Homogeneity in infinite graphs

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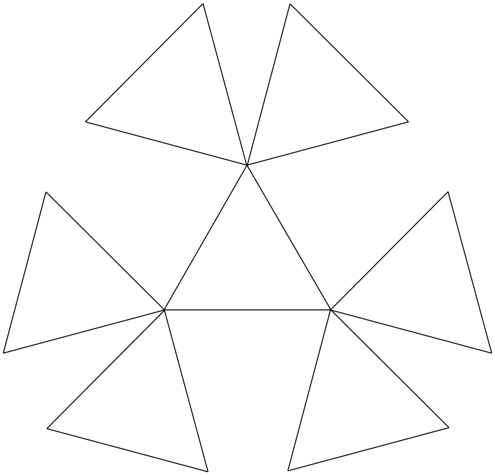
A graph is distance-transitive if it is k-distance-transitive for every $k \in \mathbb{N}$.

The graphs $X_{k,\ell}$

$X_{k,\ell}$ is the graph of connectivity 1 such that every block is a complete graph on k vertices and every vertex lies in ℓ such blocks.



The graph $X_{3,3}$.

Distance-transitive graphs

Theorem (Macpherson '82)

The connected locally finite distance-transitive graphs are the graphs $X_{k,\ell}$ for integers $k, \ell \geq 2$.

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The connected locally finite 2-distance-transitive graphs with more than one end are the connected locally finite distance-transitive graphs.

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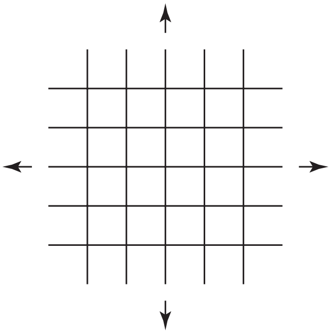
Observation

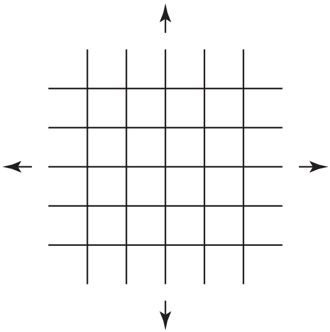
A graph has more than one end if and only if there is a finite vertex set whose deletion leaves two components each of which contains a ray.

Ends of graphs



one end

Ends of graphs



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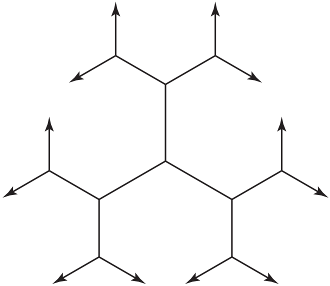

infinitely many ends

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Theorem (H + Pott)

For a connected graph G with more than one end the following assertions are equivalent:

- *G is distance-transitive;*
- *G is 2-distance-transitive;*
- *$G \cong X_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$.*

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Remark

- The 1-CS-transitive graphs are the vertex-transitive graphs.
- The 2-CS-transitive graphs are the edge-transitive graphs.

Classification results

- Gray '09: classification of all connected locally finite k-CS-transitive graphs with more than one end for $k = 3$.

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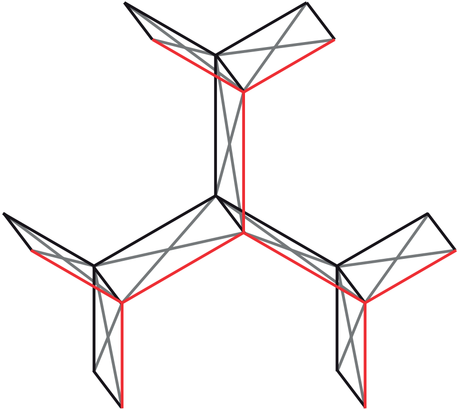
There are three distinct infinite families of connected k-CS-transitive graphs with more than one end (for $k \geq 3$).

1. The graphs $X_{\kappa,\lambda}(E)$ for certain cardinals κ, λ and some finite homogeneous graph E: replace every vertex in $X_{\kappa,\lambda}$ by a copy of E and join two vertices of distinct copies if they replace adjacent vertices in $X_{\kappa,\lambda}$.



The graph $X_{2,3}(K_2)$
(k-CS-transitive for $k \geq 3$)

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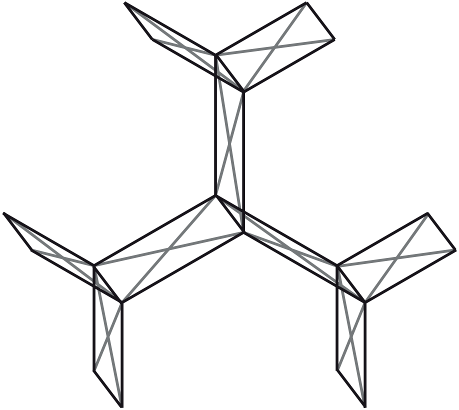

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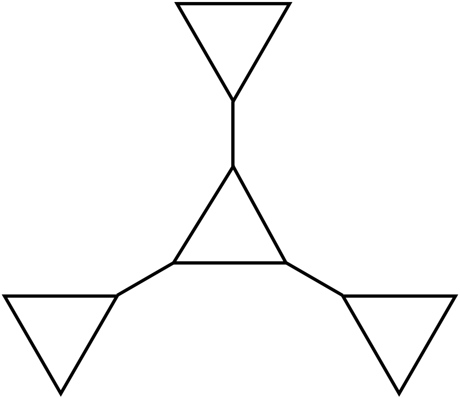
These graphs occur for any k:

- $X_{\kappa,\lambda}(K_1)$ for any $\kappa, \lambda \geq 2$;
- $X_{2,\lambda}(K_n)$ for any $\lambda \geq 2$ and $n < \frac{k}{2} + 1$;
- $X_{\kappa,2}(\overline{K_m})$ for any $\kappa \geq 2$ and $m < \frac{k+2}{3}$;
- $X_{2,2}(E)$ for certain finite homogeneous graphs E (depending on k).

2. The graphs Y_κ for some cardinal κ (if k is odd): graphs of connectivity 1 such that every vertex lies in precisely two blocks, one of size 2 and one complete graph on κ vertices.



The graph Y_3
(k-CS-transitive for odd k)

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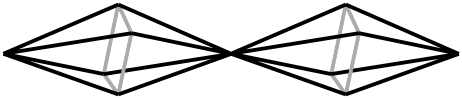
These graphs occur for any odd k:

- Y_κ for any $\kappa \geq 3$.

3. The graphs $Z_{\kappa,\lambda}(E_1, E_2)$ for certain cardinals κ, λ and finite homogeneous graphs E_1, E_2 (if k is even):
   replace in a semi-regular tree with degrees κ and λ every second vertex by a copy of E_1 and the other vertices by a copy of E_2. Then join two vertices in distinct copies by an edge if these copies replace adjacent vertices of the tree.



The graph $Z_{2,2}(K_1, C_4)$
(k-CS-transitive for even $k \geq 4$)

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   replace in a semi-regular tree with degrees κ and λ every second vertex by a copy of E_1 and the other vertices by a copy of E_2. Then join two vertices in distinct copies by an edge if these copies replace adjacent vertices of the tree.

These graphs occur for any even k:

- $Z_{2,2}(\overline{K_m}, K_n)$ for any m, n with $2m + n < k + 1$;
- $Z_{\kappa,\lambda}(K_1, K_n)$ for any $n \leq k - 1$ and either $\kappa = 2$ or $\lambda = 2$;
- $Z_{2,2}(K_1, E)$ for certain finite homogeneous graphs E (depending on k).

Theorem (H + Pott)

A connected graph with more than one end is k-CS-transitive (for some $k \geq 3$) if and only if it is one of the following graphs:

1. $X_{\kappa,\lambda}(K_1)$ *for any* $\kappa, \lambda \geq 2$;
2. $X_{2,\lambda}(K_n)$ *for any* $\lambda \geq 2$ *and* $n < \frac{k}{2} + 1$;
3. $X_{\kappa,2}(\overline{K_m})$ *for any* $\kappa \geq 2$ *and* $m < \frac{k+2}{3}$;
4. $X_{2,2}(E)$ *for certain finite homogeneous graphs E (depending on k);*
5. Y_κ *for any* $\kappa \geq 3$ *(if k is odd);*
6. $Z_{2,2}(\overline{K_m}, K_n)$ *for any* m, n *with* $2m + n < k + 1$ *(if k is even);*
7. $Z_{\kappa,\lambda}(K_1, K_n)$ *for any* $n \leq k - 1$ *and either* $\kappa = 2$ *or* $\lambda = 2$ *(if k is even);*
8. $Z_{2,2}(K_1, E)$ *for certain finite homogeneous graphs E (depending on k) (if k is even).*

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A graph is C-homogeneous (or connected-homogeneous) if every isomorphism between two connected induced subgraphs extends to an automorphism of the whole graph.

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Corollary (H + Pott)

The connected C-homogeneous graphs with more than one end are the connected distance-transitive graphs with more than one end.