Accessibility in transitive graphs

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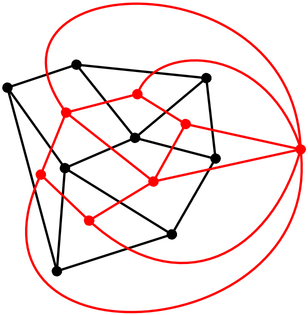

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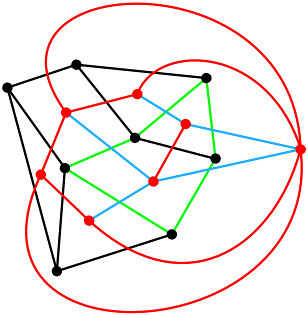

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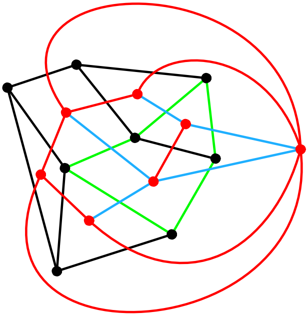


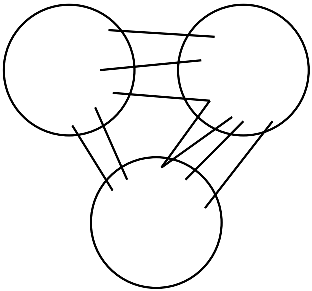
Folklore

The cycles of a planar graph are the minimal cuts of its dual.

Cut space

Definition

A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set.
The cut space is the set of all finite sums (over $\mathrm{GF}(2)$) of finite cuts.



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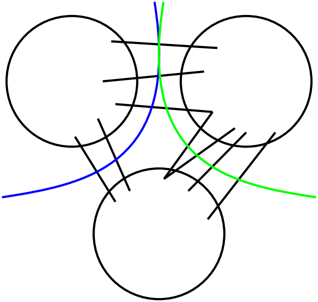

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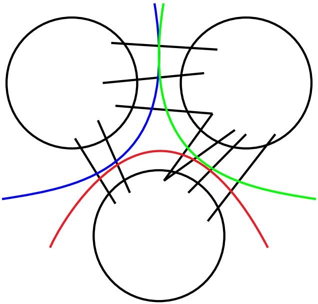
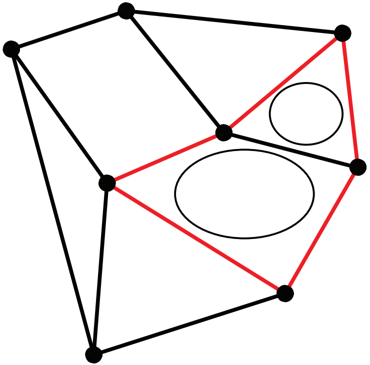


Cycle space

Definition

- The cycle space of a graph is the set of all finite sums (over GF(2)) of edge sets of finite cycles.



REMARK

(1) In a finite graph the cut space is the orthogonal space of the cycle space and vice versa.

(2) In a finite graph with n vertices and m edges, the cut space has dimension $n-1$ and the cycle space has dimension $m-n+1$.

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Is (2) interesting for infinite graphs?

A first result for infinite graphs

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

A finitely presented group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ has a locally finite Cayley graph Γ whose fundamental group is generated by $\{C^g \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of closed walks corresponding to the relators in $\mathcal{R}$

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The cut space of a locally finite Cayley graph G of a finitely generated accessible group is a finitely generated $\mathrm{Aut}(G)$*-module.*

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Theorem (Dunwoody 1985)

Let G be a locally finite Cayley graph. If its fundamental group is a finitely generated $\mathrm{Aut}(G)$*-module, then so is its cut space.*

THEOREM

Let G be a 2-edge-connected transitive graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$-module, then so is its cut space.

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Remark

Bieri and Strebel (1980) gave an example of a finitely generated accessible group that is not finitely presentable, that is, of a Cayley graph G whose cut space is a finitely generated $\mathrm{Aut}(G)$-module but whose fundamental group is not.

Brief sketch of the proof

Theorem (Dicks & Dunwoody 1989)

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Let $\mathcal{C}$ be a set of finitely many cycles with their $\mathrm{Aut}(G)$-images that generates the cycle space.

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If $\mathcal{E}'$ has *many* orbits, one of them has never a minimal or maximal element of any such chain with $C \in \mathcal{C}$.

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If $\mathcal{E}'$ has *many* orbits, one of them has never a minimal or maximal element of any such chain with $C \in \mathcal{C}$.
But such a bipartition cannot exist. □

Going to infinity: accessibility

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Theorem (Dunwoody 1985)

Every locally finite Cayley graph G whose fundamental group is a finitely generated $\mathrm{Aut}(G)$-module is accessible.

A conjecture

Conjecture (Diestel 2010)

Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.

A conjecture is confirmed

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Applications

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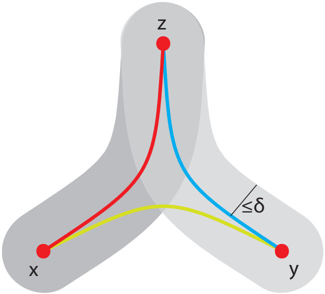
Theorem (Dunwoody 2007)

Every locally finite transitive planar graph is accessible.

Definition

A connected graph G is called hyperbolic if there exists some $\delta \geq 0$ such that for any three vertices x, y, z of G and for any three shortest paths, one between every two of the vertices, each of those paths lies in the δ-neighbourhood of the union of the other two.



Application II: hyperbolic graphs

Theorem (Gromov 1987)

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