Splitting quasi-transitive infinite graphs

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Outline

1. Canonical tree-decompositions
2. Tree amalgamations
3. Accessibility
4. Applications and Outlook

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For a connected graph G, its block graph has the cutvertices of G and its blocks as vertices – i. e. its maximal 2-connected subgraphs and separating edges. Every cutvertex is adjacent to the blocks it is contained in.

Proposition

For every connected graph, its block graph is a tree.

Block graphs

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Proposition

For every connected graph, its block graph is a tree.

Roughly saying, Tutte proved a similar theorem for the 3-connected pieces of 2-connected graphs.

Tree-decompositions

A tree-decomposition of a graph G is a pair $(T, \mathcal{V})$ of a tree T and a set $\mathcal{V} = \{V_t \mid t \in V(T), V_t \subseteq V(G)\}$ such that

1. $\bigcup_{t \in V(T)} V_t = V(G)$;
2. for every edge in G there is some V_t that contains both its incident vertices;
3. for every t on a $t_1 - t_2$ path in T we have $V_{t_1} \cap V_{t_2} \subseteq V_t$.

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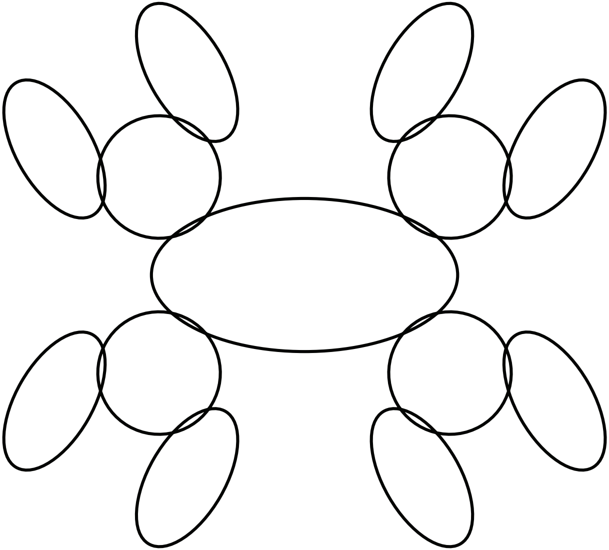
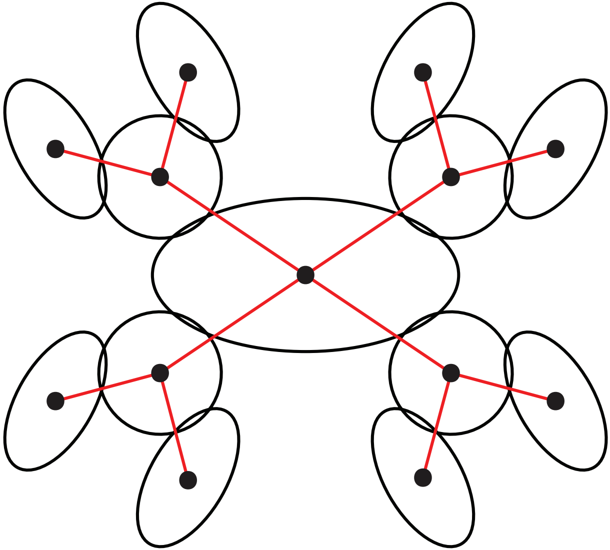
For $t \in V(T)$, the set V_t is a part of the tree-decomposition and its torso is the graph induced by V_t with additional edges between every two vertices that lie in $V_t \cap V_{t'}$ for any t' adjacent to t.



Tree-decompositions



Theorem (Tutte)

Every finite 2-connected graph admits a (canonical) tree-decomposition whose torsos are either 3-connected or cycles.

Richter extended this theorem to infinite graphs.

Tutte's theorem

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How does this extend to to higher connectivity?

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- Based upon a notion of Mader, Dunwoody and Krön used k-blocks as highly connected pieces and showed that they can be distinguished under certain circumstances by a canonical tree-decomposition.

Works of Carmesin, Diestel, H, Hundermark, Lemanczyk, Miraftab and Stein resulted in:

Theorem

Let G be a locally finite graph and let $\mathcal{P}$ be a set of distinguishable robust profiles such that for every $P \in \mathcal{P}$ there is some $\ell \in \mathbb{N} \cup \{\infty\}$ such that P is an ℓ-profile. Then there is a canonical tree-decomposition that distinguishes $\mathcal{P}$ efficiently.

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For graphs of arbitrary degree, the (direct) analogue is no longer true, but there is a result for them as well.

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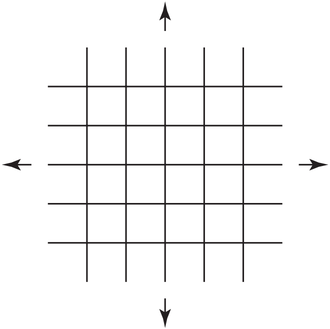
Theorem (Folklore)

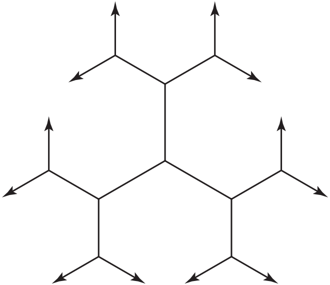
Every locally finite quasi-transitive connected graph has either 0, 1, 2 or infinitely many ends.

Ends of graphs



one end

two ends



infinitely many ends

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- Quasi-transitive connected graphs with exactly two ends are quasi- isometric to $\mathbb{Z}$.
- Can we construct all quasi-transitive locally finite connected graphs with infinitely many ends by taking as building blocks only quasi-transitive locally finite connected graphs with at most two ends?

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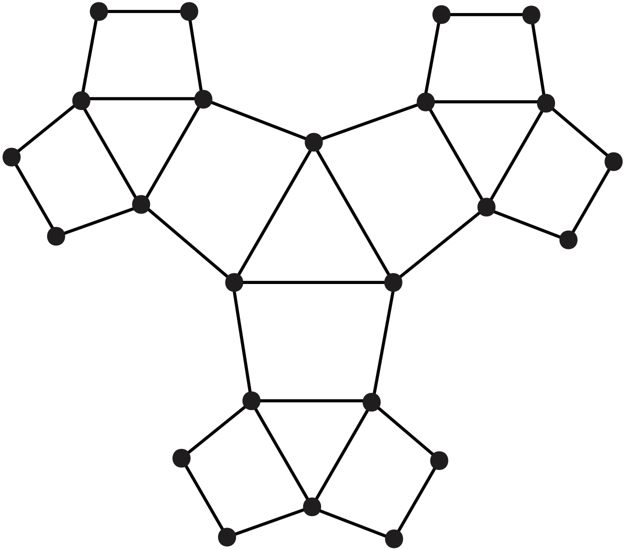
Repeat this for each of the new copies of H (except for the subgraph F_H) and so on.
The resulting graph $G * H$ is the tree amalgamation of G and H and we call G and H its factors.

A construction: example $G = C_3$, $H = C_4$



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- Thus, not every canonical tree-decomposition is induced by a tree amalgamation.
- If for G and H there are at least two Γ_G-, Γ_H-images of F_G, of F_H, respectively, then $G * H$ has more than one end.

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1. Is every quasi-transitive locally finite graph with more than one end the tree amalgamation of two quasi-transitive locally finite graphs?
2. If we start with the class of all finite and one-ended locally finite quasi-transitive graphs and construct tree amalgamations iteratively, do we end up with the class of all locally finite quasi-transitive graphs?

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Every connected, quasi-transitive, locally finite graph with more than one end is a non-trivial tree amalgamation of finite adhesion of two connected, quasi-transitive, locally finite graphs.

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Among those take one with exactly one $\mathrm{Aut}(G)$-orbit on $E(T)$ and thus at most two $\mathrm{Aut}(G)$-orbits on the set $\{V_t \mid t \in V(T)\}$.

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Furthermore, pick the tree-decomposition such that the graphs $G[V_t]$ are connected.
This tree-decomposition then gives rise to a tree amalgamation $G = G[V_t] * G[V_{t'}]$ for adjacent $t, t' \in V(T)$. □

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A splitting theorem

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Our result implies:

Theorem (Stallings)

Every finitely generated group with more than one end splits over a finite group as free product with amalgamation or HNN-extension.

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The Cayley graph of a finitely generated group is accessible if and only if the group is accessible.
Dunwoody constructed inaccessible groups. Thus, there are inaccessible quasi-transitive connected locally finite graphs.

Thomassen and Woess defined accessibility via the existence of some $n \in \mathbb{N}$ such that every two ends can be separated by at most n edges.

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3. This generalises Dunwoody's accessibility theorem for finitely presented groups.

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Theorem (H, Miraftab)

For every accessible quasi-transitive locally finite connected graph, each of its factorisation processes stops after finitely many steps.

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Further results were obtained in the following areas:

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3. (H) A bound on the asymptotic dimension of the tree amalgamation of quasi-transitive locally finite connected graphs in terms of the asymptotic dimension of their factors.
   This generalises results on finitely generated groups by Bell and Dranishnikov, by Dranishnikov and by Tselekidis.

So far, the results mentioned here for quasi-transitive graphs were always generalisations of results for groups.

Due to the geometric nature of graphs, several proofs are simpler than the corresponding ones for groups.

It would be interesting to obtain a result for graphs whose group-theoretic counterpart has not been known, yet.