Coadjoint Orbits and Representations of the Jacobi Group

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The semidirect product of the symplectic group with an adequate Heisenberg group gives the framework for the introduction of the (Segal-Shale-)Weil (or oscillator) representation and as such appears in principle in all texts treating this subject. The name *Jacobi group* G^J came up later, as the book by Eichler and Zagier [EZ] started the systematic study of Jacobi forms. We will stick to this name here, though Kirillov used the name St in [Ki] p. 237 and TSp in [Ki1] p. 466.
M. Vergne says in [Ve] p. 271 that the Weil representation, while having a canonical character, is still a "mysterious" representation without a canonical model. Thus, though the representation theory of the Jacobi group is well understood by Mackey's theory (see Duflo [Du] for the general and [BeS] p. 21 for a special treatment) and though the orbit method has been refined to sophisticated stages (see for instance the articles by Kirillov [Ki1] , Vogan [Vo], or Torasso [To]), we hope it may be not in vain to go back to a very elementary level and discuss the application of the orbit method for the construction of unitary irreducible representations of the Jacobi group.

The diploma thesis of D. Pfaue [Pf], one of the last ones directed by Peter Slodowy before his death, and a preprint as well as a paper by J.-H.Yang [Ya1,2] independently contain approaches to the explicit determination of the adjoint and coadjoint orbits of the Jacobi group G^J. In this text we will report on (parts of) both sources, fill in some gaps, and examine the standard procedure from the books of Kirillov [Ki], Woodhouse [Wo] or the article by Vogan[Vo] and Kirillov [Ki1] trying to construct explicitly representations starting from the coadjoint orbits in this special example. A first aim of this is to get a bit more material for an insight into the question when and how to attach a representation to a nilpotent orbit, as there is a lot of refined work on unipotent and minimal representations (see for instance Vogan [Vo1], Ohta [Oh], Lee [L], and Li [Li]). Another goal, however lying still far away, is to learn more about the correspondence between automorphic representations of the symplectic group Sp(2) and the Jacobi group G^J as its subgroup. This topic has already been investigated by Ralf Schmidt and the autor in [BeS], [BeS1], [Be], and [RS].

1. The Jacobi Group and its Coadjoint Action

The Jacobi group

$$G = G^J := G_1 \ltimes H,\ G_1 = SL(2,\mathbb{R}),\ \ H = \mathrm{Heis}(3)$$

is coordinatized in the following ways. As in [BeS] or [Pf], we write

$$G^J \ni g = (M, v, \kappa)$$

with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}),\ v = (\lambda, \mu) \in \mathbb{R},\ \kappa \in \mathbb{R},$$

and the multiplication law

$$gg' = (MM', vM'+v', \kappa+\kappa'+\omega(vM', v')),\ \omega(w,v) = wJ^tv,\ J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

or, as in [EZ] and [Ya1] for its realization as a subgroup of the symplectic group $G_2 = \mathrm{Sp}(2,\mathbb{R})$

$$g^{-1} = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Starting from these two (different) coordinatizations, we come to the following two descriptions of the coadjoint orbits

A) As in Guillemin–Sternberg [GS] p.124, for

$$g = (M, v, \kappa) \in G^J,\ (m, v_1, \kappa_1) \in \mathfrak{g}^J, (\hat{a}, \hat{b}, \hat{c}) \in (\mathfrak{g}^J)^*$$

we have the coadjoint action of G^J on $(\mathfrak{g}^J)^*$

$$
\begin{aligned}
&< \mathrm{Ad}^{\#}_{(M,v,\kappa)} \ (\hat{a}, \hat{b}, \hat{c}),\ (m, v_1, \kappa_1) > \\
&= \quad < (\hat{a}, \hat{b}, \hat{c}),\ (M, v, \kappa)(m_1, v_1, \kappa_1)(M, v, \kappa)^{-1} > \\
&= \quad < \hat{a}, MmM^{-1} > + < \hat{b}, (vm + v_1)m^{-1} > - < \hat{c},\ \omega(vm + 2v_1, v) - v_1 > .
\end{aligned}
$$

B) As in [BeS] or [Ya1,2], we describe the Lie algebra $\mathfrak{g}^J$ as a subalgebra of $\mathfrak{g}_2 = \mathfrak{sp}(2,\mathbb{R})$ by

$$G(x,y,z,p,q,r) = \begin{pmatrix} x & 0 & y & q \\ p & 0 & q & r \\ z & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and denote

$$X := G(1,0,\dots,0),\dots,R := G(0,\dots,0,1).$$

We get the commutators

$$\begin{array}{lll} [X,Y] = \ 2Y, & [X,Z] = -2Z, & [Y,\ Z] = X, \\ [X,\ P] = -P, & [X,Q] = \ \ Q, & [P,Q] = 2R, \\ [\,Y,P] = -Q, & [\,Z,Q] = -P, & \end{array}$$

all others are zero. As usual, we identify the dual $\mathfrak{sp}^*$ of $\mathfrak{sp}$ by the $\mathrm{Sp}(2,\mathbb{R})$ invariant isomorphism $\eta \mapsto X_\eta$ given by

$$\eta(\check{Y}) = \mathrm{tr}(X_\eta \check{Y}) =:< X_\eta, \check{Y} > \ \text{ for all } \check{Y} \in \mathfrak{sp}$$

$\eta_1,\dots,\eta_6$ denotes a basis of $(\mathfrak{g}^J)^*$ dual to

$$(X_1,\dots,X_6) := (X,\dots,R).$$

Following closely Yang, we realize $(\mathfrak{g}^J)^*$ as a subspace of $\mathfrak{sp}$ by the matrices

$$M(x,y,z,p,q,r) = \begin{pmatrix} x & p & z & 0 \\ 0 & 0 & 0 & 0 \\ y & q & -x & 0 \\ q & r & -p & 0 \end{pmatrix}, x,\dots,r \in \mathbb{R},$$

and put

$$X_* := M(1,0,\dots,0),\dots,R_* := M(0,\dots,0,1).$$

Then

$$X_{\eta_1} := X_*,\dots,X_{\eta_6} := R_*$$

is a basis of $(\mathfrak{g}^J)^*$ with

$$\begin{array}{l} < X_*, X >= 2, \ < Y_*, Y >= 1, \ < Z_*, Z >= 1 \\ < P_*, P >= 2, \ < Q_*, Q >= 2, \ < R_*, R >= 1 \end{array}$$

If g^{-1} is denoted as above, the coadjoint action of $\mathfrak{g}$ on $M(x,\dots,r)$ is given by

$$\mathrm{Ad}^*(g)M(x,\dots,r) = M(\tilde{x},\dots,\tilde{r})$$

with

$$
\begin{aligned}
\tilde{x} &= (ad+bc)x + bdy - acz + (2ac\mu - (ad+bc)\lambda)p \\
&\qquad + ((ad+bc)\mu - 2bd\lambda)q + r(a\mu - b\lambda)(c\mu - d\lambda), \\
\tilde{y} &= 2dcx + d^2y - c^2z + 2(c\mu - d\lambda)(cp+dq) + r(c\mu - d\lambda)^2, \\
\tilde{z} &= -2abx - b^2y + a^2z - 2(a\mu - b\lambda)(ap+bq) - r(a\mu - b\lambda)^2, \qquad (*) \\
\tilde{p} &= ap + bq + r(a\mu - b\lambda), \\
\tilde{q} &= cp + dq + r(c\mu - d\lambda), \\
\tilde{r} &= r.
\end{aligned}
$$

2. The Coadjoint orbits

By a simple calculation, one verifies for $M = M(x, \ldots, r)$ the relation

$$M^4 = (x^2 + yz)M^2 \ .$$

Hence, the nilpotent elements in $(\mathfrak{g}^J)^*$ are exactly those with $x^2 + yz = 0$. In particular each element in $\mathfrak{h}^*$ is nilpotent.

From both descriptions A and B given above in section **1.**, one sees that the coadjoint orbits are fibered over the last "central" variable: $(\hat{a}, \hat{b}, \hat{c}) \in (\mathfrak{g}^J)^*$ and $\mathrm{Ad}^*_{(M,v,\kappa)}(\hat{a}, \hat{b}, \hat{c})$ applied to $(m = 0, v_1 = 0, \kappa) \in \mathfrak{g}^J$ have the same effect.

1. For the **case** $\hat{c}$ **resp.** $r =: m \neq 0$ each orbit contains all $\hat{b}$ resp. $(p, q) \in \mathbb{R}^2$. So the classification of the orbits over $r \neq 0$ is just the usual classification of the coadjoint $\mathrm{SL}(2, \mathbb{R})$–orbits, i.e. we have

1.1 the hyperbolic orbits

$$\mathcal{O}_{mR_* + \alpha X_*}, \ \alpha \in \mathbb{R}_{>0},$$

1.2 the elliptic orbits

$$\mathcal{O}_{mR_* + \alpha(Y_* - Z_*)}, \ \alpha \in \mathbb{R}_{>0},$$

1.3 the nilpotent orbits

$$\mathcal{O}_{mR_* + Y_*}, \ \mathcal{O}_{mR_* + Z_*}, \ \mathcal{O}_{mR_*}.$$

All these orbits have dimension 4, except the last one having dimension 2.

2. For the **case** $\hat{c}$ **resp.** $r = 0$ one falls back into the standard case of classifying coadjoint orbits in semidirect products $G_1 \ltimes V$, where G_1 is a group acting on a vector space V as in [GS] p.124f. Here we have the orbits of $G_1 \ltimes V$ fibered over the G_1–orbits in V^*. And for each such orbit Ω the fibers are classified by the coadjoint orbits of the stabilizer group $(G_1)_\eta$ for an element $\eta \in \Omega$. In our simple case, we have $G_1 = \mathrm{SL}(2,\mathbb{R})$ and $V = \mathbb{R}^2$. So there are exactly two orbits:

2.1 $\Omega = \mathbb{R}^2 \backslash \{0, 0\}$. Here we have for $\eta = {}^t(1, 0)$ the isotropy group

$$(G_1)_\eta = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} =: n(b),\ b \in \mathbb{R} \right\} =: N$$

which has just the trivial coadjoint orbit. Hence, we have one 4–dimensional (nilpotent) coadjoint G^J–orbit which can be recognized as $\mathcal{O}_{P_*}$.

2.2 $\Omega = \{0, 0\}$. Here we have

$$(G_1)_{(0,0)} = \mathrm{SL}(2,\mathbb{R}) = \mathrm{G}_1$$

and we come up again with the classification of the coadjoint $\mathrm{SL}(2,\mathbb{R})$–orbits, i.e. we have the two dimensional orbits

$$\mathcal{O}_{\alpha X_*},\ \mathcal{O}_{\alpha(Y_* - Z_*)}(\alpha > 0),\ \mathcal{O}_{Y_*},\ \mathcal{O}_{Z_*}$$

and the point $0 \in \mathbb{R}^6 \simeq (\mathfrak{g}^J)^*$.

3. Discussion of the Orbits

If we want to compare with the usual descriptions, we sometimes replace the coordinates y, z in $(\mathfrak{g}^J)^*$ by $\check{y}, \check{z}$ with

$$y = \check{y} + \check{z},\ z = \check{y} - \check{z}.$$

All coadjoint orbits $\mathcal{O}_{U_*}$, $U_* \in \mathfrak{g}^J$ are symplectic manifolds with the Kostant–Kirillov form given by

$$B_{U_*}(\hat{X}, \hat{Y}) := < U_*,\ [\hat{X}, \hat{Y}] > .$$

Using the basis fixed in section 1, we have

$$
\begin{aligned}
[\hat{X},\hat{Y}] &= (yz'-y'z)X+2(xy'-x'y)Y-2(xz'-x'z)Z \\
&+(xq'-x'q-yp'+y'p)Q+(px'-p'x+qz'-q'z)P+2(pq'-p'q)R
\end{aligned}
$$

for

$$\hat{X}=xX+\cdots+rR,\ \hat{Y}=x'X+\cdots+r'R.$$

3.1 $U_*=mR_*+\alpha X_*,\ m\neq 0,\ \alpha>0.$

From the equations $(*)$ and the end of section 1, we can deduce (as Yang did in [Ya1,2]) that the orbit $\mathcal{O}_{U_*}$ is given by

$$
\begin{aligned}
x^2+\check{y}^2-(\check{z}^2+\alpha^2) &= (2/m)pqx-(1/m)(p^2-q^2)\check{y}+((1/m)(p^2+q^2)\check{z}, \\
r &= m.
\end{aligned}
$$

The stabilizer group of U_* comes out as

$$G_{U_*}=MA\times C,\ A=\left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix},\ a>0\right\},\ M=\{\pm E\},\ C=\{\kappa\in\mathbb{R}\}$$

(identified with their injections into G^J). The Kostant–Kirillov form is

$$B_{U_*}(\hat{X},\hat{Y})=2\alpha(yz'-y'z)+2m(pq'-p'q)$$

and its kernel

$$\mathfrak{g}_{U_*}=\mathrm{Lie}\ G_{U_*}=<X,R>.$$

3.2 $U_*=mR_*+\alpha(Y_*-Z_*),\ m\neq 0,\alpha>0.$

$\mathcal{O}_{U_*}$ is defined as a subvariety in $\mathbb{R}^6$ by the equations

$$
\begin{aligned}
x^2+\check{y}^2-(\check{z}^2-\alpha^2) &= (2/m)pqx-(1/m)(p^2-q^2)\check{y}+(1/m)(p^2+q^2), \\
r &= m.
\end{aligned}
$$

We have

$$G_{U_*}=K\times C,\ K=\mathrm{SO}(2)$$

(again SO(2) identified with its image in G^J), and the Kostant –Kirillov form

$$B_{U_*}(\hat{X},\hat{Y})=2\alpha(x(z'+y')+x'(z+y))+2m(pq'-p'q)$$

with kernel

$$\mathfrak{g}_{U_*}=<Y-Z,R>.$$

3.3 $U_* = mR_* + Z_*,\ m \neq 0.$

$\mathcal{O}_{U_*}$ is defined by the equations

$$\begin{array}{rcl} x^2 + yz & = & (1/m)(q^2 z - p^2 y + 2pqx) \\ r & = & m. \end{array}$$

We have

$$G_{U_*} = MN \times C,\ N = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} =: n(b),\ b \in \mathbb{R} \right\}$$

(with the identifications as above), and

$$B_{U_*}(\hat{X}, \hat{Y}) = -2(xz' - x'z) + 2m(pq' - p'q),$$

with kernel

$$\mathfrak{g}_{U_*} = < Y, R > .$$

3.4 $U_* = mR_* + Y_*.$

This is analogous to the preceeding case. We have

$$\begin{array}{rcl} G_{U_*} & = & M\overline{N} \times C \qquad \overline{N} = \left\{ \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} =: \overline{n}(c),\ c \in \mathbb{R} \right\}, \\ \mathfrak{g}_{U_*} & = & < Z, R > \end{array}$$

and

$$B_{U_*}(\check{X}, \check{Y}) = 2(xy' - x'y) + 2m(pq' - p'q).$$

3.5 $U_* = mR_*,\ m \neq O$

$\mathcal{O}_{U_*}$ is defined as a variety by the equations

$$mx = pq,\ my = q^2,\ mz = -p^2,\ r = m$$

(so that $x^2 + yz = 0$ holds with $y \geq 0$ for $m \geq 0$).
We have

$$G_{U_*} = \mathrm{SL}(2, \mathbb{R}) \times C,\ \mathfrak{g}_{U_*} = < X, Y, Z, R >$$

and

$$B_{U_*}(\check{X}, \check{Y}) = 2m(pq' - p'q).$$

3.6 $U_* = P_*$

We get a 4–dimensional orbit $\mathcal{O}_{P_*}$ defined in $\mathbb{R}^6$ by the equations

$$2pqx = yp^2 - zq^2,\ r = 0$$

The isotropy group is the identity component of the one in **3.3**

$$G_{U_*} = N \times C \text{ with } \mathfrak{g}_{U_*} = < Y, R >$$

and the Kostant–Kirillov form

$$B_{U_*}(\check{X}, \check{Y}) = 2(px' - p'x + qz' - q'z).$$

3.7 $U_* = \alpha X_*,\ \alpha(Y_* - Z_*)$ (for $\alpha > 0$), $Y_*,\ Z_*$, and 0.

We get exactly the $\mathrm{SL}(2, \mathbb{R})$–theory with two–dimensional orbits having all of them $p = q = r = 0$ and the trivial orbit $0 \in \mathbb{R}^6$.

4. Orbits and Representations

The theory of representations of the Jacobi group G^J with non–trivial central character reduces via the "Schrödinger–Weil representation" (a product of the Schrödinger with its associated Weil representation) to the theory of the metaplectic group $\mathrm{Mp}(2, \mathbb{R})$, the double cover of $\mathrm{SL}(2, \mathbb{R})$ (see for instance [BeS] p.15 ff, or, for the general theory, [Du]). So it's to be expected that a part of the relations between the coadjoint orbits of G^J and its unitary dual $\hat{G}^J$ are the same as for $\mathrm{SL}(2, \mathbb{R})$, resp. $\mathrm{Mp}(2, \mathbb{R})$. We will give here an explicit discussion of these and the remaining cases, thus, also give a foundation to propositions made by Yang in [Ya1,2] without further comment.

To construct representations from a coadjoint orbit of a group, the orbit has to be "admissible", i.e. "integral" and been provided with a "nice" polarization. Then one can construct a complex line bundle, whose polarized sections lead to the space for a unitary (and perhaps even irreducible) representation of the group. Or, in another formulation, this representation is to be got by an induction procedure from a subgroup arising from the polarized orbit. Though there are now several more refined procedures using and discussing Duflo's version of admissibility as for instance in Vogan [Vo] p.193 ff or in Torasso [To], we will follow the procedure designed by Kirillov [Ki] p.235 ff and [Ki1] (though in some places adapting our notation to the one used by Vogan):

For each orbit $\mathcal{O}_{U_*} \subset \mathfrak{g}^*$ we look for real subalgebras $\mathfrak{n} \subset \mathfrak{g}$ or complex ones in $\mathfrak{g}_c = \mathfrak{g} \otimes \mathbb{C}$ which are *subordinate* to U_*.
This means that

$$B_{U_*}|_{\mathfrak{n}} \equiv 0$$

holds or, equivalently

$$\check{Y} \longmapsto 2\pi i < U_*,\ \check{Y} >$$

defines a one–dimensional representation $\varrho = \varrho_{U_*}$ of $\mathfrak{n}$. Moreover, we suppose $\mathfrak{q}$ is Ad G_{U_*}–invariant and, in the complex case, $\mathfrak{m}_c = \mathfrak{n} + \overline{\mathfrak{n}}$ is a subalgebra of $\mathfrak{g}_c$ such that we have closed subgroups Q and M of G with Lie algebras $\mathfrak{q}$ and $\mathfrak{m}$ fulfilling the relations

$$L := G_{U_*} \subset Q = G_{U_*} Q^\circ \subset M,\ \mathfrak{n} \cap \overline{\mathfrak{n}} = \mathfrak{q}_c,\ \mathfrak{n} + \overline{\mathfrak{n}} = \mathfrak{m}_c.$$

In the real case, we have $\mathfrak{q} = \mathfrak{n}$.

Finally, we suppose that ϱ_{U_*} can be intergrated to a unitary character χ_Q of Q with $d\chi_Q = \varrho_{U_*}$. Then we get a unitary representation π of our group G on (the completion of) the space $\mathcal{C}^\infty(G, \mathfrak{n}, Q, \varrho, \chi_Q)$ of smooth $\mathbb{C}$–valued functions ϕ on G with

$$\phi(lg) = \triangle_Q(l)^{1/2} \chi_Q(l) \phi(g) \quad \text{for all } l \in L,\ g \in G$$

and

$$\mathcal{L}_{\check{Y}} \phi = 0 \quad \text{for all } \check{Y} \in \mathfrak{u},\ \mathfrak{n} = (\mathfrak{g}_{U_*}) + \mathfrak{u},$$

where $\triangle_Q$ is the usual modular function and $\mathcal{L}_{\check{Y}} \phi(g) = \frac{d}{dt} \phi(g \exp tX)|_{t=0}$. (Here we follow [Ki] p.199ff; for another essentially equivalent approach see [Vo] p.194 and p.199 or Barchini [Ba]).

In more detail, in our cases this leads to the following.

4.1 For the hyperbolic case $U_* = mR_* + \alpha X_*$ $(m \neq 0, \alpha > 0)$ with $L = G_{U_*} = MA \times C$ we have Ad L–invariant real polarizations

$$\mathfrak{q}_1 =< X, Y, Q, R > \text{ and } \mathfrak{q}_2 =< X, Z, P, R >$$

with associated characters

$$\varrho_j(\check{X}) = 2\pi i < mR_* + \alpha X_*, \check{X} >= 2\pi(mr + 2\alpha x),\ j = 1, 2,$$

for $\check{X} = xX + yY + qQ + rR \in \mathfrak{q}_1$ as well as $\check{X} = xX + zZ + pP + rR \in \mathfrak{q}_2$. Both characters of $\mathfrak{q}_j$ integrate to characters $\chi_{m,\alpha}$ of respective groups

$$Q_1 = MN^J \text{ and } Q_2 = M\overline{B}^J$$

given by

$$\chi^\pm_{m,\alpha}(b) = e(m\kappa) y^{2\pi i \alpha} \gamma_\pm(\varepsilon)$$

for

$$B^J = \left\{ b = \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} (0, \mu, \kappa) \right), \ y \in \mathbb{R}_{>0}, x, \mu, \kappa, \in \mathbb{R} \right\}$$

$$\gamma_\pm(\varepsilon) = 1 \text{ or } \operatorname{sgn} \varepsilon \text{ when } M = \left\{ \begin{pmatrix} \varepsilon & \\ & \varepsilon \end{pmatrix}, \ \varepsilon = \pm 1 \right\},$$

and analogously for $\overline{B}^J$, where N is replaced by $\overline{N} = \left\{ \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \right\}$.
The usual normalized parabolic induction leads for $2\pi\alpha =: s$ to the principal series representations of G^J as in [BeS] p. 48 .

4.2 For the elliptic case $U_* = mR_* + \alpha(Y_* - Z_*)$ $(m \neq 0, \alpha > 0)$ with $L = G_{U_*} = \mathrm{SO}(2) \times C =: K^J$ we have two Ad L–invariant complex polarizations

$$\mathfrak{n}_\pm =< \tilde{Z}, X_\pm, Y_\pm, Z_0 >$$

where as in [BeS]

$$\tilde{Z} = -i(Y - Z), \ Z_0 = -iR,$$

$$X_\pm = (1/2)(X \pm i(Y + Z)), \ Y_\pm = (1/2)(P \pm iQ).$$

The associated characters are

$$\varrho_\pm(\check{X}) = 2\pi i < \alpha(Y_* - Z_*) + mR_*, \check{X} >= 2\pi(mz_0 + 2\alpha\tilde{z})$$

for

$$\check{X} = \tilde{z}\tilde{Z} + x_+X_+ + y_+Y_+ + z_0Z_0 \in \mathfrak{q}_+$$

resp. analogous in $\mathfrak{q}_-$. Here we have $\mathfrak{q} = \mathfrak{n}_\pm \cap \mathfrak{g}^J =< Y - Z, R >$, i.e. $Q = L$. As one wants to integrate $\varrho_\pm$ to characters χ of Q, the integrality condition $4\pi\alpha =: k \in \mathbb{Z}$ comes up. Then we arrive at the characters $\chi_{m,k}$ of $K^J = \mathrm{SO}(2) \times C$ with

$$\chi_{m,k}((r(\vartheta))(0,0,\kappa)) = e(m\kappa)e^{ik\vartheta}$$

and by the usual holomorphic resp. antiholomorphic induction we get the discrete series representations $\pi^\pm_{m,k}$ of G^J (see [BeS] p. 47).

4.3 For the 4–dimensional nilpotent orbits $\mathcal{O}_{U_*}$, $U_* = mR_* + Z_*$, $m \neq 0$, with $L = G_{U_*} = MN \times C$ we have the Ad L–invariant real polarization

$$\mathfrak{q} =< X, Y, Q, R >$$

with character

$$\varrho(\check{X}) = 2\pi i < mR_* + Z, \check{X} > 2\pi imr \text{ for } \check{X} = xX + yY + qQ + rR \in \mathfrak{q}.$$

Similar to **4.1** this integrates to the character $\chi^{\pm}_{m,0}$ of the group $MB^J =: Q_1$. The modular factor $\triangle_{B^J}(b) = y^{3/2}$ necessary for the normalized parabolic induction creates a situation similar to the one described in [Vo] p.195, so that here half densities on G^J/Q_1 come in. One is tempted to see here the outcome of the Schrödinger–Weil representation π^m_{SW} (in the notation of [BeS]).

4.4 The case $U_* = mR_* + Y_*, m \neq 0$, is analogous to **4.3** ,

4.5 The two-dimensional nilpotent orbits $\mathcal{O}_{mR_*}$

$$L = G_{U_*} = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{C}$$

don't carry a representation by the construction used so far. As already stated in [Ki] p. 237 (where our G^J is called $\mathrm{St}(1, \mathbb{R})$), there is **no** admissible subalgebra $\mathfrak{q}$ subordinate to mR_*: In our notation, we have

$$B_{mR_*}(\check{X}, \check{Y}) = 2(pq' - p'q).$$

Hence $B_{U_*}\mid_{\mathfrak{q}} \equiv 0$ on $\mathfrak{q} =< X, Y, Z, P, R >$ or $< X, Y, Z, Q, R >$, but none of these is a subalgebra of $\mathfrak{g}^J$.
But here we can locate the usual construction of the Weil representation as Kirillov does in [Ki1] p.167. In our notations, this comes out as follows. $\mathcal{O}_{mR_*}$ projects on the coadjoint orbit $\mathcal{O}^0_{mR_*}$ of the Heisenberg group in $\mathfrak{h}^*$. By the orbit method, the associated representation is the Schrödinger representation π^m_S in $\mathcal{H}_m = L^2(\mathbb{R})$. This representation space $\mathcal{H}_m$ admits a unique action π^m_W of $\mathrm{Mp}(2, \mathbb{R})$. π^m_S and π^m_W together give, for $m \neq 0$ fixed, a unique action π^m_{SW} of $\tilde{G}^J = \mathrm{Mp}(2, \mathbb{R}) \ltimes \mathrm{H}$ on $\mathcal{H}_m$. These representations are equivalent for all $m < 0$ resp. $m > 0$. Each π^m_W decomposes into two irreducible parts (for even and odd elements in $\mathcal{H}_m$), so that we get the four usual classes of Weil (or oscillator -) representations.

4.6 A very interesting case is the nilpotent orbit $\mathcal{O}_{\mathcal{P}_*}$. Here we have the isotropy group

$$L = G_{P_*} = N \times C$$

with

$$\mathfrak{g}_{P_*} =< Y, R >$$

and

$$B_{P_*}(\check{X},\check{Y}) = 2(px' - p'x + qz' - q'z).$$

There are two Ad L- invariant polarizations

$$\mathfrak{q}_1 =< X, Y, Q, P > \text{ and } \mathfrak{q}_2 =< Y, P, Q, R >$$

with characters

$$\rho_1(\check{X}) = 2\pi i \ < P_*, \check{X} >= 0, \text{ for } \check{X} = xX + yY + qQ + rR,$$

and

$$\rho_2(\check{X}) = 2\pi i \ < P_*, \check{X} >= 2\pi ip, \text{ for } \check{X} = yY + pP + qQ + rR.$$

So in the first case, we have as a variant of **4.3** the trivial representation of $Q_1 = B^J$ and thus only a bundle of half densities on

$$G^J/B^J \simeq \mathrm{SO}(2) \times \mathbb{R}$$

(leading to representations of the metaplectic cover $\tilde{G}^J$).
In the second case, we can integrate ρ_2 to the character χ_{Q_2} of the group

$$Q_2 := \{q = n(b)(\lambda, \mu, \kappa);\ b, \lambda, \mu, \kappa \in \mathbb{R}\} =: G_\psi$$

defined by

$$\chi_0 = e(\lambda).$$

By the usual parabolic induction, we get the representation

$$\pi = \mathrm{ind}_{Q_2}^{G^J} \chi_0,$$

which is the special case $r = 0$ of the unique series $\tau_r (r \in \mathbb{R})$ of representations of G^J with trivial central character which don't reduce to representations of $\mathrm{SL}(2, \mathbb{R})$ and which is given by

$$\tau_r = \mathrm{ind}_{Q_2}^{G^J} \chi_r,\ \chi_r(q) = e(rb + \lambda),\ r \in \mathbb{R}.$$

4.7 The remaining cases

$$U_* = \alpha X_*, \alpha(Y_* - Z_*), Y_*, Z_* \text{ and } 0$$

reduce to the usual $\mathrm{SL}(2, \mathbb{R})$ theory and thus are well known. For a thorough treatment of the details, we refer to the thesis of P. Renouard [Re] (he discusses the orbit method for the group $SU(1, 1)$).

5. Summary

For the representations of $G^J = G_0 \ltimes H$ with **non-trivial central character** we get the same $1-1$ correspondence as in the $G_0 = \mathrm{SL}(2,\mathbb{R})$ theory between
– hyperbolic orbits and principal series representations,
– elliptic orbits and discrete series representations.
We don't get the complementary series.
For the nilpotent orbits in the $\mathrm{SL}(2,\mathbb{R})$ theory it seems still not clear how the correspondence is to be fixed (see the discussion in [Vo1] p.386). To me it seems most probable (see in particular [Re] p.I.110f) to associate to the nilpotent cone the principal series $\mathcal{P}^{+,0}$ and the (decomposing) $\mathcal{P}^{-,0} = D_1 \oplus D_{-1}$. All this can be carried over to the G^J–theory with non-trivial central character.

For the representations of G^J with **trivial central character** we have the phenomenon of the appearance of a nilpotent orbit carrying a special representation belonging to a single value of a continuous series.

Let's finally point to Kirillov's remark in [Ki1] 8.4 on the possibility to fit in even the complementary series in this correspondence framework. This should have significance here also.

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