QUASI-TRANSITIVE K_∞-MINOR FREE GRAPHS

MATTHIAS HAMANN

Abstract. We prove that every locally finite quasi-transitive graph that does not contain K_∞ as a minor is quasi-isometric to some planar quasi-transitive locally finite graph. This solves a problem of Esperet and Giocanti and improves their recent result that such graphs are quasi-isometric to some planar graph of bounded degree.

1. Introduction

Recently, Esperet and Giocanti [2] proved a theorem for quasi-transitive graphs, where a graph is *quasi-transitive* if its automorphism group acts on its vertex set with only finitely many orbits. Before we state their theorem , let us briefly introduce quasi-isometries. A graph G is *quasi-isometric* to another graph H if there exists $\gamma \geq 1$ and $c \geq 0$ and a map $\varphi\colon V(G) \to V(H)$ such that the following holds.

(i) $\frac{1}{\gamma} d_G(u,v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma d_G(u,v) + c$ for all $u, v \in V(G)$ and
(ii) $d_H(w, \varphi(V(G))) \leq c$ for all $w \in V(H)$.

Then φ is a *quasi-isometry*. If the constants γ and c are important, we call φ also a (γ, c)-*quasi-isometry* and say that G and H are (γ, c)-*quasi-isometric*.

Now we are able to state the theorem of Esperet and Giocanti.

Theorem 1.1. [2, Theorem 1.3] *Every locally finite quasi-transitive graph that does not contain K_∞ as a minor is quasi-isometric to some planar graph of bounded degree.*

Esperet and Giocanti proved their theorem as a first step towards a more general conjecture by Georgakopoulos and Papasoglu [4]. In order to state their conjecture, let us introduce the notion of asymptotic minors.

For $K \in \mathbb{N}$, a graph H is a K-*fat minor* of a second graph G if there exists a family $(B_v)_{v \in V(H)}$ of connected subsets of $V(G)$ and a family $(P_e)_{e \in E(H)}$ of paths in G such that

(1) for all $uv \in E(H)$, the path P_{uv} intersects $\bigcup_{w \in V(H)} B_w$ in exactly its end vertices, one of which lies in B_u, the other in B_v,
(2) $d(P_{uv}, B_w) \geq K$ for all $uv \in E(H)$ and $w \in V(H) \smallsetminus \{u, v\}$,
(3) $d(B_u, B_v) \geq K$ for all distinct $u, v \in V(H)$, and
(4) $d(P_e, P_{e'}) \geq K$ for all distinct $e, e' \in E(H)$.

We call H an *asymptotic minor* of G if for every $K > 0$, H is a K-fat minor of G. Now we can state Georgakopoulos' and Papasolgu's conjecture.

Conjecture 1.2. [4, Conjecture 9.3] *Let G be a locally finite transitive graph. Then either G is quasi-isometric to a planar graph, or it contains every finite graph as an asymptotic minor.*

The obvious question regarding Conjecture 1.2 is whether we can ask the planar graph to be transitive, too. Indeed, Esperet and Giocanti [2, Section 6] raised the problem whether the planar graph in their theorem can be asked to be quasi-transitive, too. We will prove that this is possible. That is, we will prove the following theorem.

Theorem 1.3. *Every locally finite quasi-transitive graph that does not contain K_∞ as a minor is quasi-isometric to some planar quasi-transitive locally finite graph.*

This result indicates that a possible positive solution of the following problem might be expectable.

Problem 1.4. *If G is a quasi-transitive locally finite graph quasi-isometric to a planar graph, then is G quasi-isometric to a quasi-transitive locally finite planar graph?*

Another hint that this might be true is that MacManus [7] recently proved the following analogous statement for finitely generated groups.

Theorem 1.5. [7, Corollary D] *The following are equivalent for every finitely generated group G.*

(1) *G is quasi-isometric to a planar graph.*
(2) *G is quasi-isometric to a planar Cayley graph.*

Furthermore, he proved a structural result for quasi-transitive locally finite graphs that are quasi-isometric to planar graphs, see [7, Corollary C], in terms of canonical tree-decompositions: the parts are either finite or quasi-isometric to complete Riemannian planes. We refer to Section 2 for the definition of (canonical) tree-decompositions. This structural result might be useful for Problem 1.4.

2. Preliminaries

Let G be a graph. A *tree-decomposition* of G is a pair $(T,\mathcal{V})$ of a tree T, the *decomposition tree*, and a family $\mathcal{V}=(V_t)_{t\in V(T)}$ of vertex sets of G, one for every $t\in V(T)$, such that

(T1) $V(G)=\bigcup_{v\in V(T)} V_t$,
(T2) for every $e\in E(G)$ there exists $t\in V(T)$ with $e\subseteq V_t$, and
(T3) $V_{t_1}\cap V_{t_2}\subseteq V_{t_3}$ for all t_3 on the t_1-t_2 path in T.

The sets V_t are the *parts* of the tree-decomposition and the intersection $V_{t_1}\cap V_{t_2}$ for adjacent t_1 and t_2 are the *adhesion sets*. The *adhesion* of $(T,\mathcal{V})$ is the supremum of the sizes of the adhesion sets. The *width* of $(T,\mathcal{V})$ is $\sup_{t\in V(T)}|V_t|-1$, seen as an element of $\mathbb{N}\cup\{\infty\}$, if all V_t are finite and ∞ otherwise. The *tree-width* of G is the minimum width among all tree-decompositions of G.

If the automorphism group of G induces an action on the family $\mathcal{V}$ and thereby also an action on T then we call the tree-decomposition *canonical*.

If V_t is a part of $(T,\mathcal{V})$, then the subgraph of G induced by V_t together with all (possibly new) edges uv for all distinct u,v that lie in a common adhesion set in V_t is a *torso* of $(T,\mathcal{V})$.

A *separation* of G is a pair (A,B) with $A,B\subseteq V(G)$ such that $A\cup B=V(G)$ and such that $e\subseteq A$ or $e\subseteq B$ for all edges of G. We call $|A\cap B|$ its *order*. The separation is *tight* if there are components C_A in $A\setminus B$ and C_B in $B\setminus A$ with $N(C_A)=A\cap B=N(C_B)$.

For a tree-decomposition $(T, \mathcal{V})$ and an edge $e \in E(T)$, the *edge-separation* of e is the separation

$$\Big(\bigcup_{t \in V(T_1)} V_t, \bigcup_{t \in V(T_2)} V_t\Big),$$

where T_1 and T_2 are the two components of $T - e$.

The following result by Thomassen and Woess [9, Corollary 4.3] was stated for transitive graphs, but its proof carries over almost verbatim to quasi-transitive graphs.

Lemma 2.1. [9, Corollary 4.3] *Let G be a connected quasi-transitive locally finite graph and let $k \in \mathbb{N}$. Then there are only finitely many $\mathrm{Aut}(G)$-orbits of tight separations of order k.*

The major tool in our proof of Theorem 1.3 is the following result by Esperet et al. [3].

Theorem 2.2. [3, Theorem 4.3] *Let G be a quasi-transitive locally finite graph without K_∞ as a minor and let Γ be a group acting quasi-transitively on G. Then there exists $k \in \mathbb{N}$ and a Γ-invariant tree-decomposition $(T, \mathcal{V})$ of adhesion at most 3, and such that for every $t \in V(T)$ the torso of V_t is a minor of G that is either planar or has tree-width at most k and such that Γ_t acts quasi-transitively on that torso. Furthermore, the edge-separations of $(T, \mathcal{V})$ are all tight.*

One-way infinite paths are *rays* and two rays in a graph G are *equivalent* if, for every finite vertex set $S \subseteq V(G)$, both rays have all but finitely many vertices in the same component of $G - S$. This is an equivalence relation whose equivalence classes are the *ends* of G. An end is *thick* if it contains infinitely many pairwise disjoint rays and it is *thin* otherwise. By a result of Halin [5], for every thin end, there exists $n \in \mathbb{N}$ such that there are n but not $n + 1$ pairwise disjoint rays in that end.

Two ends are *k-distinguishable* for some $k \in \mathbb{N}$ if there exists a vertex set S of size at most k such that no component of $G - S$ contains all but finitely many vertices from rays from both ends. A tree-decomposition *distinguishes* two ends *efficiently* if there is an edge-separation (A, B) such that all rays from one of the ends lie eventually in A, all rays from the other end lie eventually in B and the ends are not $(|A \cap B| - 1)$-distinguishable.

The following is a special case of [1, Theorem 7.3].

Theorem 2.3. *Let G be a locally finite graph and let $k \in \mathbb{N}$. Let $\mathcal{E}$ be a set of ends of G that are pairwise k-distinguishable. Then there is a canonical tree-decomposition distinguishing all end in $\mathcal{E}$ efficiently.*

While the following statement follows from results about factorisations and tree amalgamations of quasi-transitive graphs, we offer here a proof that avoids most of the definitions that we would need, if we conclude it from [6, Theorem 7.5].

Theorem 2.4. *Let G be a locally finite graph of finite tree-width. Then there exists a canonical tree-decomposition of finite width distinguishing all ends of G efficiently.*

Proof. A ray R of G *lies in* an end of any decomposition tree of a tree-decomposition of finite width of G if there is a ray in that end whose parts combined contain infinitely many vertices from R and each of those parts contains at least one vertex of R. It is easy to see that equivalent rays in G must lie in the same end of the decomposition tree. Thus, every end of G is thin and contains at most k distinct rays. In particular, the ends of G are pairwise k-distinguishable. So let $(T, \mathcal{V})$ be a canonical tree-decomposition distinguishing all ends of G efficiently. We may assume that every edge-separation distinguishes some pair of ends efficiently. In particular, there is an upper bound on the adhesion sets. By Lemma 2.1, there are only finitely many orbits of tight separations of bounded order. Thus, there are only finitely many orbits on $E(T)$ and hence on $V(T)$. If we show that all parts are finite, then this implies that the tree-decomposition has finite width. So let us suppose that some part is infinite. Since $(T, \mathcal{V})$ distinguishes all ends, there is a unique end in this part[1] and hence also in this torso. Note that the torso is locally finite, since it follows from Lemma 2.1 that every vertex lies in only finitely many separators of tight separations. Since the stabiliser of that part acts quasi-transitively on the torso by a results of Esperet and Giocanti [3, Lemma 3.13], it is a one-ended quasi-transitive graph. By a result of Thomassen [8, Proposition 5.6], this end must be thick, a contradiction since all ends are thin. Thus, all parts are finite, which finishes the proof as mentioned above. □

For a finite tree T, we call a vertex of T *central* if it is the middle vertex of a longest path in T. Similarly, an edge of T is *central* if it is the middle edge of a longest path in T. Note that every finite tree has either a central vertex or a central edge and that this is always fixed the automorphism group of the tree.

3. Proof of Theorem 1.3

Let G be a quasi-transitive locally finite graph that omits K_∞ as a minor. By Theorem 2.2, there exist $k \in \mathbb{N}$ and a canonical tree-decomposition $(T, \mathcal{V})$ of G of adhesion at most 3 such that the torsos are minors of G and each torso is either planar or has tree-width at most k and such that the stabiliser of each torso acts quasi-transitively on that torso. Furthermore, the edge-separations of $(T, \mathcal{V})$ are tight. Thus, there are only finitely many orbits of them by Lemma 2.1 and hence there are only finitely many $\mathrm{Aut}(G)$-orbits on $V(T)$.

We distinguish three types of torsos (finite torsos, infinite torsos of tree-width at most k and infinite planar torsos) and prepare them for our final quasi-isometry: we find for each torso of the first two kinds quasi-isometries to planar quasi-transitive locally finite graphs and, in the last situation, we have to prepare them such that separations of order 3 whose separator is also an adhesion set in $(T, \mathcal{V})$ does not leave three distinct components. We do this by adding additional separators of size 1.

If there are finite torsos, then there is an upper bound B_1 on the number of vertices in each such torso as there are only finitely many $\mathrm{Aut}(G)$-orbits on $V(T)$. Thus, each of those torsos is $(1, B_1)$-quasi-isometric to a single vertex.

Let us now consider an infinite torso H_t of tree-width at most k. Since it is locally finite, H_t has a canonical tree-decomposition of finite width. Again, since there are only finitely many $\mathrm{Aut}(G)$-orbits on $V(T)$, there exists an upper bound B_2 on the width of the canonical tree-decompositions of such torsos. Let $(T_t, \mathcal{V}_t)$ be a canonical tree-decomposition of H_t of width at most B_2 distinguishing all ends and such that all of its edge-separations are tight, which exists by Theorem 2.4. Since there are only finitely many orbits on $V(T_t)$ under the stabiliser of H_t by

[1] An end ω *lies* in a part V_t if some ray $R \in \omega$ meets V_t infinitely often.

the same argument that we have only finitely many $\mathrm{Aut}(G)$-orbits on $V(T)$, there exists an upper bound B_3 on the diameter of the parts of $(T_t, \mathcal{V}_t)$ and an upper bound B_4 on the number of parts that contain a vertex v. Again, we may assume that these bounds B_3 and B_4 hold for all torsos of this type, i. e. all infinite torsos of tree-width at most k. Thus, any map that maps each vertex u of H_t to some $s \in V(T_t)$ such that u lies in the part of s is a $(1, B_3B_4)$-quasi-isometry from H_t to T_t. Since every adhesion set S of $(T, \mathcal{V})$ in H_t is a clique and thus must lie in some common part of $(T_t, \mathcal{V}_t)$, there exists a non-empty subtree T_t^S of T_t all of whose parts contain S. As all edge-separations of $(T_t, \mathcal{V}_t)$ are tight, Lemma 2.1 implies that every T_t^S is finite. So it has a central vertex v_S or a central edge e_S.

For every infinite planar torso H_t and every adhesion set S in H_t of size 3, there are at most two components C of $H_t - S$ with $N(C) = S$, since H_t is planar and thus does not contain $K_{3,3}$ as a minor. Let $(T_t, \mathcal{V}_t)$ be a canonical tree-decomposition of adhesion at most 3 distinguishing all 3-distinguishable ends of H_t such that all of its edge-separations are tight. This exists by Theorem 2.3. We contract all edges whose edge-separations do not have one of the adhesion sets of size 3 from $(T, \mathcal{V})$ as separator and join their parts. Thereby, we obtain a tree-decomposition $(T'_t, \mathcal{V}'_t)$ that has as adhesion sets only adhesion sets of size 3 that are also adhesion sets in $(T, \mathcal{V})$. Note that the torsos are the subgraphs of H_t induced by the parts. Let G_s be a torso of $(T'_t, \mathcal{V}'_t)$. If there is an adhesion set $S \subseteq V(G_s)$ of $(T, \mathcal{V})$ that is not an adhesion set in $(T'_t, \mathcal{V}'_t)$, then $G_s - S$ has a unique infinite component that is completely attached to S, i. e. has all vertices from S in its neighbourhood, and perhaps one finite component. We delete that finite one. By doing this for all choices of S, we obtain a new graph G'_s. As there are only finitely many orbits on the adhesion sets in $(T, \mathcal{V})$, there exists B_5 such that G_s is $(1, B_5)$-quasi-isometric to G'_s for all choices of G_s.

Now we are ready to define the graph H that will be quasi-transitive, locally finite, planar and quasi-isometric to G. For that, we take the disjoint union H' of the following graphs:

(i) one vertex x_S for every adhesion set S in $(T, \mathcal{V})$;
(ii) one vertex x_t for every finite torso H_t of $(T, \mathcal{V})$;
(iii) one copy of the decomposition tree T_t for every infinite torso of tree-width at most k and
(iv) the disjoint union of all graphs G_s obtained from torsos G'_s in the tree-decomposition $(T'_t, \mathcal{V}'_t)$ of the infinite planar torsos of $(T, \mathcal{V})$ that do not have tree-width at most k.

In order to form the graph H, we add some edges to H':

(v) an edge x_Sx_t for all adhesion sets S and finite torsos H_t with $S \subseteq V_t$;
(vi) an edge x_Sv_S or two edges from x_S to the vertices incident with e_S for all adhesion sets S and infinite torsos of tree-width at most k that contain S and
(vii) edges from all $s \in S$ to x_S for all adhesion sets S in $(T, \mathcal{V})$ and the graphs G_s that contain S.

The resulting graph is denoted by H. By construction, G is connected and $(1, B)$-quasi-isometric to H, where B is the maximum of B_1, B_3B_4 and B_5. Since we made no choices during the construction of H that were not invariant under the automorphisms, the automorphism group of G acts on H. By the choices during the construction, the stabiliser of each torso of $(T, \mathcal{V})$ still acts quasi-transitively on the graph that replaces this torso and as a result, H is a quasi-transitive graph.

Obviously, it is locally finite. Since all components in H' are planar and since the vertices x_S are 1-separators and attached to either at most two adjacent vertices in a component of H' or to all vertices from the adhesion set S of $(T, \mathcal{V})$ whose removal from each component of H' leave exactly one component with all of S in its neighbourhood, we obtain that H is planar, too. This finishes the proof of Theorem 1.3. □

Acknowledgement

I thank Ugo Giocanti for comments on an earlier version of this paper.

References

[1] J. Carmesin, M. Hamann, and B. Miraftab. Canonical trees of tree-decomposition. *J. Combin. Theory (Series B)*, 152:1–26, 2022. 2

[2] L. Esperet and U. Giocanti. Coarse geometry of quasi-transitive graphs beyond planarity. arXiv:2312.08902. 1, 1.1, 1

[3] L. Esperet, U. Giocanti, and C. Legrand-Duchesne. The structure of quasi-transitive graphs avoiding a minor with applications to the domino problem. arXiv:2304.01823. 2, 2.2, 2

[4] A. Georgakopoulos and P. Papasoglu. Graph minors and metric spaces. arXiv:2305.07456. 1, 1.2

[5] R. Halin. Über die Maximalzahl fremder unendlicher Wege. *Math. Nachr.*, 30:63–85, 1965. 2

[6] M. Hamann, F. Lehner, B. Miraftab, and T. Rühmann. A Stallings type theorem for quasi-transitive graphs. *J. Combin. Theory (Series B)*, 157:40–69, 2022. 2

[7] J. MacManus. Accessibility, planar graphs and quasi-isometries. arXiv:2310.15242. 1, 1.5, 1

[8] C. Thomassen. The Hadwiger number of infinite vertex-transitive graphs. *Combinatorica*, 12:481–491, 1992. 2

[9] C. Thomassen and W. Woess. Vertex-transitive graphs and accessibility. *J. Combin. Theory (Series B)*, 58(2):248–268, 1993. 2, 2.1

Matthias Hamann, Department of Mathematics, University of Hamburg, Hamburg, Germany