A semi-implicit SPH scheme for the two-dimensional shallow water equations

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***Abstract*—This work focuses on the extension of the semi-implicit SPH scheme [2] to two dimensions for the shallow water equations. The scheme in [2] was first presented at last year's SPHERIC workshop, following the semi-implicit finite volume and finite difference approach of Casulli [1]. In standard explicit numerical methods, there is often a severe limitation on the time step due to the stability restriction imposed by the CFL condition. To this end, a new semi-implicit SPH scheme is derived, which leads to an unconditionally stable method. The discrete momentum equation is substituted into the discrete continuity equation to obtain a symmetric positive definite linear system for the free surface elevation. The resulting system can be easily solved by a matrix-free conjugate gradient method. Once the new free surface location is known, the velocity at the new time level can be directly computed and the particle positions can subsequently be updated. The method is validated on a smooth inviscid hydrostatic free surface flow for the two dimensional shallow water equations.**

I. Introduction

This paper focuses on the extension to two dimensions of the semi-implicit SPH scheme applied to the shallow water equations [2]. We consider two-dimensional inviscid hydrostatic free surface flows. These flows are governed by the *shallow water equations* which can be derived from the three dimensional incompressible Navier-Stokes equations with the assumption of a hydrostatic pressure distribution (see [4], [13]).

A considerable amount of work has been done for both structured and unstructured meshes using finite difference, finite volume and finite element schemes ([4], [13], [19], [20], [21]). A major problem of explicit schemes in numerical methods is their severe time step restriction, where the Courant-Friedrichs-Lewy (CFL) condition imposes the time step size in terms of the wave propagation speed and the mesh size. Hence, the major advantage of a semi-implicit approach is that stable schemes are obtained which allow large time step sizes at a reasonable computational cost. In a staggered mesh approach for finite differences and volumes, discrete variables are often defined at different (staggered) locations. The pressure term, which is the free surface elevation, is defined in the cell center, while the velocity components are defined at the cell interfaces. In the momentum equation, pressure terms that are due to the gradients in the free surface elevations and the velocity in the mass conservation are both discretized implicitly, whereas the nonlinear convective terms are discretized explicitly. For mesh-based schemes, the semi-Lagrangian method is one of the techniques to discretize these terms explicitly (see [14], [15], [16]).

In recent years, some authors have worked on a semi-implicit method for particle methods. In the specific, Koshizuka and Oka ([22], [29]) presented the moving-particle semi-implicit method (MPS). Ataie-Ashtiani and Farhadi ([26], [27]) worked in the same direction and presented a stable MPS method for free surface flows using a fractional step idea of discretization to split the time step into two steps. A number of authors modified, extended and improved on the MPS method of Koshika and Oka (see [23]–[25], [28], [30]), even more for the enhancement of performance, stability and accuracy of the MPS method.

In this paper the new semi-implicit *Smoothed Particle Hydrodynamics* (SPH) scheme presented [2] for the numerical solution of the one dimensional shallow water equations at the 2014 SPHERIC workshop will be extended to the shallow water equations in two space dimensions. The method is proposed, derived and discussed. The flow variables in this present study are the particle free surface elevation, particle total water depth and the particle velocity. The discrete momentum equations are substituted into the discretized mass conservation equation to give a discrete equation for the free surface leading to a system in only one single scalar quantity, the free surface elevation location. Solving for one scalar quantity in a single equation distinguishes our method in terms of efficiency from other methods. The system is solved for each time step as a linear algebraic system. The components of the momentum equation at the new time level can be directly computed from the new free surface. This can be conveniently solved by a matrix-free version of the conjugate gradient (CG) algorithm [5]. Consequently, the particle velocities at the new time level are computed and the particle positions are updated. In this semi-implicit SPH method, the stability is independent of the wave celerity. Hence, relatively large time steps can be permitted to enhance the numerical efficiency [4].

The remainder of this paper is structured as follows: In Section II, the numerical models for the two-dimensional shallow water equations and models used for the particle approximations are presented. In Section III, the key ideas of smoothed particle hydrodynamics (SPH) and the proposed semi-implicit SPH scheme are presented and derived. Two dimensional numerical results to validate the scheme are presented in Section IV. Section V provides concluding remarks along with an outlook to future research.

II. Numerical Model

This section details the computational models and their accompanying particle approximations. Vectors are defined by reference to Cartesian coordinates. The latin subscript is used to identify particle locations, where subscripts i denote the focal particle whereas the subscript j denotes the neighbor of particle i. Einstein's summation will be employed for repeated superscripts.

A. The Kernel Function

We shall use a regular mollifying function W which is a positive non-increasing, axially symmetric shaped function with compact support of the generic form

$$W(r,h) = \frac{1}{h^d} W\left(\frac{\|r\|}{h}\right). \tag{1}$$

In the specific, the classical B-spline kernel function of degree 3 is used in this study [7], given as

$$W(r,h) = W_{ij} = K \times \begin{cases} 1 - \frac{3}{2}\left(\frac{r}{h}\right)^2 + \frac{3}{4}\left(\frac{r}{h}\right)^3 & 0 \le \frac{r}{h} \le 1, \\ \frac{1}{4}\left(2 - \frac{r}{h}\right)^3 & 1 \le \frac{r}{h} \le 2, \\ 0 & \frac{r}{h} > 2, \end{cases} \tag{2}$$

where the normalisation coefficient K takes the value $\frac{2}{3}$, $\frac{10}{7\pi}$, $\frac{1}{\pi}$ according to the dimension of the space for $(d = 1, 2, \text{or } 3)$, respectively. In the function $W \in W^{3,\infty}(\mathbb{R}^d)$, h is the *smoothing length* which is related to the particle spacing Δ_P by the relation $h = 2\Delta_P$ for constant $h > 0$. The smoothing length h can vary locally according to the relations:

$$h_{ij} = \frac{1}{2}[h_i + h_j] \quad \text{where} \quad h_i = \sigma \sqrt[d]{\frac{m_j}{\rho_j}}. \tag{3}$$

In this study, the smoothing length relation in (3) is used, σ is taken from $[1.5, 2.0]$ which ensures approximately a constant number of neighbors of between $40 - 50$ in the compact support of each kernel. A popular and efficient approach based on the Shepard interpolation technique [3]

$$W'_{ij} = \frac{W_{ij}}{\sum_{j=1}^{N} \frac{m_j}{\rho_j} W_{ij}}$$

is used for the kernel function normalisation, which is especially useful for particles close to free surfaces. This technique remedies problems such as numerical instabilities, partition of unity which affect the convergence of this method.

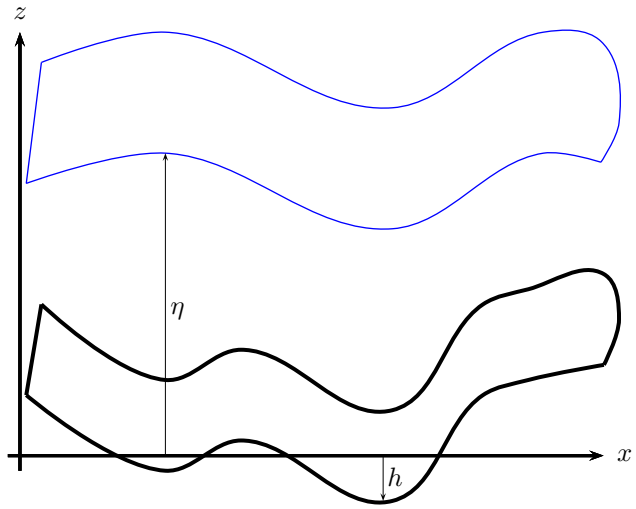

Fig. 1. Sketch of the free surface (light blue) and the bottom bathymetry (thick black)

The gradient of the kernel function is corrected using the formulation proposed by Belytschko et al. [17]. Hereafter by notation, the kernel function W'_{ij} and its gradient $\nabla W'_{ij}$ will be taken as W_{ij} and ∇W_{ij}, respectively.

B. Governing Equations

The governing equations considered in this paper can be written as a nonlinear hyperbolic conservation law of the form

$$L_b(\mathbf{\Phi}) + \nabla \cdot (\boldsymbol{F}(\mathbf{\Phi}, \boldsymbol{x}, t)) = 0, \qquad t \in \mathbb{R}^+, \mathbf{\Phi} \in \mathbb{R}, \tag{4}$$

together with the initial condition

$$\mathbf{\Phi}(\boldsymbol{x}, 0) = \mathbf{\Phi}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \subset \mathbb{R}^d, \mathbf{\Phi}_0 \in \mathbb{R}, \tag{5}$$

where L_b is the transport operator given by

$$L_b(\mathbf{\Phi}) = \frac{\partial \mathbf{\Phi}}{\partial t} + \nabla \cdot ((\boldsymbol{b}\mathbf{\Phi}))$$

and

$$\boldsymbol{x} = (x^1, ..., x^d), \boldsymbol{F} = (F^1, ..., F^d), \boldsymbol{b} = (b^1, ..., b^d),$$

where $\boldsymbol{b}$ is a regular vector field in $\mathbb{R}^d$, $\boldsymbol{F}$ is a flux vector in $\mathbb{R}^d$, and $\boldsymbol{x}$ is the position.

Fig. 1 depicts the sketch of the flow domain, i.e., the free surface elevation and the bottom bathymetry in the present study. In this configuration, the vertical variation is much smaller when compared to the horizontal variation, typical of rivers flowing over long distances, typically hundreds or thousands of kilometers. We consider the frictionless, inviscid two dimensional shallow water equations in Lagrangian derivatives given as

$$\frac{D\eta}{Dt} + \nabla \cdot (H\boldsymbol{v}) = 0, \tag{6}$$

$$\frac{D\boldsymbol{v}}{Dt} + g\nabla\eta = 0, \tag{7}$$

$$\frac{D\boldsymbol{r}}{Dt} = \boldsymbol{v}, \tag{8}$$

where $\eta = \eta(x, y, t)$ is the free surface location, and $H = H(x, y, t)$ denotes the total water depth which is given as

$$H(x, y, t) = h(x, y) + \eta(x, y, t), \tag{9}$$

where $h(x, y)$ denotes the bottom bathymetry, $\boldsymbol{v} = v(x, y, t)$ denotes the particle velocity, $\boldsymbol{r} = r(x, y, t)$ denotes the particle position, and g denotes the constant of gravity acceleration.

III. Numerical Method

There are several numerical methods that can be employed to solve equations (6) - (7). These methods can be finite differences or finite elements, explicit or implicit, conservative or non-conservative or meshless methods. In this section, following the semi-implicit finite volume and finite difference approach of Casulli [1], we will delve into the derivation of the semi-implicit SPH scheme applied to the two dimensional shallow water equations.

In standard explicit numerical methods, there is a severe limitation due to the stability restriction imposed by the CFL condition. The restriction requires a much smaller time step size than permitted by accuracy considerations. Fully implicit discretization often leads to unconditionally stable methods but they typically lead to the simultaneous solution of a large number of coupled nonlinear equations. For accuracy, the time step cannot be chosen arbitrarily large. To this effect, a stable, efficient, robust and simple semi-implicit SPH numerical method is derived in this section.

A. Classical SPH formulation

The standard SPH formulation discretizes the computational domain $\Omega(t)$ by a finite set of N particles, with positions $\boldsymbol{r}_i$. According to Gingold and Monaghan [10], the SPH discretization of the shallow water equations (6) - (7) reads:

$$\frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} + \sum_{j=1}^{N} \frac{m_j}{\rho_j} H_{ij} \boldsymbol{v}_j \nabla W_{ij} = \boldsymbol{0}, \tag{10}$$

$$\frac{\boldsymbol{v}_i^{n+1} - \boldsymbol{v}_i^n}{\Delta t} + g \sum_{j=1}^{N} \frac{m_j}{\rho_j} \eta_j \nabla W_{ij} = \boldsymbol{0}, \tag{11}$$

$$\frac{D\boldsymbol{r}_i}{Dt} = \boldsymbol{v}_i, \tag{12}$$

and the particles are moved by (12), where Δt is the time step, m_j denotes the particle mass, ρ_j denotes the particle density, and ∇W_{ij} is the gradient of the interpolation kernel W_{ij} with respect to x_i. In the scheme [10] by Gingold and Monaghan, $\nabla \cdot (H\boldsymbol{v})$, $\nabla \eta$ are explicitly computed.

The gradient formulation used in (10) - (11) follows by substituting the flow variable with corresponding derivatives, using integration by parts, the divergence theorem and some elementary transformations.

B. SPH formulation of Vila and Ben Moussa

Towards the derivation of our semi-implicit SPH scheme, the SPH formalism of Vila and Ben Moussa ([6], [9]) is used. The basic idea in Vila and Ben Moussa in the scheme comprises of replacing a centered approximation

$$(F(v_i, x_i, t) + F(v_j, x_j, t)) \cdot n_{ij}$$

of (4) by a numerical flux of a finite difference scheme in conservation form $2G(n_{ij}, v_i, v_j)$ which should satisfy the equations

$$G(n(x), v, v) = F(v, x, t) \cdot n(x)$$

$$G(n, v, u) = -G(-n, u, v).$$

With this formalism, the SPH discretization reads

$$\frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} + \sum_{j=1}^{N} \frac{m_j}{\rho_j} 2H_{ij} \boldsymbol{v}_{ij} \nabla W_{ij} = \boldsymbol{0}, \tag{13}$$

$$\frac{\boldsymbol{v}_i^{n+1} - \boldsymbol{v}_i^n}{\Delta t} + g \sum_{j=1}^{N} \frac{m_j}{\rho_j} 2\eta_{ij} \nabla W_{ij} = \boldsymbol{0}, \tag{14}$$

$$\frac{D\boldsymbol{r}_i}{Dt} = \boldsymbol{v}_i. \tag{15}$$

In this formalism, in Fig. 2, for a pair of particles i and j, we define the free surface elevation $\boldsymbol{\eta}_i$, $\boldsymbol{\eta}_j$ and velocity $\boldsymbol{v}_i$, $\boldsymbol{v}_j$ at each particle i and j, respectively. In our approach, we artificially define a staggered like velocity $\boldsymbol{v}_{ij}$ between two interacting particles i and j as

$$\boldsymbol{v}_{ij} = \frac{1}{2}(\boldsymbol{v}_i + \boldsymbol{v}_j) \cdot \boldsymbol{n}_{ij} \tag{16}$$

in the normal direction $\boldsymbol{n}_{ij}^{d=1,2}$ at the midpoint of the two interacting particles, where n_{ij} is a vector given as

$$n_{ij}^1 = \frac{x_j - x_i}{\|x_j - x_i\|} \qquad \text{and} \qquad n_{ij}^2 = \frac{y_j - y_i}{\|y_j - y_i\|}$$

in both x and y directions, respectively where we have written

$$\delta_{ij}^1 = \|x_j - x_i\| \qquad \text{and} \qquad \delta_{ij}^2 = \|y_j - y_i\|$$

denoting the distance between a pair of particles i and j and d is the spatial dimension. Since we know the value of the velocities at the midpoint of the particles, we use kernel summation to update the velocity at the next location.

C. Semi-implicit SPH Scheme

To start with the derivation of the semi-implicit SPH scheme, let us consider some characteristic analysis of the governing equations (6) - (7). Writing equations (6) - (7) in a non conservative quasi-linear form by expanding derivatives in the continuity equation and momentum equations (assuming smooth solutions), we obtain

$$u_t + uu_x + vu_y + g\eta_x = 0, \tag{17}$$

$$v_t + uv_x + vv_y + g\eta_y = 0, \tag{18}$$

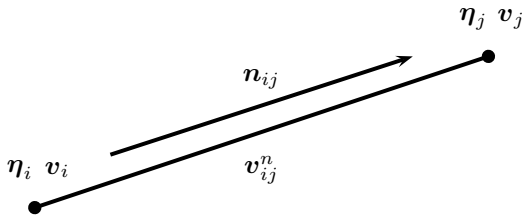

Fig. 2. Staggered velocity defined at the midpoint of a pair of interacting particles i and j.

$$\eta_t + u\eta_x + v\eta_y + H(u_x + v_y) = -uh_x - vh_y. \quad (19)$$

Writing (17) - (19) in matrix form, we obtain

$$\mathbf{Q}_t + \mathbf{A}\mathbf{Q}_x + \mathbf{B}\mathbf{Q}_y = \mathbf{C} \quad (20)$$

where

$$\mathbf{A} = \begin{pmatrix} u & 0 & \boxed{g} \\ 0 & u & 0 \\ \boxed{H} & 0 & u \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} v & 0 & 0 \\ 0 & v & \boxed{g} \\ 0 & \boxed{H} & v \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ -uh_x - vh_y \end{pmatrix}.$$

Equation (20) is a strictly hyperbolic system with real and distinct eigenvalues. The characteristic equation is given by

$$det(q\mathbf{I} + r\mathbf{A} + s\mathbf{B}) = 0. \quad (21)$$

The characteristic equation is simplified as

$$(q + ru + sv)\left[(q + ru + sv)^2 - gH(r^2 + s^2)\right] = 0, \quad (22)$$

where the solution (r, s, q) of equation (22) are the directions normal to the characteristic cone at the cone's vertex. We split equation (22) and we obtain

$$q + ru + sv = 0 \quad (23)$$

and

$$(q + ru + sv)^2 - gH(r^2 + s^2) = 0, \quad (24)$$

where $\dfrac{dx}{dt} = u$, $\dfrac{dy}{dt} = v$ are the characteristic curves. if the characteristic cone has a vertex at the point $(\overline{x}, \overline{y}, \overline{t})$, then this cone consist of the line passing through vertex $(\overline{x}, \overline{y}, \overline{t})$ and parallel to the vector $(u, v, 1)$ which satisfies the equation

$$((x-\overline{x}) - u(t-\overline{t}))^2 + ((y-\overline{y}) - v(t-\overline{t}))^2 - gH(t-\overline{t})^2 = 0. \quad (25)$$

In particlar, the gradient of the left hand side of equation (25) satisfies equation (24) on the cone surface. After solving (21), the solution yields

$$\lambda_{1,2} = \boldsymbol{v} \pm \sqrt{gH}.$$

When the particle velocity v is far smaller than the particle celerity $\sqrt{gH}$, i.e., $|\boldsymbol{v}| \ll \sqrt{gH}$, the particle flow is said to be strictly subcritical and thus the characteristic speeds λ_1 and λ_2 have opposite directions. The maximum wave speed is given as

$$\lambda_{\max} = \max(\sqrt{gH_i}, \sqrt{gH_j}).$$

In this case, $\sqrt{gH}$ represents the dominant term which originates from the off diagonal terms g and H in the matrix **A** and **B**.

We now have tracked back where the term $\sqrt{gH}$ originates from in the governing equations. We wish to further point out that the first part of the characteristic cone in equation (22) depends only on the particle velocity u and v. Equation (24) defines the second part depends only on the celerity $\sqrt{gH}$. As we can see, gH in Equation (22) comes from the off-diagonal terms g and H in the matrices **A** and **B**. The terms g and H represent the coefficient of the derivative of the free surface elevation η_x in equation (17), the coefficient of the derivative η_y in equation (18) in the momentum equations and the coefficient of velocity u_x and v_y in the volume conservation equation (19). Since we do not want the stability of this method to be dependent on the celerity $\sqrt{gH}$, we discretize the derivatives η_x, η_y and u_x, v_y implicitly.

Following the characteristic analysis presented above, we want to derive the semi-implicit SPH scheme for the two dimensional shallow water equations. The derivatives of the free surface elevation η_x and η_y in the momentum equation and the derivative of the velocity in the continuity equation are discretized implicitly. The remaining terms such as the nonlinear advective terms in the momentum equation are discretized explicitly so that the system to be solved eventually will be linear.

Let us consider the continuity equation in the original conservative form given as

$$\eta_t^n + \nabla \cdot (H^n \boldsymbol{v}^{n+1}) = 0. \quad (26)$$

The velocity **v** will be discretized implicitly, the total water depth H is discretized explicitly. For the sake of notation, by implicitly and explicitly we mean $n+1$ and n in the superscript, respectively:

$$\boldsymbol{v}_t^n + g \cdot \nabla \eta^{n+1} = 0$$

$$\eta_t^n + \nabla \cdot (H^n \boldsymbol{v}^{n+1}) = 0.$$

Furthermore, we have discretized the particle velocities and free surface elevation in time by the theta method for the sake of time accuracy and computational efficiency i.e $n+1 = n+\Theta$. So we have

$$\boldsymbol{v}_t^n + g \cdot \nabla \eta^{n+\Theta} = 0 \quad (27)$$

$$\eta_t^n + \nabla \cdot (H^n \boldsymbol{v}^{n+\Theta}) = 0, \quad (28)$$

where the theta method notation reads:

$$\eta^{n+\Theta} = \Theta \eta^{n+1} + (1-\Theta)\eta^n$$

$$\boldsymbol{v}^{n+\Theta} = \Theta \boldsymbol{v}^{n+1} + (1-\Theta)\boldsymbol{v}^n.$$

The factor Θ is called the implicitness factor which should be taken from $\left[\frac{1}{2}, 1\right]$ see Casulli and Cattani [4] for details.

The general semi-implicit SPH discretization of (27) - (28) assumes the form

$$\begin{aligned}\frac{\boldsymbol{v}_{ij}^{n+1} - \boldsymbol{F}\boldsymbol{v}_{ij}^{n}}{\Delta t} &+ \frac{g}{\delta_{ij}}\Theta(\eta_j^{n+1} - \eta_i^{n+1}) \\ &+ \frac{g}{\delta_{ij}}(1-\Theta)(\eta_j^{n} - \eta_i^{n}) \\ &= 0,\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} &+ \Theta \sum_{j=1}^{N} \frac{m_j}{\rho_j}(2H_{ij}^n \boldsymbol{v}_{ij}^{n+1})\nabla \boldsymbol{W}_{ij} \cdot \boldsymbol{n}_{ij} \\ &+ (1-\Theta)\sum_{j=1}^{N} \frac{m_j}{\rho_j}(2H_{ij}^n \boldsymbol{v}_{ij}^{n})\nabla \boldsymbol{W}_{ij} \cdot \boldsymbol{n}_{ij} \\ &= 0,\end{aligned} \tag{30}$$

where

$$H_{ij}^n = \max(0, h_{ij}^n + \eta_i^n, h_{ij}^n + \eta_j^n). \tag{31}$$

In a Lagrangian formalism, the explicit operator $\boldsymbol{F}\boldsymbol{v}_{ij}^n$ takes the simple form in (29)

$$\mathbf{F}\mathbf{v}_{ij}^n = \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_j), \tag{32}$$

where $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ denotes the velocity of particles i and j at time t^n. The new velocity is computed through simple kernel summation:

$$\mathbf{v}_i^{n+1} = \mathbf{v}_i^n + \sum_{j=1}^{N} \frac{m_j}{\rho_j}(\mathbf{v}_{ij}^{n+1} - \mathbf{v}_i^n)W_{ij}. \tag{33}$$

We should note that in (29) we have not used the gradient of the kernel function for the discretization of the gradient of η. We rather used a finite difference discretization for the pressure gradient. This increases the accuracy, in (29) $\boldsymbol{F}$ corresponds to an explicit spatial discretization of the advective terms. Since SPH is a Lagrangian scheme, the nonlinear convective term is discretized automatically, using the Lagrangian (material) derivative contained in the particle motion in Eqn. (12). Equation (32) is used to interpolate the particle velocities from the particle location to the staggered velocity location.

D. The Free Surface Equation

Let the particle volume ω_i in (30) be given as $\omega_i = \frac{m_i}{\rho_i}$. Irrespective of the form imposed on $\boldsymbol{F}$, equations (29) - (30) constitute a linear system of equations with unknowns $\boldsymbol{v}_i^{n+1}$ and η_i^{n+1} over the entire particle configuration. We solve this system at each time step for the particle variables from the prescribed initial and boundary conditions. The cardinal feature of this present numerical method from the computational point of view is that the discrete momentum equation is substituted in the discrete continuity equation. The model is reduced into a smaller model in η_i^{n+1} as the only unknown.

Multiplying (30) by ω_i and inserting (29) into (30) we obtain

$$\begin{aligned}\omega_i\eta_i^{n+1} - g\Theta^2\frac{\Delta t^2}{\delta_{ij}}\sum_{j=1}^{N} 2\omega_i\omega_j\left[H_{ij}^n(\eta_j^{n+1} - \eta_i^{n+1})\nabla \boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij}\right] \\ = \boldsymbol{b}_i^n,\end{aligned} \tag{34}$$

where the right hand side $\boldsymbol{b}_i^n$ represents the known values at time level t^n given as

$$\begin{aligned}\boldsymbol{b}_i^n &= \omega_i\eta_i^n - \Delta t\sum_{j=1}^{N} 2\omega_i\omega_j H_{ij}^n \boldsymbol{F}\boldsymbol{v}_{ij}^{n+\Theta}\nabla \boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij} \\ &+ g\Theta(1-\Theta)\frac{\Delta t^2}{\delta_{ij}}\sum_{j=1}^{N} 2\omega_i\omega_j\left[H_{ij}^n(\eta_j^n - \eta_i^n)\nabla \boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij}\right],\end{aligned} \tag{35}$$

where $\boldsymbol{F}\boldsymbol{v}_{ij}^{n+\Theta} = \Theta\boldsymbol{F}\boldsymbol{v}_{ij}^n + (1-\Theta)\boldsymbol{v}_{ij}^n$. Since H_{ij}^n, ω_i, ω_j are non-negative numbers, equations (34) - (35) constitute a linear system of N equations for η_i^{n+1} unknowns.

The resulting system is symmetric and positive definite (SPD). Because of the SPD property, this system admits a unique solution which can be efficiently obtained by an iterative method. We obtain the new free surface location by (34), equation (29) gives readily and uniquely the new particle velocity $\boldsymbol{v}_i^{n+1}$.

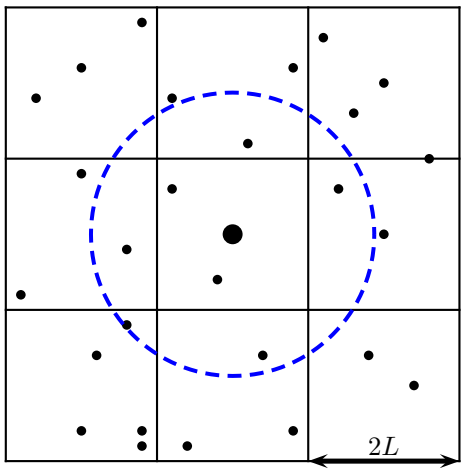

Fig. 3. Fictitious Cartesian Grid: Neigboring search is done within the 9 cells in a two-dimensional space. The smoothing length is constant and the support domain for the particles is $2L$

E. Neighboring Search Technique

The search for surrounding particles j of the focal particle i at the position $\boldsymbol{x}_i$ is a major challenge that must be solved efficiently, hence we have used the following strategy. We define a background fictitious Cartesian grid in Fig. 3. This grid contains the fluid with a mesh size of $2L$, the grid is kept fixed all through the simulation. The grid comprises macrocells which consist of particles, (see details in [33].) The idea is analogous to the book-keeping cells as used by Monaghan in

[31]. To compute the free surface elevation η and the fluid velocity $\boldsymbol{v}$, only particles inside the same macro-cell or in the immediate surrounding macro-cells will contribute. Ferarri et al [32] explained the efficiency of the neigboring search. The idea is building the list of particles in a given macro cell and also the indices pointing to macro-cell containing the particle. We store the coordinates of each particle to reduce the time for accessing data in the neighbor search technique. So in our neighboring search, a particle can only interact with particles in its macro-cell or in the neigboring macro-cells. In our two dimensional case in the present study then we will loop over the bounding box of 9 macro-cells, as we can see in Fig. 3.

IV. Numerical Example

In this section, the semi-implicit SPH scheme that has been derived in Section III will be validated on a simple smooth test problem for the two-dimensional shallow water equations. In this section, an academic numerical example will be validated that has a smooth solution, i.e., a collapsing Gaussian bump. In the subsequent test problem, the acceleration due to gravity constant g is set to $g = 9.81$.

A. Smooth Surface Wave Propagation

In this example, we consider a smooth free surface wave propagation. We consider the initial value problem

$$\eta(x,y,0) = 1 + 0.1e^{-\frac{1}{2}\left(\frac{r^2}{\sigma^2}\right)},$$
$$u(x,y,0) = v(x,y,0) = h(x,y) = 0,$$

in the domain $\Omega = [-1,1] \times [-1,1]$ with a prescribed flat bottom bathymetry, i.e., $h(x,y) = 0$, where $\sigma = 0.1$ and $r^2 = x^2 + y^2$. The computational domain Ω is discretized with $124,980$ particles. The final simulation time $t = 0.15$ is used and the time step is chosen to be $\Delta t = 0.0015$. We have used the implicitness factor $\Theta = 0.65$. The smoothing length is taken as $l_i = \alpha(\omega_i)^{\frac{1}{d}}$, where $\alpha = [1.5, 2]$ and $d = 2$. The numerical solution is shown in Fig. 5. The profiles in Fig. 4 depicts the three dimensional surface plot of the free surface elevation at times $t = 0s, 0.05s, 0.10s, 0.15s$. Due to the radial symmetry of the problem, we obtain a reference solution by solving the one-dimensional shallow water equations with a geometric source term in radial direction: a method based on the high order classical shock capturing total variation diminishing (TVD) finite volume scheme is employed for computing the reference solution using $5,000$ points and the Osher-type flux for the Riemann solver, see [12] for details. The comparison between our numerical results obtained with semi-implicit SPH scheme and the reference solution is shown. A good agreement between the two solutions is observed in Fig. 5. We attribute the difference in the plot to the fact that the SPH method has a larger effective stencil, which might increase the numerical viscosity, and moreover, since a low order accurate time integration scheme has been used.

The radial cut solutions of the free surface elevation and the velocity in the $x-$ direction is given in Fig. 5

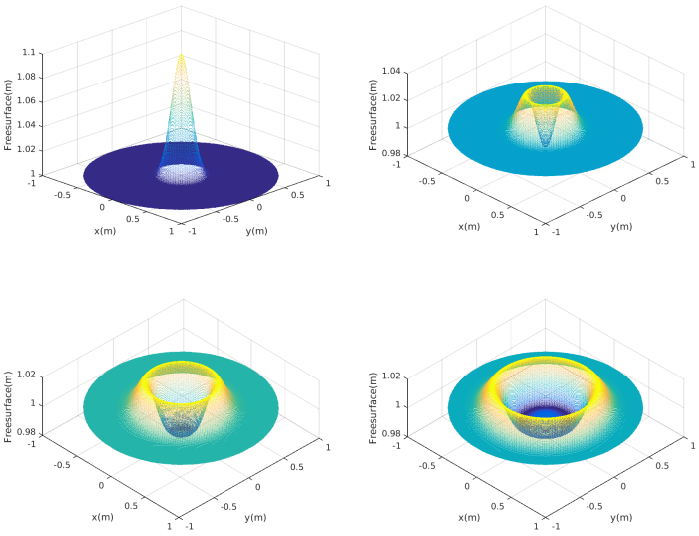

Fig. 4. 3-D surface plot of the free-surface: SISPH solution at times $t = 0.0s, 0.05s, 0.10s, 0.15s$ with $124,980$ particles.

V. Conclusion

The paper presents a new SPH formulation based on a novel semi-implicit SPH discretization. The semi-implicit algorithm applied to the two dimensional shallow water equations has been derived and discussed. The momentum equation is discretized by a finite difference approximation for the gradient of the free surface and SPH appoximation for the mass conservation equation.

Because we substituted the discrete momentum equations into the discrete mass conservation equations, our scheme reduces to a linear sparse system for the free surface elevation. We therefore have one linear scalar value for the free surface to be solved, we conveniently solve this with the matrix-free version of the conjugate gradient (CG) algorithm.

This method possesses some key features: the method is mass conservative, the time step is not restricted by the stability condition that is dictated by the surface wave speed thus relatively large timesteps are permitted.

Future research will be devoted to the extension of this scheme to 3D numerical examples, wetting and drying problems, application to shock problems and extension to nonhydrostatic free surface flows.

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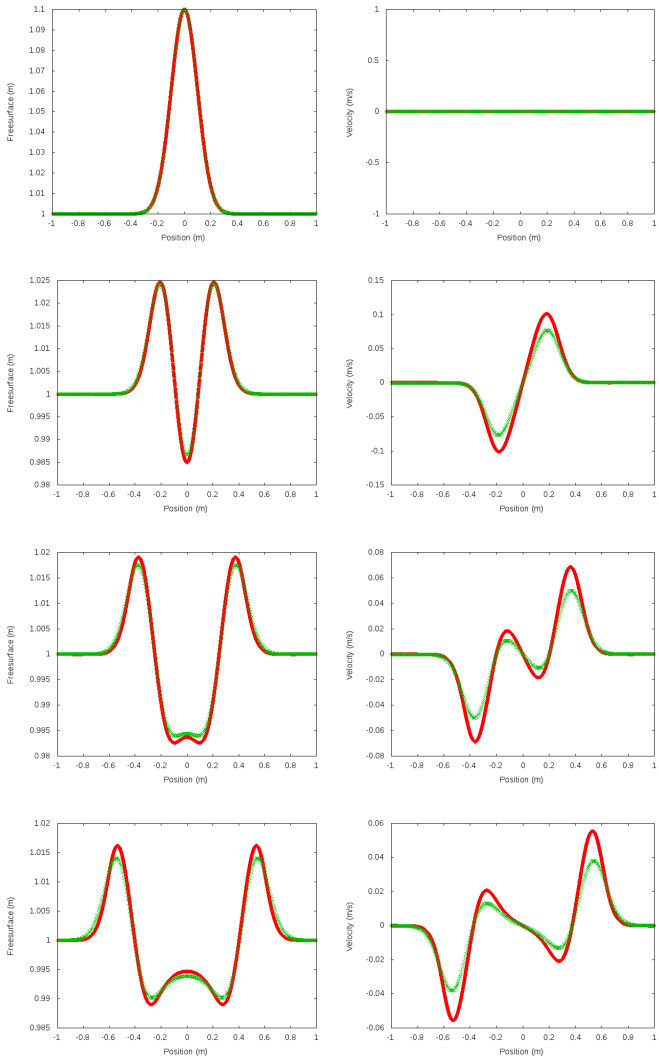

Fig. 5. The radial cut of semi-implicit solution (green) versus reference solution (red): Free-surface(left), velocity(right) in the $x-$ direction at times $t = 0.0s, 0.05s, 0.10s, 0.15s$.

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