Connected-homogeneous digraphs

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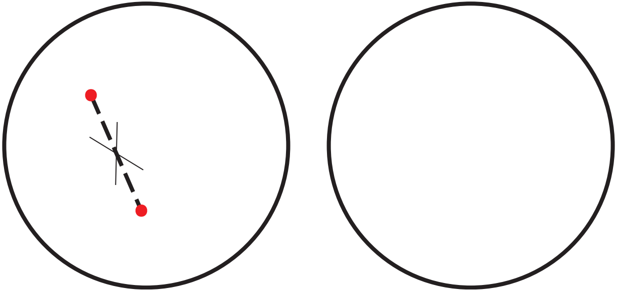
Homogeneous (di)graphs

Definition

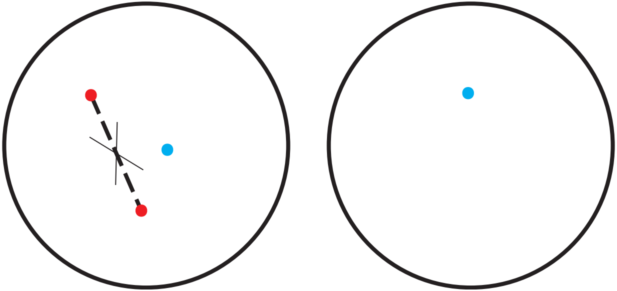
A (di)graph is homogeneous if any isomorphism between each two finite induced sub(di)graphs extends to an automorphism of the whole (di)graph.

- countable homogeneous graphs are classified: Gardiner '76, Lachlan&Woodrow '80
- countable homogeneous digraphs are classified: Lachlan '82,'84, Cherlin '98

Disconnected homogeneous graphs



Disconnected homogeneous graphs



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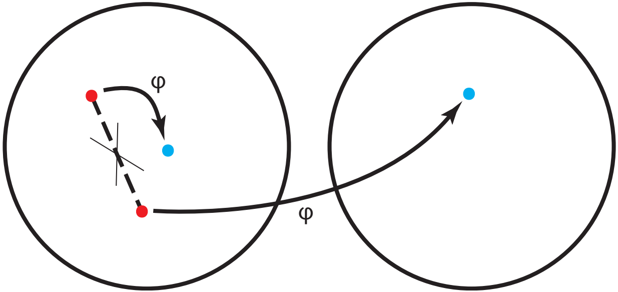

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Remark

- Homogeneous (di)graphs are C-homogeneous.
- There are C-homogeneous (di)graphs that are not homogeneous.

Ends of graphs

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In a graph G, two rays are equivalent if for any finite vertex set S of G both rays lie eventually in the same component of $G - S$.

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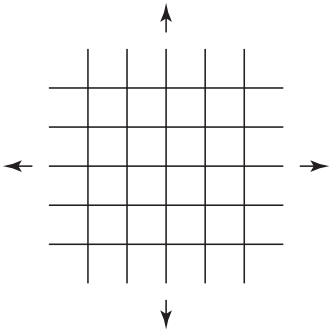
The ends of a digraph are the ends of its underlying undirected graph.

Theorem (Diestel, Jung, Möller '93)

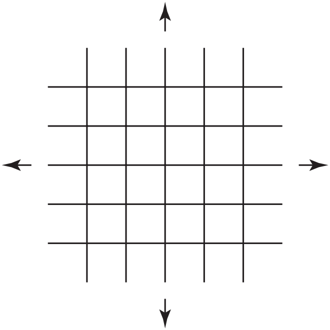
Every connected transitive infinite graph has either one, two, or infinitely many ends.

Ends of graphs



one end

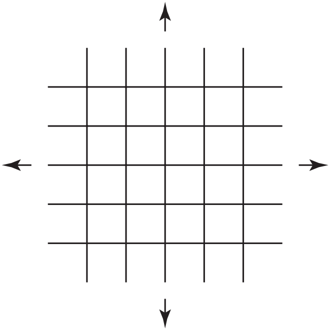
Ends of graphs



one end

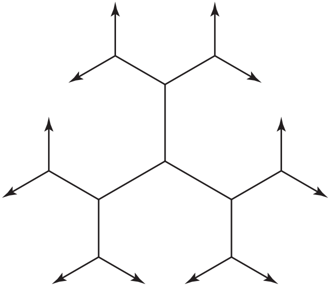
two ends

Ends of graphs



one end

two ends



infinitely many ends

C-homogeneous graphs

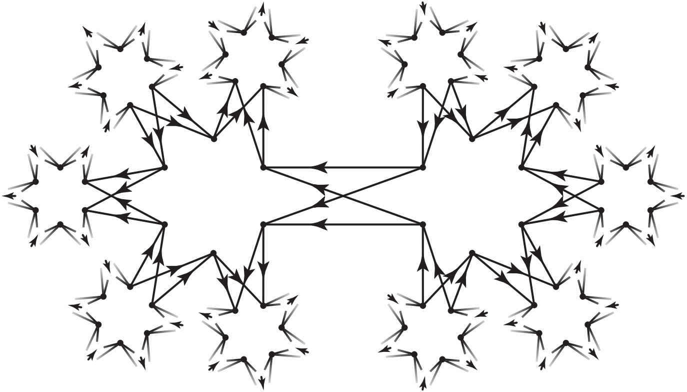
- countable C-homogeneous graphs are classified: Gardiner '78, Enomoto '81, Hedman&Pong '10, Gray&Macpherson '10

C-homogeneous digraphs

There are classification results for

- locally finite connected C-homogeneous digraphs with more than one end an several other restrictions:
  Gray&Möller '11
- all connected C-homogeneous digraphs of arbitrary cardinality with more than one end:
  H.&Hundertmark
- all finite C-homogeneous digraphs:
  H.
- all locally finite C-homogeneous digraphs:
  H.

Example of a C-homogeneous digraph



C-homogeneous graphs vs. C-homogeneous digraphs

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2. There are C-homogeneous graphs such that no orientation of its edges gives rise to a C-homogeneous digraph.
   E.g. those graphs that arise from finite complete graphs with more than 3 vertices by gluing together such graphs in a tree-like way (the graphs $X_{k,\ell}$ for finite $k \geq 3$): every vertex lies in precisely ℓ copies of the K_{k+1} and separates the graph into ℓ components.

C-homogeneous digraphs with more than one end

Theorem (H.&Pott)

A graph with more than one end is C-homogeneous if and only if it is a regular tree or an $X_{\kappa,\lambda}$ *for cardinals* $\kappa, \lambda \geq 2$.

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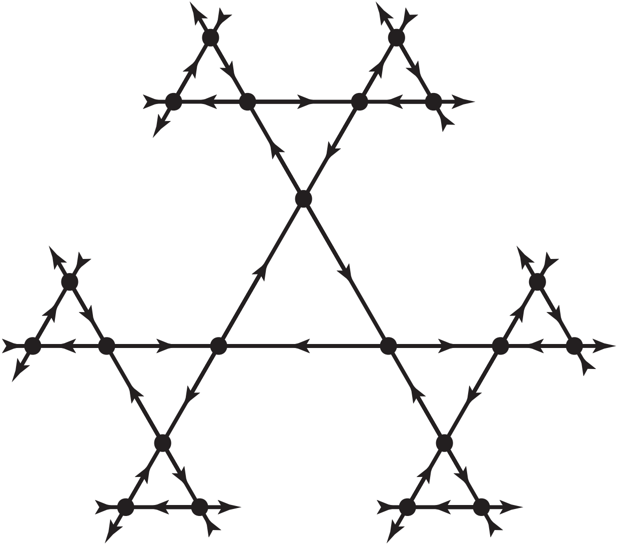
Theorem (H.&Hundertmark)

A digraph with infinitely many ends whose underlying undirected graph is C-homogeneous is C-homogeneous if and only if it is a regular tree or if the blocks are isomorphic homogeneous tournaments.

Example of a C-homogeneous digraph: $T(2)$

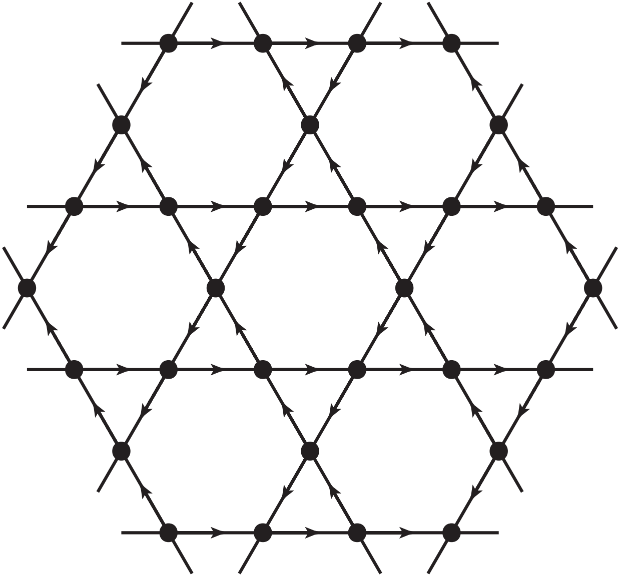


Theorem (H.)

For a finite or locally finite connected digraph D the following statements are equivalent.

1. *The digraph D is C-homogeneous, contains directed triangles and every vertex has in- and out-degree 2;*
2. *there is an $\mathrm{Aut}(T(2))$-invariant equivalence relation $\sim$ on $VT(2)$ such that D is isomorphic to the quotient digraph of $T(2)$ with respect to $\sim$.*

Example of a C-homogeneous one-ended digraph

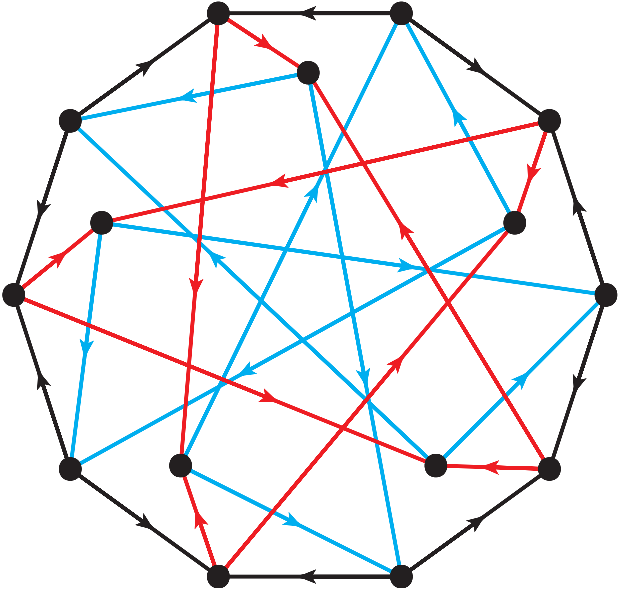


One-ended C-homogeneous digraphs

Theorem (H.)

A locally finite one-ended digraph is C-homogeneous if and only if it is a quotient of $T(2)$.

Example of a finite C-homogeneous digraph



Finite C-homogeneous digraphs

Theorem (H.)

A connected finite digraph is C-homogeneous if and only if it is

1. *a composition of a homogeneous and an empty digraph or*
2. *a composition of directed cycle and an empty digraph or*
3. *a quotient of $T(2)$ or*
4. *the tripartite complement of a disjoint union of a finite number of directed triangles.*

The digraphs $DL(\Delta)$

For a connected edge-transitive bipartite digraph Δ let $DL(\Delta)$ be the digraph constructed by gluing copies of Δ together such that

- every vertex lies in precisely two copies of Δ, once in each set of the bipartition;
- every vertex is a cut vertex and separates the two copies of Δ that contains it.

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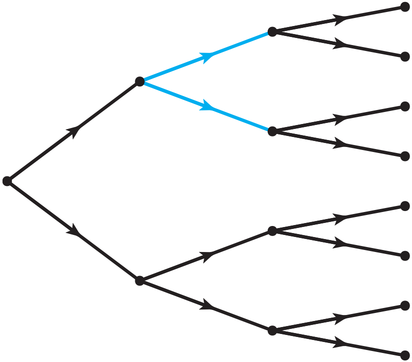
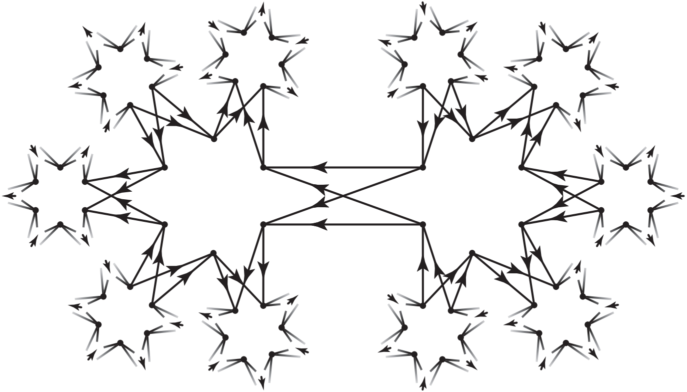


Theorem (H.&Hundertmark)

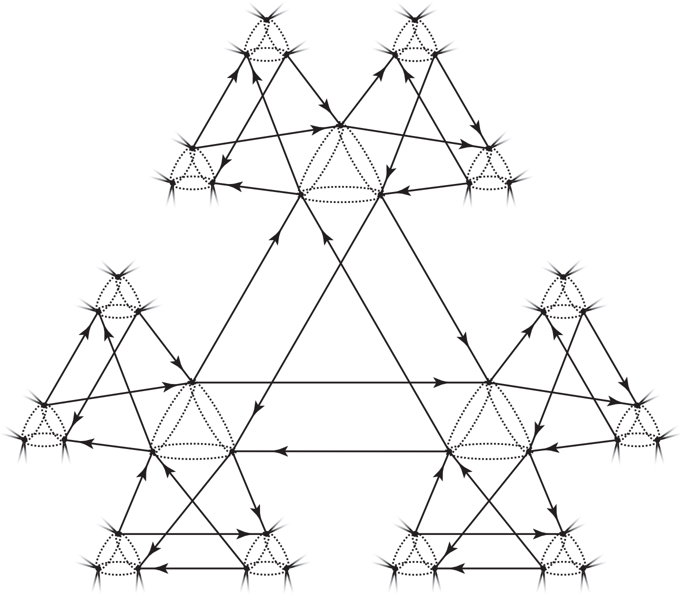
A digraph with infinitely many ends whose underlying undirected graph is not C-homogeneous is C-homogeneous if and only if it belongs to one of the following digraphs.

1. *It is $DL(\Delta)$ for Δ being one of*
   - *semi-regular tree,*
   - C_{2m},
   - *complete bipartite digraph* $K_{m,m}$,
   - *bipartite complement of a perfect matching,*
   - *generic homogeneous bipartite digraph;*
2. *it is an* $M'(2m)$*;*
3. *it is an* $M(\kappa, m)$.

The digraph M'(6)



The digraph M(3,3)



Theorem (Gray& Möller '11)

A connected two-ended digraph is C-homogeneous if and only if it is the composition of the directed double ray with a finite empty digraph.

Open problem

Problem

Classify the infinite one-ended C-homogeneous digraphs that are not locally finite.