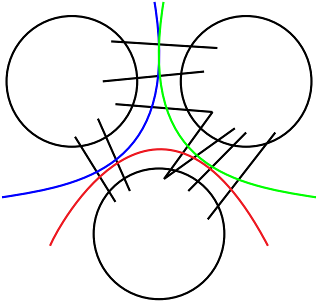

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Theorem (Dicks & Dunwoody 1989)

The cut space of a locally finite Cayley graph G of a finitely generated accessible group is a finitely generated $\mathrm{Aut}(G)$*-module.*

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Let G be a locally finite Cayley graph. If its first homology group is a finitely generated $\mathrm{Aut}(G)$*-module, then so is its cut space.*

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Theorem (?)

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Cycle space

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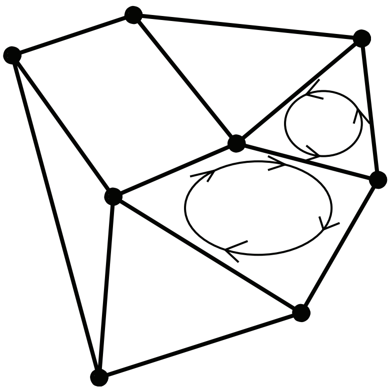

Theorem (H. 2015$^+$)

Let G be a 2-edge-connected transitive graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$-module, then so is its cut space.

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Remark

Bieri and Strebel (1980) gave an example of a finitely generated accessible group that is not finitely presentable, that is, of a Cayley graph G whose cut space is a finitely generated $\mathrm{Aut}(G)$-module but its first homology group is not.

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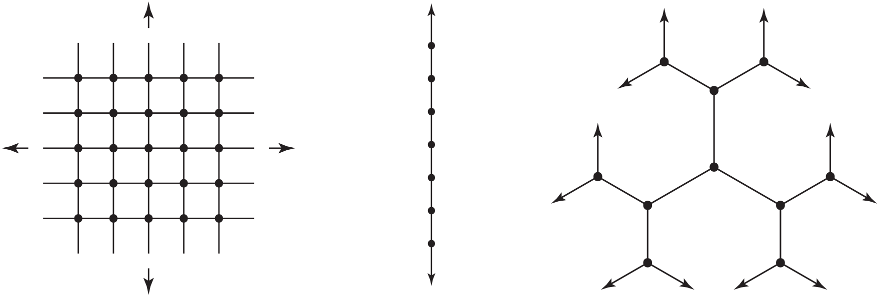

Definition

A graph is *accessible* if there is some $k \in \mathbb{N}$ such that for any two distinct ends, there an edge set of size at most k separating them.

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Theorem (Thomassen & Woess 1993)

A finitely generated group is accessible if and only if one (and hence every) of its locally finite Cayley graphs is accessible.

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Theorem (Dunwoody 1985)

Every locally finite Cayley graph G whose first homology group is a finitely generated $\mathrm{Aut}(G)$*-module is accessible.*

Conjecture (Diestel 2010)

Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.

A conjecture is confirmed

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Applications I

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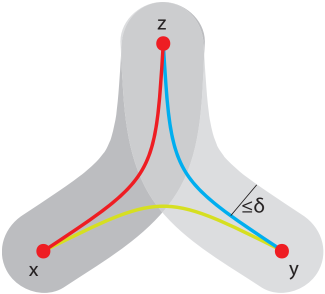
Theorem (Dunwoody 2007)

Every locally finite transitive planar graph is accessible.

Application II: hyperbolic graphs

Definition

A connected graph G is called hyperbolic if there exists some $\delta \geq 0$ such that for any three vertices x, y, z of G and for any three shortest paths, one between every two of the vertices, each of those paths lies in the δ-neighbourhood of the union of the other two.



Application II: hyperbolic graphs

Theorem (Gromov 1987)

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QUESTION

How can we translate Stallings's theorem in graph theoretic notions?