ON THE TREE-LIKENESS OF HYPERBOLIC SPACES

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ABSTRACT. Inside any proper hyperbolic geodesic space X we construct a rooted topological $\mathbb{R}$-tree T that reflects the geometry of X in the following sense. All rays in T are quasi-geodesic in X. Every geodesic ray in X lies eventually close to a ray of T. The embedding of T in X extends continuously to their boundaries in a finite-to-one way, the number of boundary points of T mapping to a given boundary point of X being bounded if the (Assouad) dimension of the boundary of X is finite.

1. INTRODUCTION

Since Gromov's article on hyperbolic groups [15] appeared, there have been various attempts to describe a given hyperbolic space by comparing it with the simplest form of a hyperbolic space, an $\mathbb{R}$-tree.

On the one hand, there are results that construct for a given hyperbolic space an $\mathbb{R}$-tree whose local structure resembles the local structure of the hyperbolic space. The best known among these are results attributed to Gromov (see [11, Chapitre 8] and [14, §2.2]) that construct for a finite subset of the completion of a δ-hyperbolic space an $\mathbb{R}$-tree in the space whose completion contains the given set and such that all of its geodesics between elements of the finite set are quasi-geodesics in the hyperbolic space for constants that depend only on the size of the set and on δ. There is also a result by Benjamini and Schramm [4, Theorem 1.5] for locally finite hyperbolic graphs which states that if they have exponential growth then there exists a subtree with exponential growth such that the embedding is a bilipschitz map.

On the other hand there are constructions of trees that try to capture the entire hyperbolic boundary of a given hyperbolic space. When the space is a locally finite graph, then several ideas for such constructions can already be found in Gromov's article [15, Sections 7.6, 8.5.B, and 8.5.C]. They have been elaborated on in [12, Chapter 5]. These trees capture the hyperbolic boundary of the hyperbolic graph in that there are continuous maps from their own boundary onto that of the graph. However, these trees are not necessarily subtrees of the hyperbolic graph. If the hyperbolic graph has bounded degree, then some of these maps are finite-to-one. In [17] the author showed that inside every hyperbolic graph of bounded degree there exists a spanning tree whose boundary maps continuously and finite-to-one onto the hyperbolic boundary of the graph.

Both approaches can be combined in the case of hyperbolic groups. In that situation there is a geodesic rooted spanning tree of any of its locally finite Cayley graphs that has a finite-to-one continuous surjection from its boundary to the one

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of the Cayley graph, cp. [20, p. 10].[1] But in general, the same approach does not work for arbitrary proper hyperbolic geodesic spaces: Example 4.5 of [17] discusses a locally finite hyperbolic graph with precisely one hyperbolic boundary point all of whose geodesic spanning trees have infinitely many boundary points.

In this article, we show that it is possible also in the general situation to combine these two approaches except that the tree we construct is only quasi-geodesic eventually: we shall construct in Section 4 inside every proper hyperbolic geodesic space X whose boundary has finite Assouad dimension a rooted topological $\mathbb{R}$-tree T such that

- all rays are eventually quasi-geodesic rays (for the same global constants) and
- the embedding $T \to X$ induces a continuous bounded-to-one map from the boundary of the tree onto the one of the hyperbolic space.

If the hyperbolic space is visual, that is roughly speaking that every point has bounded distance to some geodesic double ray (see Section 5 for more details), then every point of the space has distance at most some constant κ from the constructed tree (Theorem 6.2). If we consider an arbitrary proper hyperbolic geodesic space, then there is some κ with the property that every geodesic outside the described set has finite length (Theorem 6.3).

The assumption that the hyperbolic boundary has finite Assouad dimension is not a strong assumption. For example, if the space is a locally finite hyperbolic graph of bounded degree, e.g. if the graph is the Cayley graph of a finitely generated hyperbolic group with a finite set of generators, then it has finite Assouad dimension. More generally, Bonk and Schramm [5, Theorem 9.2] showed that the hyperbolic boundary of every proper hyperbolic geodesic space with bounded growth at some scale has finite Assouad dimension, where a metric space X has *bounded growth at some scale* if there are constants $N \in \mathbb{N}$ and $r, R \in \mathbb{R}$ with $R > r > 0$ such that every open ball of radius R in X can be covered by N open balls of radius r.

Other approaches exhibiting the tree-likeness of hyperbolic spaces include quasi-isometric embeddings of visual hyperbolic spaces into the product of binary metric trees, see Buyalo et al. [8], or sub-cones at infinity, see [14, Proposition 2.1.11] and [24, Lemme 5.6].

In the final section we give a proof that, for every proper hyperbolic geodesic space X and every topological $\mathbb{R}$-tree $T \subseteq X$ whose embedding into X induces a continuous surjection from the boundary of T to the boundary of X, the number of inverse images of a boundary point of X is bounded from below by a function that depends only on the topological dimension of the hyperbolic boundary of X.

2. Hyperbolic spaces

In this section we define properties for metric spaces, in particular, for hyperbolic spaces and cite some of their properties. For a more detailed introduction to hyperbolicity, we refer to [1, 11, 14, 15, 23] as well as [7, Chapter III.H] and [25, Chapter 22].

[1] A rooted spanning tree is *geodesic* if the distance between any vertex and the root is the same in the tree as in the graph.

Let X be a metric space. A *geodesic* between two points $x, y \in X$ is the image of an isometric map $\varphi : [0, d(x, y)] \to X$ with $\varphi(0) = x$ and $\varphi(d(x, y)) = y$. By $[x, y]$ we denote a geodesic between x and y. If we want to specify the particular metric space X, then we write $[x, y]_X$. The space X is *geodesic* if for any two points $x, y \in X$ there exists a geodesic in X between them. It is *proper* if for every $r \in \mathbb{R}$ and $x \in X$ the closed ball $\overline{B}_r(x)$ is compact. If there is a $\delta \geq 0$ such that for any three points x, y, z and any geodesics $[x, y], [y, z], [z, x]$ between each two of the points $[x, z]$ lies in $\overline{B}_\delta([x, y] \cup [y, z])$ then we call the space *(δ-)hyperbolic* and δ is the *hyperbolicity constant*.

Homeomorphic images of $[0, 1]$ are called *paths*. A *ray* is a homeomorphic image R of $[0, \infty)$ such that for every ball of finite diameter R lies eventually outside that ball. *Double rays* are homeomorphic images of $\mathbb{R}$ such that the restrictions to $\mathbb{R}_{\geq 0}$ and to $\mathbb{R}_{\leq 0}$ are both rays. A (double) ray is *geodesic* if it is an isometric image of $[0, \infty)$ (of $\mathbb{R}$). A ray R is *eventually geodesic* if there is a ball B of finite diameter such that $R \setminus B$ is geodesic.

Since we are looking at the hyperbolic boundary from distinct viewpoints, we state here three different definitions of the hyperbolic boundary all of which are equivalent. Two geodesic rays π_1, π_2 are *equivalent* if for any sequence $(x_n)_{n\in\mathbb{N}}$ of points on π_1 we have $\liminf_{n\to\infty} d(x_n, \pi_2) \leq M$ for an $M < \infty$. In hyperbolic geodesic spaces, this is an equivalence relation, compare with [9, Section 2.4.2]. The *hyperbolic boundary* ∂X is the set of all equivalence classes of this relation. By $\widehat{X}$ we denote $X \cup \partial X$.

The *Gromov-product* (with respect to $o \in X$) of two elements $x, y \in X$ is

$$(x, y)_o := \frac{1}{2}(d(x, o) + d(y, o) - d(x, y)).$$

If it is obvious by the context which point we use as the base-point for the product, we simply write (x, y).

Now we give the second definition of the hyperbolic boundary. A sequence $(x_i)_{i\geq 0}$ *converges to a point* x if $\lim_{i\to\infty}(x_i, x) = 0$. A sequence $(x_i)_{i\geq 0}$ *converges to infinity* if $\lim_{i,j\to\infty}(x_i, x_j) \to \infty$. Two sequences $(x_i)_{i\geq 0}$, $(y_j)_{j\geq 0}$ that converge to infinity are *equivalent* if $\lim_{i,j\to\infty}(x_i, y_j) = \infty$. In hyperbolic geodesic spaces this is an equivalence relation. The *hyperbolic boundary* is the set of equivalence classes of this equivalence relation. A sequence $(x_i)_{i\geq 0}$ *converges* to a boundary point η if it is in the equivalence class η (notation: $(x_i)_{i\geq 0} \to \eta$). In [14] the equivalence of this definition with the first one given is shown.

A third way to define the hyperbolic boundary is via the completion defined by a metric d_ε. Let $\varepsilon > 0$ with $\varepsilon' := \exp(\varepsilon\delta) - 1 < \sqrt{2} - 1$. Let

$$d_\varepsilon(x, y) := \inf\{\sum_{i=1}^{n} \exp(-\varepsilon(x_{i-1}, x_i)) \mid x_i \in X, x_0 = x, x_n = y\}.$$

Then d_ε is a metric on X. The *hyperbolic boundary* is the completion of X with respect to this metric without X. For a proof of the equivalence of this definition with the previous ones see [14, Proposition 7.3.10].

An important theorem about the hyperbolic boundary is the following. For references see [5, Section 6], [11, Proposition 2.3.2], and [23, Corollary 2.65].

Theorem 2.1. *If X is a proper geodesic hyperbolic space, then the hyperbolic boundary is compact for all metrics d_ε with $\varepsilon > 0$ and $\exp(\varepsilon\delta) < \sqrt{2}$.*

Furthermore, for all $\eta, \mu \in \partial X$ and with $\varepsilon' = \exp(\varepsilon\delta) - 1$ we have

$$\varepsilon' \exp(-\varepsilon\,(\eta, \mu)) \leq d_\varepsilon(\eta, \mu) \leq \exp(-\varepsilon\,(\eta, \mu)). \qquad \square$$

Geodesic (double) rays play an important role in the context of hyperbolic geodesic spaces, as we already saw in the first definition of the hyperbolic boundary. The following proposition shows that there are plenty of them.

Proposition 2.2. [23, Proposition 2.60], [11, Proposition 2.2.1] *Let X be a proper hyperbolic geodesic space. For every $x \in X$ and every hyperbolic boundary point η there is a geodesic ray from x to η, and for every two distinct hyperbolic boundary points there is a geodesic double ray between these two boundary points.* $\square$

Let $\gamma \geq 1, c \geq 0$. A (γ, c)-*quasi-isometry* from X to another metric space Y is a map $f : X \to Y$ with

$$\gamma^{-1} d_X(x, y) - c \leq d_Y(f(x), f(y)) \leq \gamma d_X(x, y) + c$$

for all $x, y \in X$ and with $\sup\{d_Y(y, f(X)) \mid y \in Y\} \leq c$. Then X is *quasi-isometric* to Y. A (double) ray R in X is (γ, c)-*quasi-geodesic* if it is the image of a (γ, c)-quasi-isometry from $\mathbb{R}_{\geq 0}$ ($\mathbb{R}$, resp.) to R. Hence a (double) ray is geodesic, if it is a $(1, 0)$-quasi-geodesic (double) ray. If the constants γ, c are unimportant, then we just speak of *quasi-geodesics*.

The next proposition shows that in every proper hyperbolic geodesic space the geodesics and quasi-geodesics lie close to each other.

Proposition 2.3. [23, Theorem 2.31], [11, Théorème 3.1.4] *Let X be a proper δ-hyperbolic geodesic space. For all $\gamma_1 \geq 1, \gamma_2 \geq 0$ there is a constant $\kappa = \kappa(\delta, \gamma_1, \gamma_2)$ such that for every two points $x, y \in X$ every (γ_1, γ_2)-quasi-geodesic between them lies in a κ-neighborhood around every geodesic between x and y and vice versa.*

Furthermore, this extends to (γ_1, γ_2)-quasi-geodesic and geodesic (double) rays. $\square$

Proposition 2.4. [25, (22.4)] *Let X be a proper δ-hyperbolic geodesic space. Then for all $x, y, z \in X$ we have*

$$(x, y)_z \leq d(z, [x, y]) \leq (x, y)_z + 2\delta. \qquad \square$$

We extend to definition of the Gromov-product to $\widehat{X}$: for $a, b \in \widehat{X}$ let

$$(a, b) := \inf \liminf_{i,j \to \infty} (x_i, y_j)$$

where the infimum is taken over all sequences $(x_i)_{i \geq 0} \to a$ and $(y_i)_{i \geq 0} \to b$.

Combining Proposition 2.3 and [9, Lemma 2.2.2] we obtain the following proposition.

Proposition 2.5. *Let X be a proper δ-hyperbolic geodesic space, let $\eta, \nu \in \partial X$, and let $o \in X$. For all geodesic double rays π from η to ν we have*

$$(\eta, \nu)_o \leq d(o, \pi) \leq (\eta, \nu)_o + 4\delta. \qquad \square$$

Proposition 2.6. *Let X be a proper geodesic hyperbolic space with a metric d_ε as in Theorem 2.1 with $\varepsilon > 0$ and $\varepsilon' := \exp(\varepsilon\delta) - 1 < \sqrt{2} - 1$. Let $o \in X$ be the base-point for the Gromov-product of X. Then, for every $q \geq 1$, there exists a $\beta = \beta(\delta, q, \varepsilon) > 0$ such that for all $\eta_1, \eta_2, \mu_1, \mu_2 \in \partial X$ with $\frac{1}{q} \leq d_\varepsilon(\eta_1, \mu_1)/d_\varepsilon(\eta_2, \mu_2) \leq q$ we have $|d(o, [\eta_1, \mu_1]) - d(o, [\eta_2, \mu_2])| \leq \beta$.*

Proof. By Theorem 2.1 we have

$$\varepsilon' \exp(-\varepsilon(\eta_1, \mu_1)) \leq d_\varepsilon(\eta_1, \mu_1) \leq q d_\varepsilon(\eta_2, \mu_2) \leq q \exp(-\varepsilon(\eta_2, \mu_2)).$$

As a consequence we have by symmetry

$$|(\eta_1, \mu_1) - (\eta_2, \mu_2)| \leq \frac{1}{\varepsilon} \ln(\frac{q}{\varepsilon'}).$$

The claim follows immediately with Proposition 2.5. □

An $\mathbb{R}$-*tree* is a metric space T such that for any two points $x, y \in T$ there exists a unique arc between them, which has length $d(x, y)$, and a *topological* $\mathbb{R}$-*tree* is a homeomorphic image of an $\mathbb{R}$-tree. An easy observation is that $\mathbb{R}$-trees are 0-hyperbolic geodesic spaces. The converse direction – that 0-hyperbolic geodesic spaces are $\mathbb{R}$-trees – is a bit more difficult. But proofs can be found in nearly every of the introductory books or articles on hyperbolic spaces. For more details on $\mathbb{R}$-trees see for example [10, 18, 19].

3. The Assouad dimension

In this section we introduce the Assouad dimension, which is the main dimension concept in this article. Furthermore, we compare it with a related concept. For a more detailed introduction to the Assouad dimension we refer to [2] and in particular to [22, Appendix A].

Let X be a metric space. For $\alpha, \beta > 0$ let $S(\alpha, \beta)$ be the maximal cardinality of a subset V of X such that each two distinct elements of V have distance at least α and at most β. Let n be the infimum of all $s \geq 0$ such that there is a $C \geq 0$ with $S(\alpha, \beta) \leq C(\frac{\beta}{\alpha})^s$ for all $0 < \alpha \leq \beta$. Then n is called the *Assouad dimension* of the metric space X (notation: $\dim_A(X) = n$).

A metric space X is *doubling* if there exists a $\kappa \geq 1$ such that every ball of radius r can be covered by at most 2^κ balls of radius at most $\frac{r}{2}$. By $\dim_2(X)$ we denote the infimum of all these κ. A subset Y of X has *diameter* $\sup\{d(x, y) \mid x, y \in Y\}$ (notation: $\operatorname{diam}(Y)$), and a set $\mathcal{Y} \subseteq \mathcal{P}(X)$ has *diameter* $\operatorname{diam}(\mathcal{Y}) = \sup\{\operatorname{diam}(Y) \mid Y \in \mathcal{Y}\}$. The *radius* of a subset Y of X is $\operatorname{rad}(Y) := \inf\{\sup\{d(x, y) \mid x \in Y\} \mid y \in Y\}$ and the *radius* of a set $\mathcal{Y} \subseteq \mathcal{P}(X)$ is $\operatorname{rad}(\mathcal{Y}) := \sup\{\operatorname{rad}(Y) \mid Y \in \mathcal{Y}\}$. For every $r \geq 0$, a family $\mathcal{B} = (B_i)_{i \in I}$ of subsets of X has r-*multiplicity* at most n if every subset of X with diameter at most r intersects with at most n members of the family. A point $x \in X$ has r-*multiplicity* at most n in $\mathcal{B}$ if $\overline{B}_r(x)$ intersects with at most n members of the family $\mathcal{B}$ non-trivially.

One assumption in our main result is that the Assouad dimension of the hyperbolic boundary of the proper hyperbolic geodesic space is finite. It is easier to use the doubling property instead. The following theorem guarantees that we treat the same hyperbolic spaces.

Theorem 3.1. [22, Theorem A.3] *Let X be a metric space. Then, X is doubling if and only if it has finite Assouad dimension.* □

It is easy to adapt the proof of [21, Lemma 2.3] for Lemma 3.2, see [17, Lemma 3.2] for details.

Lemma 3.2. *Let X be a doubling metric space, let $N = 2^{\dim_2(X)}$, and let $r > 0$. Then X has a covering $\mathcal{B}$ of closed balls of diameter at most $2r$ such that $\mathcal{B}$ is the* *disjoint union of at most N^4 subsets $\mathcal{B}_i$ of $\mathcal{B}$ each of which has r-multiplicity at most 1; so $\mathcal{B}$ has r-multiplicity at most N^4.*

Furthermore, it is possible to choose $\mathcal{B}$ so that a given subset Y of X with $d(x,y) > r$ for all $x, y \in Y$ is a subset of the set of centers of the balls in $\mathcal{B}$, so that each two centers have distance more than r and so that every center has $3r$-multiplicity at most N^4 in $\mathcal{B}$.

Additionally, if Y is finite and X bounded, then we may choose $\mathcal{B}$ finite. $\square$

Let us briefly compare the Assouad dimension with another dimension concept. A metric space X has *asymptotic dimension* n (notation: $\operatorname{asdim}(X) = n$) if n is the smallest natural number such that for every $\varrho > 0$ there exists an open cover $\mathcal{U}$ of X such that every $x \in X$ lies in at most $n+1$ elements of $\mathcal{U}$, such that $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$, and such that

$$\inf_{x \in X} \sup_{U \in \mathcal{U}} d(x, X \smallsetminus U) \geq \varrho.$$

In the main theorems (Theorem 6.2 and Theorem 6.3) we are talking about proper hyperbolic geodesic spaces whose hyperbolic boundary has finite Assoud dimension. Since the hyperbolic boundary is a doubling space, we conclude from [9, Corollary 10.2.4] that the hyperbolic space itself has finite asymptotic dimension as soon as the space is visual hyperbolic (see Section 5 for the definition). We refer to [9] for a broader overview of the distinct dimension concepts for hyperbolic spaces and to [3, 16] for more about the asymptotic dimension.

4. Construction of the topological $\mathbb{R}$-tree

In this section we construct a rooted topological $\mathbb{R}$-tree T inside a geodesic proper hyperbolic space X whose hyperbolic boundary has finite Assouad dimension and whose hyperbolic constant is not 0.

The idea of the construction is similar to the construction of the main result in [17] for locally finite hyperbolic graphs. But because the aim of the construction and therefore the construction itself differs from the one in [17] and because we are dealing with proper hyperbolic geodesic spaces instead of locally finite hyperbolic graphs, we have to prove some properties at the end of this section.

Let $d_h = d_\varepsilon$ be a metric such that ε satisfies the assumptions as in Theorem 2.1 and hence such that $(\widehat{X}, d_h)$ is a compact metric space. By [5, Sections 6 and 9] the property of ∂X having finite Assouad dimension does not depend on the particular choice of ε. That means if ∂X has finite Assouad dimension for one metric d_ε, then this holds for all these metrics. That is the reason why we are able just to say that ∂X has finite Assouad dimension. By Theorem 3.1, we know that ∂X is a doubling metric space. So let $N = 2^{\dim_2(\partial X)}$ and let $\varrho = \frac{\exp(7\varepsilon\delta)}{\varepsilon'}$ (with $\varepsilon' = \exp(\varepsilon\delta) - 1$).

The rooted topological $\mathbb{R}$-tree T that we shall construct will have the following properties.

(1) Every ray in T converges to a point in the hyperbolic boundary of X;
(2) for every boundary point η of X there is a ray in T converging to η;
(3) there is an $n \in \mathbb{N}$ such that for every boundary point η of X there are at most n distinct rays in T that start at the root of T and converge to η.

We construct the rooted topological $\mathbb{R}$-tree T recursively. Let $r \in X$ be the base-point of the Gromov-product which we used for the definition of the metric d_ε. The point r will be the root of T. For the construction of T we construct a strictly descending sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ in $\mathbb{R}_{>0}$, a strictly increasing sequence $(d_j)_{i\in\mathbb{N}}$ two sequences $(S_j)_{j\in\mathbb{N}}, (Y_j)_{j\in\mathbb{N}}$ of finite subsets of ∂X, two sequences $(\mathcal{U}_j)_{j\in\mathbb{N}}, (\mathcal{B}_j)_{j\in\mathbb{N}}$ of closed covers of ∂X, and a sequence $(T_j)_{j\in\mathbb{N}}$ of topological $\mathbb{R}$-trees that lie in X. Our final tree T will be the union of all the T_j. The other sequences will help us in the construction of the topological $\mathbb{R}$-trees T_j and they will satisfy the assertions (a) to (k) for every j.

(a) $\varepsilon_j = \frac{\varepsilon_{j-1}}{64\varrho N^4}$;
(b) $S_{j-1} \subseteq Y_j \subseteq S_j$;
(c) $d_h(\eta,\mu) \geq \varepsilon_j$ for all $\eta \neq \mu \in S_j$;
(d) $d_h(\eta,\mu) \geq \frac{\varepsilon_{j-1}}{32}$ for all $\eta \neq \mu \in Y_j$;
(e) the closed cover $\mathcal{U}_j$ consists of precisely the closed ε_j-balls around the elements of S_j;
(f) the set $\mathcal{U}_j$ has $\frac{\varepsilon_{j-1}}{8}$-multiplicity at most $N^{\log_2(32\varrho N^4)}$;
(g) the set $\mathcal{B}_j$ consists of all closed balls of radius $\frac{\varepsilon_{j-1}}{32}$ around the elements of Y_j and it has $\frac{\varepsilon_{j-1}}{32}$-multiplicity at most N^4;
(h) every $\eta \in Y_j$ has $(\frac{3\varepsilon_{j-1}}{32})$-multiplicity at most N^4 in $\mathcal{B}_j$;
(i) $T_{j-1} \subseteq T_j$;
(j) every ray in T_j converges to an elements of S_j and to each element of S_j converges precisely one ray in T_j that starts at the root;
(k) every ray in T_j is eventually geodesic, in particular, there is a constant c depending only on ε_j such that every ray in $T_j \smallsetminus \overline{B}_c(x)$ is a geodesic ray;
(l) d_j depends only on ε_j and we have $\overline{B}_{d_j}(r) \cap (T_j \smallsetminus T_{j-1}) = \emptyset$.

Before we start the recursion step, we first define the elements of all sequences for $j = 0$. Let $\mu^0 \in \partial X$, $S_0 = Y_0 = \{\mu^0\}$ and $\varepsilon_0 = \sup\{d_h(\mu^0,\eta) \mid \eta \in \partial X\}$ (notice that ∂X is bounded by Theorem 2.1). Let $\mathcal{B}_0 = \mathcal{U}_0 = \{\partial X\}$ and let T_0 be a geodesic ray from r to μ^0 which exists by Proposition 2.2. Then all the properties (a) – (l) are satisfied for $j = 0$.

For the recursion step we may choose ε_j so that (a) holds. Lemma 3.2 shows that there is a finite closed covering $\mathcal{B}_j$ of ∂X with balls of radius $\frac{\varepsilon_{j-1}}{32}$ such that this covering has $\frac{\varepsilon_{j-1}}{32}$-multiplicity at most N^4 and such that the set Y_j of centers of these balls contains S_{j-1} and such that every $\eta \in Y_j$ has $(\frac{3\varepsilon_{j-1}}{32})$-multiplicity at most N^4 in $\mathcal{B}_j$. Then (d), (g), (h), and the first inclusion of (b) hold.

Let S_j be a subset of ∂X with $Y_j \subseteq S_j$ such that $d_h(\mu,\nu) > \varepsilon_j$ for all $\mu,\nu \in S_j$, such that $\mathcal{U}_j := \{\overline{B}_{\varepsilon_j}(\mu) \mid \mu \in S_j\}$ is a closed cover of ∂X with $\frac{\varepsilon_{j-1}}{8}$-multiplicity at most $N^{\log_2(32\varrho N^4)}$. This set S_j exists by applying the proof of Lemma 3.2, cf. [17, Proof of Theorem 1.4]. As a consequence we have (c), (f), (e), and the remaining part of (b). The only element of any of the sequences that remains to be constructed is the topological $\mathbb{R}$-tree T_j.

We construct the topological $\mathbb{R}$-tree T_j by recursion. Let $T_j^{0,1} = T_{j-1}$. We enumerate the set $S_j \smallsetminus S_{j-1}$ in the following way. Let $\mu_1^1, \mu_2^1, \ldots$ be the elements with $8\varepsilon_{j-1}$-multiplicity 1 in $\mathcal{B}_{j-1}$, let $\mu_1^2, \mu_2^2, \ldots$ be the elements with $(2 \cdot 8\varepsilon_{j-1})$-multiplicity at most 2 in $\mathcal{B}_{j-1}$ but not $8\varepsilon_{j-1}$-multiplicity at most 1 in $\mathcal{B})j-1$, and so on. As the set $\mathcal{B}_{j-1}$ has $\frac{\varepsilon_{j-2}}{32}$-multiplicity at most N^4 and $2\varrho N^4\varepsilon_{j-1} = \frac{\varepsilon_{j-2}}{32} \geq \operatorname{rad}(\mathcal{B}_{j-1})$, there are no μ_k^i with $i > N^4$ by (g), in particular, we have $S_j \smallsetminus S_{j-1} = \{\mu_1^1, \ldots, \mu_m^M\}$ for some $m, M \in \mathbb{N}$ with $M \leq N^4$.

The topological $\mathbb{R}$-tree $T_j^{i,k}$ will be the union of the topological $\mathbb{R}$-tree $T_j^{i-1,k}$ and an eventually geodesic ray from $T_j^{i-1,k}$ to the hyperbolic boundary point μ_k^i, where we denote by $T_j^{0,k}$ the union of all $T_j^{n,k-1}$. So let $\mu_k^i \in S_j \setminus S_{j-1}$ and assume that we have already constructed the topological $\mathbb{R}$-tree $T_j^{i-1,k}$. Due to (e), there is a $\mu \in S_{j-1}$ with $d_h(\mu_k^i, \mu) \leq \varepsilon_{j-1}$. Due to Proposition 2.2, there exists a geodesic double ray R from μ_k^i to μ in X. Let Q denote the largest distance from r to any geodesic double ray between two boundary points of distance at most ε_{j-1} and at least ε_j and let q denote the smallest distance of from r to any such double ray. Then we have $\beta \geq Q - q$ for the constant $\beta = \beta(\delta, \frac{\varepsilon_{j-1}}{\varepsilon_j}, \varepsilon)$ from Proposition 2.6.

Let us first consider the case that R is totally disjoint from $T_j^{i-1,k}$ in the ball with center r and radius $Q + 5\delta$. Due to the construction of $T_j^{i-1,k}$, there is a geodesic ray P in $T_j^{i-1,k}$ that converges to μ and whose first point x_R has distance $Q + 5\delta$ to r. We consider a geodesic $\tilde{\pi}_R$ from R to P that has length at most δ with the additional property that for some point z on $R \cap B_Q(r)$ we have that $d_{R\cup\tilde{\pi}_R}(z, x_P)$ is minimal. This exists because X is proper and because every point y on P with $d(r, y) \geq Q + 3\delta$ is δ-close to a point on R, and hence the same holds for every point on P. As $\tilde{\pi}_R$ lies in the ball with center r and radius $Q+6\delta$, which is compact, there is a smallest subpath π_R of $\tilde{\pi}_R$ that contains a point of R and a point of $T_j^{i-1,k}$. Let y_R denote the intersection point of π_R and R. Then we add the subray of R from y_R to μ_k^i together with π_R to $T_j^{i-1,k}$ to obtain the new topological $\mathbb{R}$-tree $T_j^{i,k}$. If x lies during the construction on a geodesic double ray P, then we say that we have *connected* μ_k^i *to* that limit point η of P that has smaller distance to μ_k^i and if they have the same distance to μ_k^i, then we choose one of them arbitrary. If x lies on some π_P for a double ray P, then we have *connected* μ_k^i either *to* the boundary point η we constructed a new ray to with P or inductively *to* the boundary point we connected η to, depending which one has the smallest distance to μ_k^i; in the case of a tie, we choose again arbitrary. If the hyperbolic boundary point η to which μ_k^i is connected lies in S_{j-1}, then μ_k^i is *eventually connected to* η. If this is not the case, then μ_k^i is *eventually connected to* that hyperbolic boundary point to which η is eventually connected to.

Now we look at the case that there is a common point of R and $T_j^{i-1,k}$ that has distance at most $Q + 5\delta$ to r. Then there is a unique common point x of R and $T_j^{i-1,k}$ such that Rx, the subray of R from x to μ_k^i, contains no other point of $T_j^{i-1,k}$ by compactness of the ball of radius $Q + 5\delta$. In this case we just add the subray Rx to the boundary point μ_k^i to the topological $\mathbb{R}$-tree to obtain the topological $\mathbb{R}$-tree $T_j^{i,k}$. By the choice of x, the space $T_j^{i,k}$ is indeed a topological $\mathbb{R}$-tree. In preparation of the proof of Lemma 6.1 we denote by π_R the point x and we set $x_R := x$. The property for μ_k^i of being *connected to* and being *eventually connected to* is defined analogously to the first case.

Let $T_j := \bigcup_{i,k} T_j^{i,k}$. That T_j is a topological $\mathbb{R}$-tree is an easy observation, because T_j is the union of a finite chain of topological $\mathbb{R}$-trees, so it is the last element of the chain. We remark that the topological $\mathbb{R}$-tree T_j satisfies the properties (i), (j), (k) for $c = Q + 6\delta$, and (l) for $d = q$.

We just have defined all sequences as claimed. Set

$$T := \bigcup_{j\in\mathbb{N}} T_j.$$

It remains to prove that T is a topological $\mathbb{R}$-tree and satisfies the assertions (1) to (3) as claimed. Since each of the topological $\mathbb{R}$-trees T_j is connected and $T_j \subseteq T_{j+1}$, we know that T is connected. It is obvious that T contains no circle, i.e. no homeomorphic image of S^1, as a circle is a compact and hence bounded subset of (X, d_ε) and due to the choice of the constant in (l) during the construction. Furthermore, the fact that T is homeomorphic to an $\mathbb{R}$-tree follows from (l) as $(d_j)_{j\in\mathbb{N}}$ is strictly increasing, because each $T_{j-1} \cap \overline{B}_{d_j}(r)$ is homeomorphic to an $\mathbb{R}$-tree by construction.

In order to prove the assertions (1) to (3) we shall prove two claims in which we use the notations from the step j of the construction.

Claim 4.1. *Let μ_k^n and μ_l^n be two elements of $S_j \setminus S_{j-1}$ with $d_h(\mu_k^n, \mu_l^n) \leq \varrho\varepsilon_{j-1}$ for some $n \leq N^4$. Then for any $\eta \in Y_{j-1}$ with $d_h(\mu_k^n, \eta) \leq n\varrho\varepsilon_{j-1}$ we have $d_h(\mu_l^n, \eta) \leq n\varrho\varepsilon_{j-1}$.*

Proof of Claim 4.1. The $((n-1)\varrho\varepsilon_{j-1})$-multiplicity of μ_k^n and the one of μ_l^n in $\mathcal{B}_{j-1}$ must be n. Thus, for every hyperbolic boundary point η in Y_{j-1} with distance at most $n\varrho\varepsilon_{j-1}$ to μ_k^n we have $d_h(\mu_k^n, \eta) \leq (n-1)\varrho\varepsilon_{j-1}$ and hence also $d_h(\mu_l^n, \eta) \leq n\varrho\varepsilon_{j-1}$. □

Claim 4.2. *(i) If μ_i^k is connected to $\mu \in S_j$, then we have*

$$d_h(\mu, \mu_i^k) \leq \varrho\varepsilon_{j-1}.$$

(ii) If μ_i^k is eventually connected to $\eta \in S_{j-1}$ in T_j, then we have

$$d_h(\eta, \mu_i^k) \leq \varrho N^4\varepsilon_{j-1} + \mathrm{rad}(\mathcal{B}_{j-1}) = 5\varrho N^4\varepsilon_{j-1}.$$

Furthermore, η lies in some $B \in \mathcal{B}_{j-1}$ with $d_h(\mu_i^k, B) \leq \varrho N^4\varepsilon_{j-1}$.

Proof of Claim 4.2. Let us first prove (i). Let us assume $x_R \in R$. Then R meets some other double ray R' or a geodesic segment $\pi_{R'}$. If R meets some other double ray R', then μ is a limit point of R' and any geodesic double ray $[\mu_i^k, \mu]$ lies in a δ-neighborhood of $R \cup R'$, so it has distance at least $q-\delta$ to r. Let μ' be the second accumulation point of R. So we have $\varepsilon_j \leq d_h(\mu', \mu_i^k) \leq \varepsilon_{j-1}$ and we conclude by Proposition 2.5 that $(\mu', \mu_i^k) \leq (\mu, \mu_i^k) + 5\delta$. By Theorem 2.1, we have

$$\begin{array}{rcl} d_h(\mu, \mu_i^k) & \leq & \exp(-\varepsilon(\mu, \mu_i^k)) \\ & \leq & \exp(-\varepsilon(\mu', \mu_i^k) + 5\varepsilon\delta) \\ & \leq & \varrho\varepsilon' \exp(-\varepsilon(\mu', \mu_i^k)) \\ & \leq & \varrho d_h(\mu', \mu_i^k) \\ & \leq & \varrho\varepsilon_{j-1}. \end{array}$$

Let us assume that either R meets some $\pi_{R'}$, or π_R is non-trivial and meets either some ray R' or some $\pi_{R'}$. Then y_R has distance at most 2δ to $y_{R'}$. Let P' be the ray for R' that is P for R if R or π_R meets $\pi_{R'}$. Let μ' be the limit point of R' that is not the limit point of P'. Then $[\mu_{i+1}^k, \mu']$ lies in a 3δ-neighborhood of $R \cup \pi_R \cup \pi_{R'} \cup R'$. By Theorem 2.1, this gives us the following inequality, where ν is the second accumulation point of R. Notice that we have $\varepsilon_j \leq d_h(\mu_i^k, \nu) \leq \varepsilon_{j-1}$ and, due to Proposition 2.5, we have $(\nu, \mu_i^k) \leq (\mu', \mu_i^k) + 7\delta$.

$$\begin{array}{rcl} d_h(\mu', \mu_i^k) & \leq & \exp(-\varepsilon(\mu', \mu_i^k)) \\ & \leq & \exp(-\varepsilon(\nu, \mu_i^k) + 7\varepsilon\delta) \\ & \leq & \varrho\varepsilon' \exp(-\varepsilon(\nu, \mu_i^k)) \\ & \leq & \varrho d_h(\nu, \mu_i^k) \\ & \leq & \varrho\varepsilon_{j-1}. \end{array}$$

By the choice of μ, we know $d_h(\mu, \mu_k^i) \leq d_h(\mu', \mu_k^i) \leq \varrho\varepsilon_{j-1}$.

Now, we prove the second assertion. Let us choose m be minimal such that the $((m-1)\varrho\varepsilon_{j-1})$-multiplicity of μ in $\mathcal{B}_{j-1}$ is not $m-1$ but such that the $(m\varrho\varepsilon_{j-1})$-multiplicity of μ in $\mathcal{B}_{j-1}$ is m. We may assume that $\mu \neq \eta$, that is $\mu \in S_j \setminus S_{j-1}$. By induction we know that η lies in one of the elements of $\mathcal{B}_{j-1}$, say in B, that is responsible for the $(n\varrho\varepsilon_{j-1})$-multiplicity of at most n of μ in $\mathcal{B}_{j-1}$ where n denotes the corresponding value for μ that is m for μ_i^k. As μ_i^k is connected to μ, we have $n \leq m$. So we have $d_h(\mu_i^k, B) \leq m\varrho\varepsilon_{j-1}$ and hence

$$d_h(\eta, \mu_i^k) \leq m\varrho\varepsilon_{j-1} + \operatorname{diam}(\mathcal{B}_{j-1}) \leq m\varrho\varepsilon_{j-1} + 4\varrho N^4\varepsilon_{j-1}.$$

Since every element of $S_j \setminus S_{j-1}$ has $(m\varrho\varepsilon_{j-1})$-multiplicity at most m in $\mathcal{B}_{j-1}$ for some $m \leq N^4$, we have $d_h(\eta, \mu_i^k) \leq 5\varrho N^4\varepsilon_{j-1}$. □

By construction of the topological $\mathbb{R}$-trees T_j and due to Claim 4.2 and Properties (f) and (g), we have the following property.

(∗) In every step and for every closed ball $B \in \mathcal{B}_k$ a boundary point in B can only be eventually connected to elements of at most N^4 different balls in $\mathcal{B}_k$. Furthermore, there are at most $N^{\log_2(32\varrho N^4)}$ distinct boundary points in B that are eventually connected to elements of the same ball of $\mathcal{B}_k$.

Let us now show the assertions (1) to (3) for the topological $\mathbb{R}$-tree T. For a closed ball $B \in \mathcal{B}_k$ let B' be the union of B and all other (at most N^4) closed balls in $\mathcal{B}_k$ of distance at most $\varrho N^4\varepsilon_k$ to B.

Because of (j) we just have prove that any ray that we created without our knowledge in the limit step converges to some hyperbolic boundary point. Let π be such a ray in T. We remark that we do not need any of the properties (1) to (3) for the proof of Lemma 6.1. Thus, we can apply it here without creating a circular argument. The lemma says that π is eventually a quasi-geodesic ray. We deduce from Proposition 2.3 that there is a geodesic ray $\widehat{\pi}$ and a $\kappa \geq 0$ such that π lies eventually in a κ-neighborhood of $\widehat{\pi}$. Hence, π converges to the same boundary as $\widehat{\pi}$ and we have proved (1).

For the proof of (2), let $\eta \in \partial X$. In every construction step k there is at least one closed ball $B_k \in \mathcal{B}_k$ with $\eta \in B_k$ because $\mathcal{B}_k$ is a cover of ∂X. Hence there is in each step some boundary point $\eta_k \in S_k \cap B_k$ with $d_h(\eta_k, \eta) \leq \varepsilon_k$. So T_k contains a ray to η_k. Let π_k be a ray from r to η_k in T_k. For every $\varrho \in \mathbb{N}$ there is a path in $T_k \cap \overline{B}_\varrho(r)$ that is contained in infinitely many of the π_k by compactness of $\overline{B}_\varrho(r)$ and because there are only finitely many paths in T_j that starts at r and end at a point with distance ϱ from r. Thus there is a ray π such that every point on π lies on infinitely many of the rays π_k. Due to (1), the ray π has precisely one accumulation point. This accumulation point must be η because of Claim 4.2 and the choice of the rays π_k.

For every $B \in \mathcal{B}_k$ in step k there are at most N^4 closed balls in the step $k-1$ such that a boundary point in $(B \cap S_k) \smallsetminus S_{k-1}$ is eventually connected to some hyperbolic boundary point of such a ball. Furthermore, for each of these balls there are at most $N^{\log_2(32\varrho N^4)}$ many hyperbolic boundary points to which our new ones are eventually connected. Thus, we know that the number of rays to one boundary point is bounded by a function depending only on $\dim_2(\partial X)$. Hence, we have also proved the remaining assertion (3).

5. Visual hyperbolic spaces

We call a hyperbolic space X *visual* if for some $o \in X$ there is a $D > 0$ such that for every $x \in X$ there is an $\eta \in \partial X$ with

$$d(o, x) \leq (x, \eta)_o + D.$$

Remark that the property for hyperbolic spaces to be visual is independent of the choice of o.

An easy observation is that the definition of visual hyperbolic spaces is equivalent to the following. For some (and hence every) $o \in X$ there is a $D' > 0$ such that for every $x \in X$ there is an $\eta \in \partial X$ such that any geodesic between o and x lies in a D'-neighborhood of a geodesic ray from o to η.

Remark that by Corollary 1.3.5 of [9] hyperbolicity is preserved by quasi-isometries and it is not hard to see that the same holds for visual hyperbolicity.

5.1. **Hyperbolic approximations of metric spaces.** For every metric space X, there is a hyperbolic space Y whose hyperbolic boundary is homeomorphic to X. Constructions of such spaces can be found in [6, 8, 9, 13]. The hyperbolic space Y is called a *hyperbolic approximation of* X. That the constructed space Y is indeed a hyperbolic space is shown in [9, Proposition 6.2.10] and we just state the proposition without proof. Note that by looking at the construction it is easy to see that Y is visual hyperbolic since any vertex of Y lies on an infinite geodesic ray that starts at the *root* of the hyperbolic approximation.

Proposition 5.1. [9, Proposition 6.2.10] *A hyperbolic approximation Y of any metric space X is a visual hyperbolic graph with $\partial Y \cong X$.* $\square$

If we restrict the metric space X to be doubling, then the degrees of all the vertices in a hyperbolic approximation of X are uniformly bounded by [9, Proposition 8.3.3]. We combine this result with Proposition 5.1 and obtain the following proposition.

Proposition 5.2. *A hyperbolic approximation Y of any doubling metric space X is a visual hyperbolic locally finite graph of bounded degree with $\partial Y \cong X$.* $\square$

5.2. **Rough similarities.** We cite a result by Buyalo and Schroeder [9]. In order to do that we have to make some further definitions.

Let X and Y be two metric spaces. If there are a map $f : X \to Y$ and constants $k, \lambda > 0$ such that

$$|\lambda d_X(x, y) - d_Y(f(x), f(y))| \leq k$$

holds for all $x, y \in X$ and $\sup_{y \in Y} d_Y(y, f(X)) \leq k$, then X is (λ, k)-*roughly similar* to Y, or just *roughly similar* to Y, and we call f a (λ, k)-*rough similarity*, or just a *rough similarity*.

In particular, every space Y that is roughly similar to a space X is also quasi-isometric to X. As (visual) hyperbolicity is preserved by quasi-isometries, it is also preserved by rough similarities.

Theorem 5.3. [9, Corollary 7.1.5.] *Every visual hyperbolic space X is roughly similar to a subspace of a hyperbolic geodesic space Y with the same hyperbolic boundary, $\partial X = \partial Y$.* $\square$

We obtain the following corollary.

Corollary 5.4. *Let X be a proper hyperbolic geodesic space whose hyperbolic boundary is doubling. Let $\gamma_1 \geq 1, \gamma_2 \geq 0$ be constants. Then there is a subspace Y of X such that the following statements hold for Y.*

(1) *Y is a proper visual hyperbolic geodesic space;*
(2) *every (γ_1, γ_2)-quasi-geodesic ray of X lies eventually in Y;*
(3) *the identity $\iota : Y \to X$ extends to a continuous map $\hat{\iota} : \widehat{Y} \to \widehat{X}$ with $\hat{\iota}(\partial Y) = \partial X$.*

Proof. Let Z be a visual hyperbolic locally finite graph that is a hyperbolic approximation of the hyperbolic boundary ∂X. By Theorem 5.3, there is a subspace Z' of X that is (λ, k)-roughly similar to Z for some constants $\lambda \geq 1, k \geq 0$. Let Y be the subspace of X that is induced by Z' and all points with distance at most $\kappa(\delta, \gamma_1, \gamma_2) + 2\,\kappa(\delta, \lambda, k)$ to any element of Z' for the constants $\kappa(\delta, \gamma_1, \gamma_2)$ and $\kappa(\delta, \lambda, k)$ of Proposition 2.3. Since Z is locally finite and X is proper, the space Z' is proper and the same holds for Y. As Z is visual hyperbolic and this is a property that is preserved by quasi-isometries, assertion (1) holds for Z' and thus also for Y as the identity from Z' to Y is a quasi-isometry by the choice of Y. Since Z is a geodesic space, every two points of Z' can be joined by a (λ, k)-quasi-geodesic. This together with various applications of Proposition 2.3 implies (2). The assertion (3) is a direct consequence of the fact that quasi-isometries between proper hyperbolic geodesic spaces can be extended to quasi-isometries between their hyperbolic compactifications. □

6. Tree-likeness of hyperbolic spaces

We remark that, usually, the constructed tree in [17] for locally finite hyperbolic graphs is far from having only rays that are eventually quasi-geodesic. But the changes in the construction we made in this paper are strong enough to guarantee that all rays in the constructed topological $\mathbb{R}$-tree are already eventually quasi-geodesic rays in the hyperbolic space.

Lemma 6.1. *Let X be a proper hyperbolic geodesic space whose hyperbolic boundary has finite Assouad dimension and let T be the topological $\mathbb{R}$-tree that was constructed in Section 4 with root r. Then there are constants $\gamma_1 \geq 1, \gamma_2 \geq 0$ such that every ray in T starting at the root is a (γ_1, γ_2)-quasi-geodesic ray in X.*

Proof. We assume all assumptions and notations as in the construction step j of Section 4. By Proposition 2.6 there is a constant β depending only on the quotient $\frac{\varepsilon_j}{\varepsilon_{j-1}}$ and not depending on the particular j such that for every four boundary points $\eta_1, \eta_2, \eta_3, \eta_4$ with

$$\varepsilon_{j-1} \geq d_h(\eta_1, \eta_2), d_h(\eta_3, \eta_4) \geq \varepsilon_j$$

we have

$$|d(r, [\eta_1, \eta_2]) - d(r, [\eta_3, \eta_4])| \leq \beta.$$

Note that $\beta \geq Q - q$ with Q, q as defined in Section 4. Let $M := N^4 N^{\log_2(32\theta N^4)}$. In the first step of the proof we shall prove for every two points w, y with $y \in T_j^{i,k} \setminus T_j^{i-1,k}$, $w \in T_j^{i,k} \cap [r, y]_T$ the inequality

$$d_Y(w, y) \leq d(w, y) + (jM + n)(88\delta + 4\beta) \tag{1}$$

with $Y := T_j^{i,k}$ for an n that denotes the number how often we have already enlarged the tree T_{j-1} by additional rays whose intersection with $[r, y]_T$ is not empty. We conclude from $(*)$ in Section 4 that n is bounded by M and we even have $n < M$, since we add just in this step a ray.

Let R be the geodesic double ray, as in the recursion step and let P be that geodesic ray with $P \subseteq R$ that we added together with π_R to $T_j^{i-1,k}$ to obtain $T_j^{i,k}$. Let x be the unique point in $T_j^{i-1,k} \cap \pi_R$ and let x' be the unique point in $\pi_R \cap P$. By the choice of π_R we have $d(x, x') \leq \delta$. Furthermore, we have $d(x, r) \leq Q + 5\delta$ and $d(x', r) \leq Q + 6\delta$.

Let $\eta := \mu_i^k$ and let μ be the other limit point of R. Let $b \in R$ with $d(b, Q_\eta) \leq \delta$ for $Q_\eta = [r, \eta]$ and $d(b, Q_\mu) \leq \delta$ for $Q_\mu = [r, \mu]$. Such a point exists by definition of hyperbolicity and as X is proper. Let a be a point on R with minimal distance to r.

Let us prove

$$d(a, b) \leq 4\delta. \tag{2}$$

Let $c \in Q_\eta$ and $c' \in Q_\mu$ both of minimal distance to b. By the choice of b we have $d(b, c) \leq \delta$ and $d(b, c') \leq \delta$. As X is hyperbolic, the geodesic double ray R is contained in the δ-neighborhood of the subset

$$Z := \eta Q_\eta c \cup [c, b] \cup [b, c'] \cup c' Q_\mu \mu$$

of X, where $\eta Q_\eta c$ is the subray of Q_η from c to η and analogously for $c' Q_\mu \mu$. In particular, we have $d(a, Z) \leq \delta$. Let $a' \in Z$ with $d(a, a') \leq \delta$. Then we have $d(r, a') \leq d(r, a) + \delta$. By symmetry, we may assume that $a' \in \eta Q_\eta c \cup [c, b]$. If $a' \in [c, b]$, then we have $d(a', c) \leq \delta$. If $a' \in \eta Q_\eta c$, then we have $d(a', c) \leq 2\delta$, since c is the point on $Q_\eta \cap Z$ with minimal distance to r and $d(r, c) \geq d(r, a) - \delta$. The inequality

$$d(a, b) \leq d(a, a') + d(a', c) + d(c, b) \leq \delta + 2\delta + \delta = 4\delta$$

proves (2).

For any another point $\widehat{a}$ on R with distance $d(r, a)$ to r, we conclude from (2) that $d(a, \widehat{a}) \leq 8\delta$.

Let x'' be the point on $[r, x']$ with $d(r, x'') = d(r, a) - \delta$. In particular, we have $d(x', x'') \leq \beta + 7\delta$. Since X is hyperbolic, there is a point on $[r, a] \cup [a, x']$ with distance at most δ to x''. If this point lies on $[r, a]$, then we have $d(x'', a) \leq 3\delta$, and, if this point lies on $[a, x']$, then it has the same distance to r as a and hence distance at most 8δ to a. Thus, we have $d(x'', a) \leq 9\delta$. So we proved

$$d(a, x') \leq \beta + 16\delta. \tag{3}$$

Let a_w be a point on R with $d(w, a_w) = d(w, R)$. To prove

$$d(w, y) \geq d(w, a_w) + d(a_w, y) - 4\delta \tag{4}$$

let y' a point on $[w, y]$ with distance at most δ to both $[w, a_w]$ and $[a_w, y]$. Such a point exists by the same arguments as for b. Let $y_1 \in [w, a_w] \cap B_\delta(y')$ and $y_2 \in [a_w, y] \cap B_\delta(y')$. We have $d(w, y_2) \geq d(w, a_w)$ and hence

$$d(y_1, a_w) \leq d(y_1, y') + d(y', y_2) \leq 2\delta.$$

This immediately implies $d(y, y') + 3\delta \geq d(a_w, y)$. In addition, we have

$$d(w, y') \geq d(w, y_2) - \delta \geq d(w, a_w) - \delta.$$

As $y' \in [w, y]$, we conclude

$$d(w, y) = d(w, y') + d(y', y) \geq d(w, a_w) + d(a_w, y) - 4\delta.$$

So we have proved (4).

The next step is to show

$$d(a, a_w) \leq 18\delta + \beta. \tag{5}$$

As X is hyperbolic, we conclude directly that $[a, a_w]$ lies in a 2δ-neighborhood of $[a, r] \cup [r, w] \cup [w, a_w]$. But since $d(r, a) = d(r, R)$ and $d(w, a_w) = d(w, R)$, a part of length at most 4δ of $[a, a_w]$ lies in the 2δ-neighborhood of $[r, a]$ and a part of length at most 4δ lies in the 2δ-neighborhood of $[w, a_w]$. The point w lies in $[r, y] \cap T_j^{i-1,k}$ and hence $d(r, w) \leq d(r, a) + \beta + 6\delta$, so at most a part of length $10\delta + \beta$ of $[a, a_w]$ lies in a 2δ-neighborhood of $[r, w]$. Then we conclude directly (5).

Let a'_w be a point on P with minimal distance to w and with $a'_w = a_w$ if a_w lies on P. Let us show

$$a_w = a'_w \quad \text{or} \quad d(r, a_w) \leq d(r, x') + \beta + 8\delta. \tag{6}$$

Let us assume that $a_w \neq a'_w$. Let z be a point on $[a_w, w]$ with distance δ to a_w. Since X is proper, z must have a point of distance at most δ on $[w, x']$. As $[x', w]$ lies in the δ-neighborhood of $[r, w] \cup [r, x']$, it lies in $B_{Q+7\delta}(r)$. Because of $d(r, x') \geq Q - \beta$, we obtain $d(r, a_w) \leq Q + 8\delta$ and (6) follows immediately.

Since X is δ-hyperbolic and proper, there is a point z on $[x', a_w]$ with distance at most δ to $[r, x']$ and to $[r, a_w]$. By (6), we obtain

$$d(w, a'_w) \leq d(w, x') \leq d(w, a_w) + \beta + 12\delta.$$

By an analogous argumentation, we conclude

$$d(a_w, a'_w) \leq \beta + 16\delta. \tag{7}$$

Then we can conclude inductively, where j' denotes the recursion step in which we added x to the previous topological $\mathbb{R}$-tree.

$$\begin{array}{rcl}
d_Y(w, y) & = & d_Y(w, a'_w) + d_Y(a'_w, y) \\
& \leq & d_Y(w, x) + d(x, x') + d(x', a) + d(a, a_w) + d(a_w, a'_w) \\
& & +d(a'_w, y) \\
& \leq & d_Y(w, x) + \delta + \beta + 16\delta + \beta + 18\delta + \beta + 16\delta + d(a_w, y) \\
& \leq & d(w, x) + ((j - (j' + 1))M + n)(\alpha_1\delta + \alpha_2\beta) \\
& & +51\delta + 3\beta + d(a_w, y) \\
& \leq & d(w, a_w) + d(a_w, a) + d(a, x') + d(x, x') + d(a_w, y) + 51\delta + 3\beta \\
& & ((j - (j' + 1))M + n)(\alpha_1\delta + \alpha_2\beta) \\
& \leq & d(w, a_w) + d(a_w, y) + \beta + 18\delta + \beta + 16\delta + \delta + 51\delta + 3\beta \\
& & ((j - (j' + 1))M + n)(\alpha_1\delta + \alpha_2\beta) \\
& \leq & d(w, y) + 4\delta + 86\delta + 5\beta + ((j - (j' + 1))M + n)(\alpha_1\delta + \alpha_2\beta) \\
& \leq & d(w, y) + ((j - (j' + 1))M + (n + 1))(\alpha_1\delta + \alpha_2\beta)
\end{array}$$

with $\alpha_1 = 90$ and $\alpha_2 = 5$. So we have

$$d_Y(w, y) \leq d(w, y) + ((j - (j' + 1))M + (n + 1))(90\delta + 5\beta).$$

Let π be a ray in T that starts at r. Inductively, there are constants c_1 and c_2 (independent of the choice of π) such that π is a (c_1, c_2)-quasi-geodesic ray. $\square$

This lemma enables us to prove our main result. We will prove it in two steps. First, we prove the result for proper visual hyperbolic geodesic spaces and then for arbitrary proper hyperbolic geodesic spaces.

6.1. **The case: visual hyperbolic spaces.** Visual hyperbolic spaces seem to have a treelike-structure, since there is a maximal distance from each point to the union of all geodesic rays starting at the same point. This in fact is the main reason why the topological $\mathbb{R}$-tree constructed in Section 4 points out the tree-likeness of visual hyperbolic spaces. This is specified in Theorem 6.2.

For a hyperbolic space X and a subspace Y of X, we say that the *canonical map from ∂Y to ∂X exists* if the identity $\iota : Y \to X$ extends to a continuous map $\hat{\iota} : \widehat{T} \to \widehat{Y}$ and we call $\hat{\iota}|_{\partial Y}$ is the *canonical map* from ∂Y to ∂X.

Theorem 6.2. *Let X be a proper visual hyperbolic geodesic space whose hyperbolic boundary has finite Assouad dimension. Then there is a topological $\mathbb{R}$-tree $T \subseteq X$ that has the following properties:*

(i) *the canonical map γ from ∂T to ∂X exists and is surjective;*
(ii) *there is a constant $M < \infty$ such that every $\eta \in \partial X$ has at most M inverse images under γ;*
(iii) *there are constants $c_1 \geq 1$, $c_2 \geq 0$ such that every ray in T is eventually (c_1, c_2)-quasi-geodesic;*
(iv) *there is a constant $\Delta < \infty$ such that every point of X lies in a Δ-neighborhood of a point of T;*
(v) *there is a constant $\Lambda < \infty$ such that every geodesic ray of X lies eventually in the Λ-neighborhood of some ray of T.*

The constants M, c_1, c_2, Δ, and Λ depend only on the hyperbolicity constant δ and on the doubling constant of ∂X.

Proof. Let T be the topological $\mathbb{R}$-tree constructed in Section 4 with root r. We already proved in that section the properties (i) and (ii). Lemma 6.1 gives us (iii). Because X is visual hyperbolic, there is a $D > 0$ such that for every $x \in X$ there is an $\eta \in \partial X$ with $d(x, \pi) \leq D$ for every geodesic ray π from r to η. Let π_x be a point on π with $d(x, \pi_x) \leq D$. In T there is a ray π_T from r converging to η. Due to Proposition 2.3 there is a point x_T on π_T with $d(\pi_x, x_T) \leq \kappa$ for a constant κ that depends only on δ, c_1, and c_2. Hence we have $d(x, x_T) \leq \kappa + D$ and (iv) is proved with $\Delta = \kappa + D$.

To prove (v), let π be a geodesic ray in X and let π' be a geodesic ray in X that starts at r and converges to the same hyperbolic boundary point η like π. Due to (i), there is a ray π_T in T that converges to η. This ray in T is a (c_1, c_2)-quasi-geodesic ray in X due to (iii). By Proposition 2.3, there is a constant $\kappa = \kappa(\delta, c_1, c_2)$ such that π' lies in the κ-neighborhood of π_T. So (v) follows with $\Lambda = \kappa$. □

6.2. **The case: hyperbolic spaces.** The final aim of the main part of this paper is to demonstrate the tree-likeness of arbitrary proper hyperbolic geodesic spaces in terms of contained topological $\mathbb{R}$-trees. For that, we combine the result for the visual hyperbolic spaces with the theorems from Section 5.

A subset Y of a hyperbolic geodesic space X has *finite geodesic out-spread* if every geodesic in $X \setminus Y$ has finite length.

Theorem 6.3. *Let X be a proper hyperbolic geodesic space whose hyperbolic boundary has finite Assouad dimension. Then there is a topological $\mathbb{R}$-tree $T \subseteq X$ that has the following properties:*

(i) *the canonical map γ from ∂T to ∂X exists and is surjective;*

(ii) *there is a constant $M < \infty$ such that every $\eta \in \partial X$ has at most M inverse images under γ;*
(iii) *there are constants $c_1 \geq 1$, $c_2 \geq 0$ such that every ray in T is eventually (c_1, c_2)-quasi-geodesic;*
(iv) *there is a constant $\Delta < \infty$ such that the set $B_D(T)$ has finite geodesic out-spread;*
(v) *there is a constant $\Lambda < \infty$ such that every geodesic ray of X lies eventually in the Λ-neighborhood of some ray of T.*

The constants M, c_1, c_2, Δ, and Λ depend only on the hyperbolicity constant δ and on the doubling constant of ∂X.

Proof. Let T' be a topological $\mathbb{R}$-tree in X as constructed in Section 4 and let γ_1, γ_2 be the constants of Lemma 6.1. By Corollary 5.4, there is a proper visual hyperbolic geodesic subspace Y of X that has the properties that every geodesic in $X \setminus Y$ has finite length, so Y has finite geodesic out-spread in X, and that every (γ_1, γ_2)-quasi-geodesic ray lies eventually in Y. In Y we find a topological $\mathbb{R}$-tree T as in Theorem 6.2. For this topological $\mathbb{R}$-tree, also the canonical map $\partial T \to \partial X$ exists and is surjective by Theorem 6.2 (i) applied for T and Y and due to Corollary 5.4 (3). Furthermore, (ii) also holds because it holds for T and Y and because $\partial X = \partial Y$. In addition, Lemma 6.1 implies (iii). Since Y has already finite geodesic out-spread, the same holds for $\overline{B}_\Delta(T)$ because of $Y \subseteq \overline{B}_\Delta(T)$, where Δ denotes the constant from Theorem 6.2 (iv). Finally, (v) is a direct consequence of Theorem 6.2 (v), since every geodesic ray in X lies eventually in Y. This finishes the proof of Theorem 6.3. $\square$

7. The topological dimension of the boundary

Before we prove the main result of this section, we define the topological dimension of a topological space X. A *refinement* $\mathcal{U}$ of an open cover $\mathcal{V}$ of X is an open cover of X such that for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ with $U \subseteq V$. We say that X has *topological dimension at most* n if every open cover has a refinement such that each $x \in X$ lies in at most $n+1$ elements of the refinement, and X has *topological dimension* n if it has topological dimension at most n but not topological dimension at most $n-1$. We call an open cover $\mathcal{U}$ of a topological space X with topological dimension n *critical* if there exists no refinement $\mathcal{V}$ of $\mathcal{U}$ such that each $x \in X$ lies in at most n sets $V \in \mathcal{V}$. For completion, we mention that $\dim X \leq \dim_A X$ holds due to [22, Facts 3.3].

Lemma 7.1. *Let X be a compact metric space such that there exists a totally disconnected metric space Y and an equivalence relation $\sim$ on Y with at most $M \in \mathbb{N}$ elements in each equivalence class such that X and $Y/\sim$ are homeomorphic. Then X has topological dimension at most $M-1$.*

Proof. Let $\mathcal{U}$ be a finite critical open cover of X and $\varphi : X \to Y/\sim$ be a homeomorphism. Let $\mathcal{U}'$ be that open cover of Y that is induced by $\mathcal{U}$, that is for every $U' \in \mathcal{U}'$ there is a $U \in \mathcal{U}$ with $U' = \bigcup_{u \in U} \varphi(u)$. As Y is totally disconnected, it has topological dimension 0. Hence, there is a finite open cover $\mathcal{V}'$ of $\mathcal{U}'$ with pairwise disjoint elements. For any $V' \in \mathcal{V}'$ let V be the set of all $\varphi^{-1}([y])$ with $y \in V'$ and let $\mathcal{V}$ be the set of all these sets V for $V' \in \mathcal{V}'$. Since all V' are open sets so are all V and thus $\mathcal{V}$ is an open cover of X. By construction, $\mathcal{V}$ is also a refinement of $\mathcal{U}$

and every $x \in X$ lies in at most M elements of $\mathcal{U}$. Thus the topological dimension of X is at most $M-1$. □

Theorem 7.2. *Let X be a proper hyperbolic geodesic space and let T be a topological $\mathbb{R}$-tree in X such that the canonical map from ∂T to ∂X exists and is surjective and such that every hyperbolic boundary point of X has at most M inverse images in ∂T for some $M \in \mathbb{N}$. Then the topological dimension of ∂X is at most $M-1$.*

Proof. Since ∂T is compact and ∂T totally disconnected, the assertion is a direct consequence of Lemma 7.1. □

In the terms of [17], where spanning trees of locally finite hyperbolic graphs were investigated, we obtain an analogous result. A *spanning tree* of a graph is a subgraph on all vertices of the graph that is a tree.

Theorem 7.3. *Let G be a locally finite hyperbolic graph and let T be a spanning tree of G such that the canonical map from ∂T onto ∂G exists and is surjective and such that every hyperbolic boundary point of G has at most M inverse images for some $M \in \mathbb{N}$. Then the topological dimension of ∂G is at most $M-1$.* □

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