Accessibility and canonical tree-decompositions

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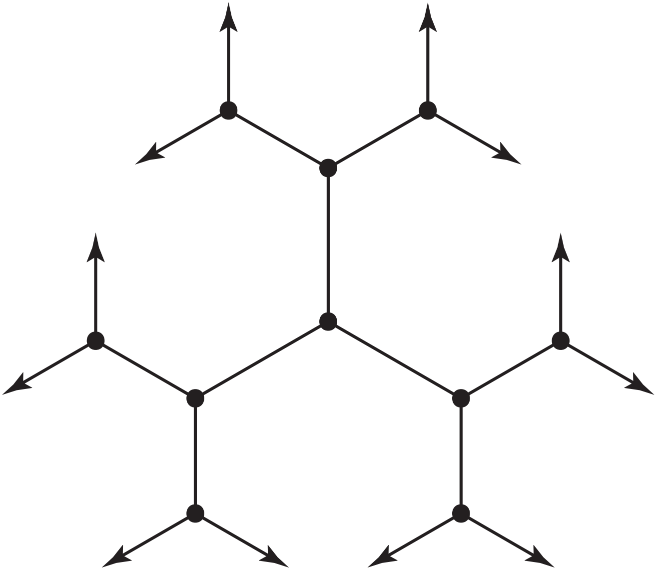
Content

A construction

Let G be a graph.
Attach to every vertex one copy of G for each type (orbit) of vertices.
Continue this process for all new vertices.

A construction: example with $G = P_2$



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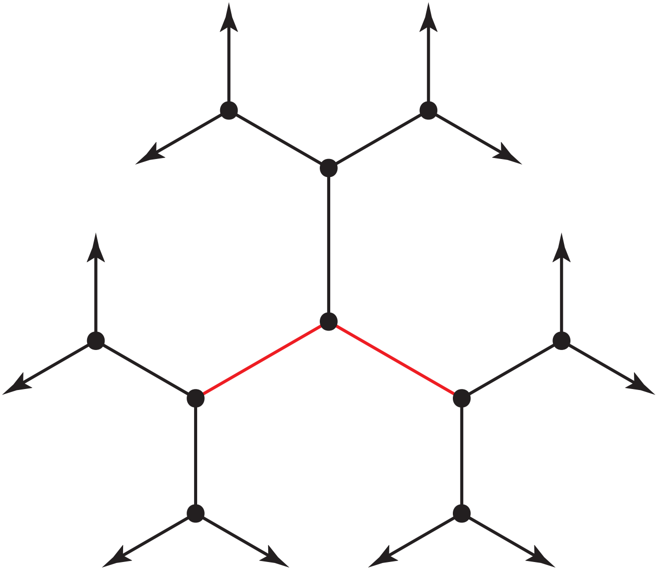

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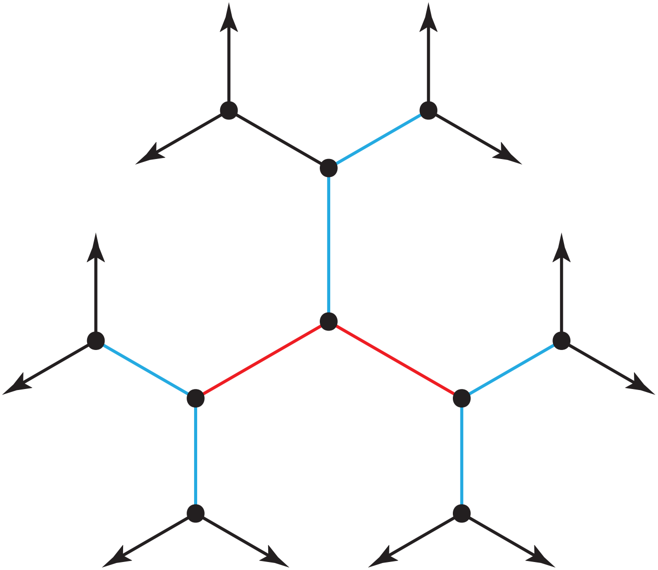

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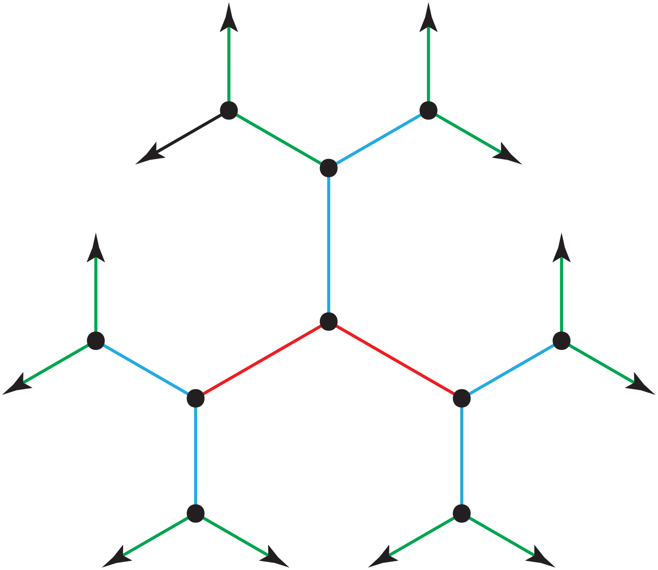
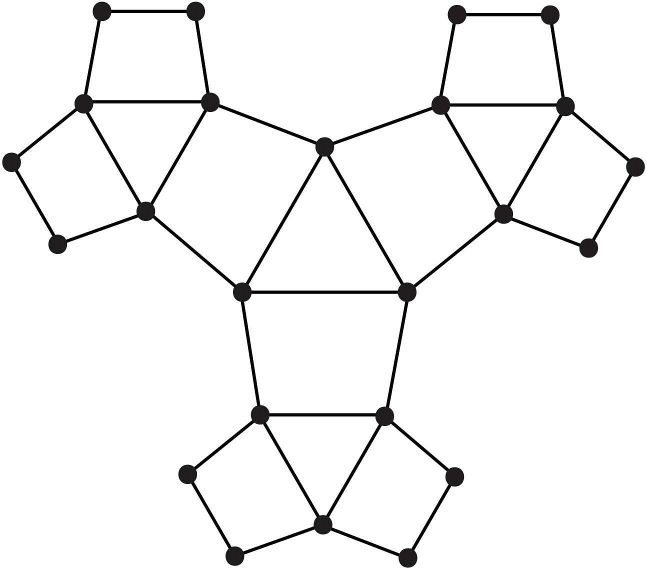


We change the construction a bit:

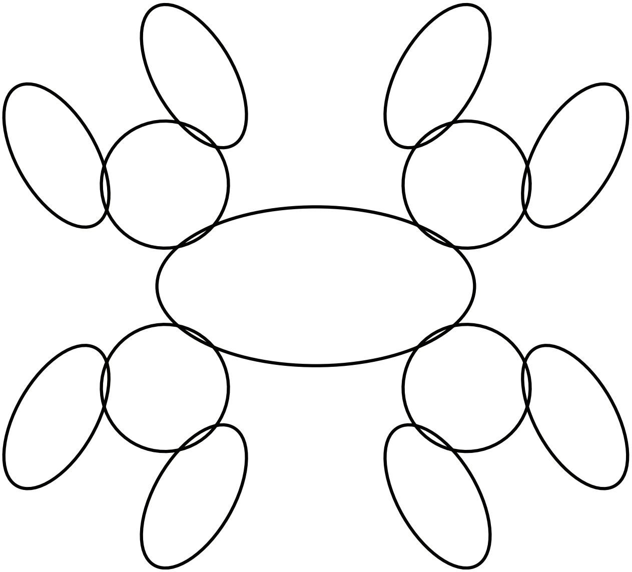
Start with two graphs G, H.
Pick isomorphic finite connected subgraphs F in G and H.
Attach to every type E of F in G a copy of H where we identify a type of F in H with E.
Now do the analogous thing for the new copies of H.
Continue this process.

A construction: example $G = C_3$, $H = C_4$



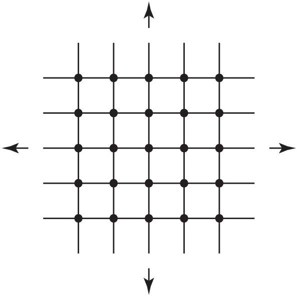
A graph obtained from two graph G_1, G_2 as in the previous construction is called a tree amalgamation of G_1 and G_2.

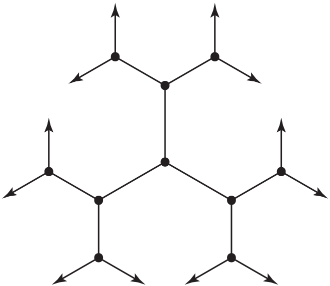
A construction: schematic picture



Definition

- A ray is a one-way infinite path.
- Two rays in a graph G are *equivalent* if for any finite vertex set $S \subseteq V(G)$ both rays lie eventually in the same component of $G - S$.
- The equivalence classes of this relation are the *ends* of the graph.





Quasi-transitive graphs

Question

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A graph is quasi-transitive if its automorphism group acts on its vertex set with only finitely many orbits.
A graph is locally finite if every vertex has finite degree.

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Question

Is $\mathcal{G}$ the class of all connected quasi-transitive locally finite graphs?

Content

1. Constructing infinite transitive graphs
2. Accessibility
3. Canonical tree-decompositions
4. k-blocks

Accessibility

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Theorem (Dunwoody 1993)

There is a connected inaccessible transitive locally finite graph.

Theorem (H, Lehner, Miraftab, Rühmann)

The class $\mathcal{G}$ is the class of all connected accessible quasi-transitive locally finite graphs.

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THEOREM (H, LEHNER, MIRAFTAB, RÜHMANN)

If G is a connected quasi-transitive locally finite graph with more than one end, then it is a non-trivial tree amalgamation of two connected quasi-transitive locally finite graphs.

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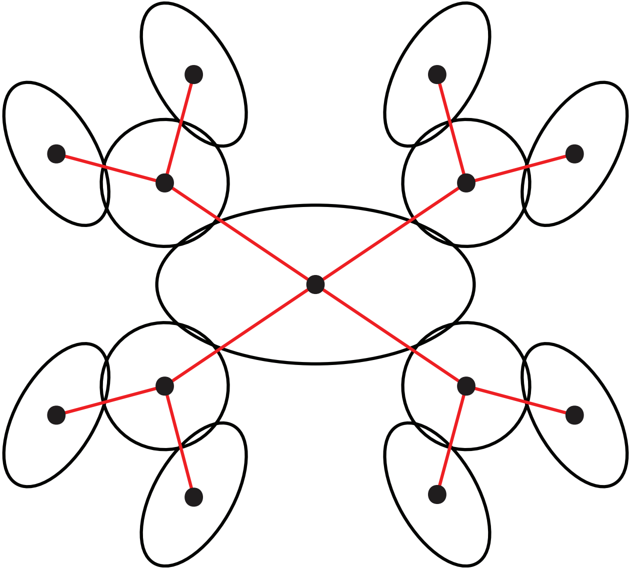
A tree-decomposition of a graph G is a pair $(T, \mathcal{V})$ of a tree T and a set $\mathcal{V} = \{V_t \mid t \in V(T), V_t \subseteq V(G)\}$ such that

1. $\bigcup_{t \in V(T)} V_t = V(G)$;
2. for every edge in G there is some V_t that contains both its incident vertices;
3. for every t on a $t_1 - t_2$ path in T we have $V_{t_1} \cap V_{t_2} \subseteq V_t$.



New formulations

Theorem (H, Lehner, Miraftab, Rühmann)

The class $\mathcal{G}$ is the class of all connected accessible quasi-transitive locally finite graphs.

Theorem (H, Lehner, Miraftab, Rühmann)

A quasi-transitive locally finite graph G is accessible if it has a tree-decompositions $(T, \mathcal{V})$ of finitely many $\mathrm{Aut}(G)$-orbits such that at most one end of G lives in each V_t.

THEOREM (H, LEHNER, MIRAFTAB, RÜHMANN)

If G is a connected quasi-transitive locally finite graph with more than one end, then it is a non-trivial tree amalgamation of two connected quasi-transitive locally finite graphs.

THEOREM (H, LEHNER, MIRAFTAB, RÜHMANN)

If G is a connected quasi-transitive locally finite graph with more than one end, then it has a non-trivial tree-decomposition $(T, \mathcal{V})$ such that

- *each V_t induces a connected quasi-transitive locally finite graph and*
- *the automorphisms of G induce an action on $(T, \mathcal{V})$ with at most two orbits on $\mathcal{V}$.*

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Similar theorems have been proved previously by

- Dunwoody/Dicks and Dunwoody (1985/1989) via edge cuts
- Dunwoody and Krön (2014)

Extension of group theoretic theorems

Our theorems generalise several group theoretic theorems to graphs:

1. Stallings' theorem of splitting multi-ended finitely generated groups (1971);
2. Dunwoody's accessibility theorem of finitely presented groups (1985);
3. Dicks' and Dunwoody's characterisation of accessible groups (1989).

Theorem (H)

A connected quasi-transitive locally finite graphs is accessible if its cycle space is generated by cycles of bounded length.

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k-BLOCKS

A k-block is a maximal set X of at least k vertices such that no set of less than k vertices separates any $x, y \in X$.

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Remark

Note that the inseparability of X is measured not in X but within the whole graph.

Distinguishing k-blocks

Theorem (Carmesin, Diestel, Hundertmark, Stein 2014)

Let $k > 0$. Every finite graph G has a canonical tree-decomposition of adhesion at most k that efficiently distinguishes all its k-blocks.

Remark

Previously, Dunwoody and Krön (2014) showed that the k-blocks are arranged in a tree-like way.

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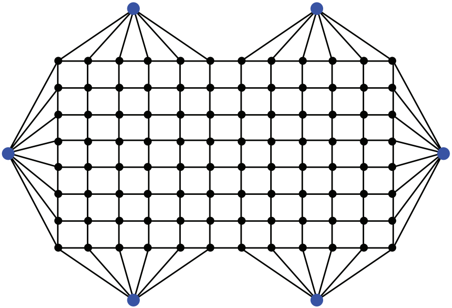
4. Every k-connected subgraph lies in a k-block.

Add at least k vertices joined to a large grid such that each new vertex has at least k-neighbours all of which lie on the boundary of the grid.



The new vertices form a k-block.

Question

When does a graph have a k-block?

Minimum degree

Theorem (Mader 1974)

Graphs with minimum degree at least $2k$ contain a $(k+1)$-block.

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A k-connected graph with minimum degree more than $\frac{3}{2}k - 1$ contains a $(k+1)$-block.

Problem

For $k \in \mathbb{N}$ find the smallest d such that graphs of minimum degree at least d contains a $(k+1)$-block.

Average degree

Theorem (Carmesin, Diestel, H, Hundertmark 2014)

Graphs with average degree at least $3k$ *contain a* $(k+1)$*-block.*

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Further directions

- Weißauer recently investigated connections between having a k-block and width parameters.
- Otherwise not much is known.