Cuts and cycles in transitive graphs

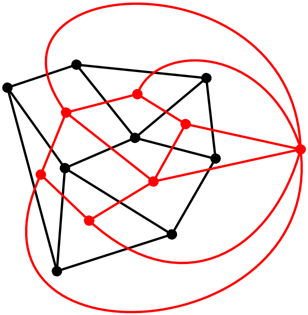
Matthias Hamann

(Universität Hamburg)

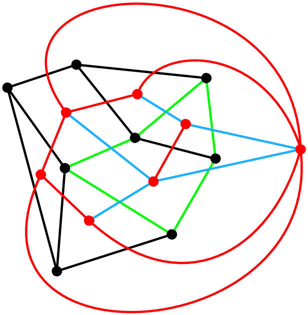
7 November 2015

We look for connections between cuts and cycles of graphs.

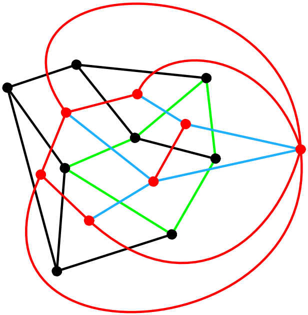
We look for connections between cuts and cycles of graphs.



We look for connections between cuts and cycles of graphs.



We look for connections between cuts and cycles of graphs.



Folklore

The cycles of a planar graph are the cuts of its dual.

Cut space

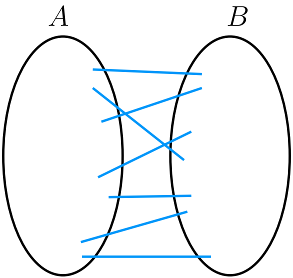
Definition

A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set.

Cut space

Definition

A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set.



Cut space

Definition

A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set.

Definition

The cut space of a graph is the set of all finite sums (over $\mathrm{GF}(2)$) of finite cuts.

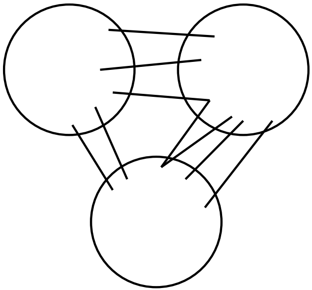
Cut space

Definition

A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set.

Definition

The cut space of a graph is the set of all finite sums (over $\mathrm{GF}(2)$) of finite cuts.



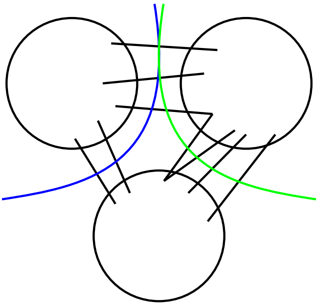
Cut space

Definition

A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set.

Definition

The cut space of a graph is the set of all finite sums (over $\mathrm{GF}(2)$) of finite cuts.



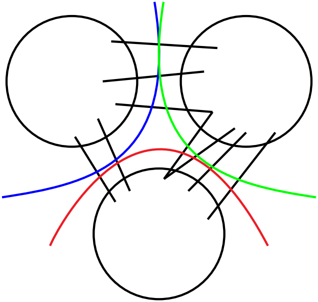
Cut space

Definition

A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set.

Definition

The cut space of a graph is the set of all finite sums (over $\mathrm{GF}(2)$) of finite cuts.



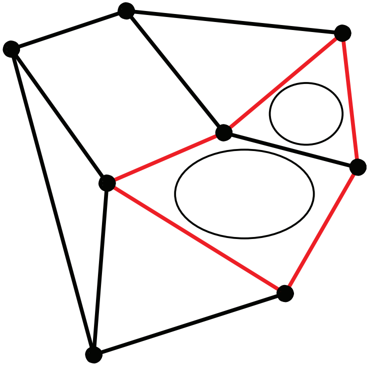
Cycle space

Definition

- The cycle space of a graph is the set of all finite sums (over GF(2)) of edge sets of finite cycles.

Definition

- The cycle space of a graph is the set of all finite sums (over GF(2)) of edge sets of finite cycles.



Remark (1)

In a finite graph the cut space is the orthogonal space of the cycle space and vice versa.

The case: finite graphs

Remark (1)

In a finite graph the cut space is the orthogonal space of the cycle space and vice versa.

Remark (2)

In a finite graph with n vertices and m edges, the cut space has dimension $n - 1$ and the cycle space has dimension $m - n + 1$.

Remark (1)

In a finite graph the cut space is the orthogonal space of the cycle space and vice versa.

Remark (2)

In a finite graph with n vertices and m edges, the cut space has dimension $n-1$ and the cycle space has dimension $m-n+1$.

• Remark (1) has a rather complicated counterpart for infinite graphs for which we have to consider 'infinite cycles' and suitable compactifications of infinite graphs.

REMARK (1)

In a finite graph the cut space is the orthogonal space of the cycle space and vice versa.

REMARK (2)

In a finite graph with n vertices and m edges, the cut space has dimension $n - 1$ and the cycle space has dimension $m - n + 1$.

- Remark (1) has a rather complicated counterpart for infinite graphs for which we have to consider ‘infinite cycles’ and suitable compactifications of infinite graphs.

- Remark (2) seems to have no counterpart at all for infinite graphs.

A first result for infinite graphs

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

A first result for infinite graphs

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

Why does it give an answer to our question?

Reformulating Dunwoody's theorem

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

A finitely presented group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ has a locally finite Cayley graph Γ whose first homology group is generated by $\{g(C) \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of closed walks corresponding to the relators in $\mathcal{R}$

Reformulating Dunwoody's theorem

Theorem (Dunwoody 1985)

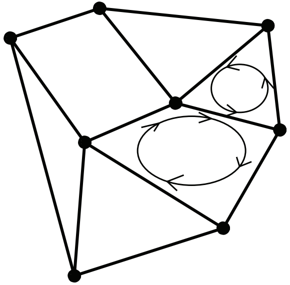
Finitely presented groups are accessible.

A finitely presented group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ has a locally finite Cayley graph Γ whose first homology group is generated by $\{g(C) \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of closed walks corresponding to the relators in $\mathcal{R}$, that is, its first homology group is a finitely generated G-module.

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

A finitely presented group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ has a locally finite Cayley graph Γ whose first homology group is generated by $\{g(C) \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of closed walks corresponding to the relators in $\mathcal{R}$, that is, its first homology group is a finitely generated G-module.



Reformulating Dunwoody's theorem

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

A finitely presented group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ has a locally finite Cayley graph Γ whose first homology group is generated by $\{g(C) \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of closed walks corresponding to the relators in $\mathcal{R}$, that is, its first homology group is a finitely generated G-module.

Theorem (Dicks & Dunwoody 1989)

The cut space of a locally finite Cayley graph G of a finitely generated accessible group is a finitely generated $\mathrm{Aut}(G)$*-module.*

Reformulating Dunwoody's theorem

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

A finitely presented group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ has a locally finite Cayley graph Γ whose first homology group is generated by $\{g(C) \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of closed walks corresponding to the relators in $\mathcal{R}$, that is, its first homology group is a finitely generated G-module.

Theorem (Dicks & Dunwoody 1989)

The cut space of a locally finite Cayley graph G of a finitely generated accessible group is a finitely generated Aut(G)*-module.*

Theorem (Dunwoody 1985)

Let G be a locally finite Cayley graph. If its first homology group is a finitely generated Aut(G)*-module, then so is its cut space.*

Theorem

Let G be a 2-edge-connected transitive graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$-module, then so is its cut space.

A new connection

Theorem

Let G be a 2-edge-connected transitive graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$*-module, then so is its cut space.*

Can we ask for ‘if and only if’?

A new connection

Theorem

Let G be a 2-edge-connected transitive graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$*-module, then so is its cut space.*

Can we ask for 'if and only if'?

Remark

Bieri and Strebel (1980) gave an example of a finitely generated accessible group that is not finitely presentable, that is, of a Cayley graph G whose cut space is a finitely generated $\mathrm{Aut}(G)$-module but its first homology group is not.

Definition

A graph is *accessible* if there is some $k \in \mathbb{N}$ such that for any two rays for which some finite edge set separates them eventually there is also one such set of size at most k.

Definition

A graph is *accessible* if there is some $k \in \mathbb{N}$ such that for any two rays for which some finite edge set separates them eventually there is also one such set of size at most k.

Theorem (Thomassen & Woess 1993)

A finitely generated group is accessible if and only if one (and hence every) of its locally finite Cayley graphs is accessible.

Going to infinity: accessibility

Definition

A graph is *accessible* if there is some $k \in \mathbb{N}$ such that for any two rays for which some finite edge set separates them eventually there is also one such set of size at most k.

Theorem (Thomassen & Woess 1993)

A finitely generated group is accessible if and only if one (and hence every) of its locally finite Cayley graphs is accessible.

Theorem (Dunwoody 1985)

Every locally finite Cayley graph G whose first homology group is a finitely generated $\mathrm{Aut}(G)$*-module is accessible.*

Conjecture (Diestel 2010)

Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.

A conjecture is confirmed

Theorem

Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.

Applications

Applications I

We obtain a combinatorial proof of

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

Applications I

We obtain a combinatorial proof of

Theorem (Dunwoody 1985)

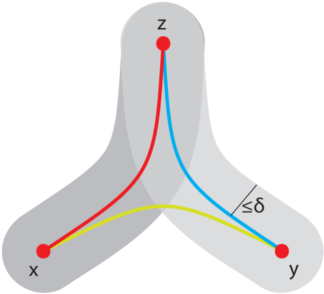
Finitely presented groups are accessible.

Theorem (Dunwoody 2007)

Every locally finite transitive planar graph is accessible.

Definition

A connected graph G is called hyperbolic if there exists some $\delta \geq 0$ such that for any three vertices x, y, z of G and for any three shortest paths, one between every two of the vertices, each of those paths lies in the δ-neighbourhood of the union of the other two.



Application II: hyperbolic graphs

Theorem (Gromov 1987)

Finitely generated hyperbolic groups are finitely presented (and hence accessible).

Application II: hyperbolic graphs

Theorem (Gromov 1987)

Finitely generated hyperbolic groups are finitely presented (and hence accessible).

Conjecture (Dunwoody 2011)

Every locally finite transitive hyperbolic graph is accessible.

Application II: hyperbolic graphs

Theorem (Gromov 1987)

Finitely generated hyperbolic groups are finitely presented (and hence accessible).

Conjecture (Dunwoody 2011)

Every locally finite transitive hyperbolic graph is accessible.

Theorem

Every locally finite transitive hyperbolic graph is accessible.

Question

Is every locally finite hyperbolic transitive graph quasi-isometric to some Cayley graph?