Panorama of Kähler’s Work

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Erich Kähler’s research covers an unusually wide area. From celestial mechanics he got into complex function theory, differential equations, analytic and complex geometry with their differential forms, and then into his main topic, joining all this with arithmetics, i.e. arithmetic geometry. His main interest was in finding the unity in the variety of mathematical themes and establishing thus mathematics as a universal language for physics, chemistry, biology and even philosophy and theology.
At least a bit more precisely, this means the following. As usual, it is difficult to range good mathematics into different baskets labelled for instance differential geometry, number theory or whatever. And as said above, it is particularly difficult in Kähler’s case, but I will enumerate some labels in five blocks following roughly the chronological order and the stations of his scientific life in Leipzig, Hamburg, Rome, Hamburg, Königsberg, and – after the war – Leipzig, Berlin and again Hamburg.

1. Influenced by having read Gauß, Weierstraß and Lagrange already at school, his first publications starting in 1926 treated classical differential equations, in particular the **three–body problem**.

2. From there, he got into the **theory of complex functions** mainly in two variables, where his topics became among others

   – Singularities and knots, their monodromy groups.

   – Four–dimensional Riemannian manifolds, their topology and in particular Betti numbers.

   – Periods, their relations and the interpretation of these relations.

   – Differential equations for functions in two and more variables, in particular Poincaré’s hyperfuchsian and Picard’s hyperabelian functions, and their monodromy groups.

3. Adapting Elie Cartan’s theory of “symbolic” **differential forms**, Kähler developed this theory further to make it a smooth tool to treat the following topics

   – Differential forms on (hyper)surfaces, their genera and plurigenera in the spirit of Severi, Enriques, Castelnuovo and the Italian school of algebraic geometry.

- Periods and their relations in the general context.
- A certain "remarkable Hermitian metric" and its potential. This is the 14–pages paper in our "Hamburger Abhandlungen vol.9" from 1933 which led to the notions "Kähler metrics", "Kähler manifolds" and more, up to "Kähler geometry", which is now an essential tool in mathematical physics and will appear here in our colloquium in at least two lectures.
- The general theory of systems of partial differential equations including the theorem of Frobenius and the existence and uniqueness theorems involved in the proofs of the main theorems of the theory of Lie Groups. This is found in the small but important booklet "Einführung in die Theorie der Systeme von Differentialgleichungen" from 1934.
- Maxwell's equations of electrodynamics in a formalism Kähler developed in 1937 and which is now standard in theoretical physics. And finally
- Dirac's equation for the electron. Here, he developed in a series of four papers in 1961 the calculus of "interior multiplication" which introduces a new ring structure on the ring of exterior differentials and is now mostly treated in the form of a Clifford algebra structure.

4. **Arithmetic Geometry**

Though the name of Kähler is mostly connected to the development of Kähler geometry, his main achievement – at least to his and also my opinion – is his work on melting arithmetic, analysis and geometry – and, if one wants, even philosophy. Independent of this, part of Kähler's work here is even a precursor to the much later outcome of the Arakelov theory:

Already in 1939, just before the war, Kähler announced in his lecture "Über die Beziehungen der Mathematik zur Physik und Astronomie" for the Copernicus Society in Königsberg an attempt to join arithmetic to the analytic and geometric methods used to construct physical models up to then. And he proposed that not "space" but "fields" should be the primary objects of study. Then, after the interruption of the war, finally in 1951 from Leipzig, he started a series of publications with his article on purely algebraic fields in the Mathematische Nachrichten, terminating in 1963 with a résumé entitled "Infinitesimal–Arithmetic", but culminating in 1958 with the article of 399 pages in the Annali di Matematica

“Geometria Aritmetica”.

I quote from André Weil’s review:

“This, in more ways than one, is an unusual piece of work. By its size, it is a book; it appears as a volume in a journal. The author is German; the book appears in Italian. The subject combines algebra and geometry with some arithmetical flavoring; but the author, instead of following in his terminology the accepted usage in either one of these subjects, or adapting it to its purposes, has chosen to borrow his vocabulary from philosophy, so that

rings, homomorphisms, factor–rings, ideals, complete local rings

appear as

“objects”, “perceptions”, “subjects”, “perspectives”,“individualities”.

One may see this work by Kähler in the line of several attempts to give a joint foundation to algebraic geometry and number theory, as done by Chevalley, Weil, Eichler and ultimately Grothendieck, Dieudonné and the French school. There is a footnote in Grothendieck’s EGA I p.8 pointing to Kähler’s work: “Parmi les travaux qui se rapprochent de notre point de vue en Géométrie algébrique, signalons l’important travail de E. Kähler...”

Besides the problem of the language, there is the fact that Kähler’s main interest was, as said above, in the study and construction of birational invariants characterizing the structure of field extensions finitely generated over a ground field or the prime ring . Thus, in his system he considered only varieties coming from primary rings and not the completely general situation EGA allowed for. As we will be fortunate enough to have the lecture by Bost on what may still be found in the Geometria Aritmetica from the modern point of view, for the sake of completeness, I will only enumerate the contents of Kähler’s main work, called the “Bible” by his former collaborators in Leipzig.

- The definition of a "differential module" RdR, resp. $\Omega^1_{R/A}$ in the modern notation, associated to a ring R with subring A. This notion is discussed in extenso also in EGA but doesn't appear under the name of module of "Kähler differentials" as it does in other publications as for instance in Kunz's book with just this title or in Hartshorne's "Algebraic Geometry".
- Applying Fitting's calculus of determinantal ideals to the differential module of R/A , Kähler defines a chain of ideals $\mathfrak{d}_i(R/A), i \in \mathbb{N}_0$, which are now called the "Kähler differents" and determines these in certain cases .
- The notion of a "variety" V/A, which can now be understood as saying that V is the ensemble of local rings of a reduced scheme X/A, and related objects like "subvarieties" ("figure" in Italian, "Gebilde" in German), "intersection multiplicities", "correspondences", ..., and finally a Hausdorff topological space $R(V)$ associated to a variety V for $A \supset \mathbb{Z}$.

  Briefly said: we have the main tools for the algebraic geometry in the "algebraic case" over a ground field $A = k$ as well as in the "arithmetic case" over a ground ring $A = \mathbb{Z}$. And these tools are used for distinguishing the following notions and objects describing the structure of a given field extension K/A:
- The **"differential integrity"** $D(K/A)$: As a joint generalization of the procedure to define the maximal order $\mathfrak{o}$ of a number field k as the ring of entire elements, i.e. those contained in all discrete valuation rings of k, and the abelian differentials of the first kind on a Riemannian surface as those which are holomophic in all points of the surface, the differential integrity associated to K/A is given by

  $$D(K/A) = \cap_S [SdS],$$

  i.e. as the intersection of the canonical images $[SdS]$ of the differential rings $\Omega_{S/A}$ in $\Omega_{K/A}$, where S is a discrete valuation ring of K/A . Together with his students in Leipzig he determined these "arithmetically entire differentials" explicitly in some examples, for instance for the "Fermat field"

  $$K = \mathbb{Q}(x, y) \quad \text{with} \quad x^m + y^m = 1, \quad m \in \mathbb{N},$$

  or certain elliptic function fields in particular

  $$K = \mathbb{Q}(\sqrt{-23}, x, y) \quad \text{with} \quad y^2 = x^3 + \sqrt{-23}.$$

Later on he proposed to compare the arithmetically entire differentials with the differentials with entire q–development. For the fields of elliptic modular functions this was done, at least to some extent, here in Hamburg (by Schramm and myself).

— **Periods and modular functions:** A space R of a field K with trgr $K = n$ is called an arithmetical space if it is of the form $R = R(V), V/\mathbb{Z}$ a proper finitely generated variety, where all $s \in V$ with $s \supset \mathbb{Q}$ are regular. Such a space decomposes into $g = \deg\, k$, k the maximal number field contained in K, connected topological spaces R_i with n–dimensional complex structure

$$R = \coprod_{i=1}^{g} R_i.$$

By integrating arithmetically entire differentials from the differential integrity $D(K)$ over R, resp. cycles on R. Kähler arrives at notions which have arithmetic meaning. In particular, for each $\omega \in D_n(K)$ the numbers

$$\int\limits_{R_i} \omega \wedge \bar{\omega} \qquad i = 1, \dots, g$$

are independent of the choice of the space R of K. With the abbreviation

$$\int\limits_{R} \omega \wedge \bar{\omega} \ := \ \sum_{i=1}^{g} \int\limits_{R_i} \omega \wedge \bar{\omega}$$

one has a differential of degree $2n$ depending only on K and the variable t

$$\Omega(t) \ := \sum_{\omega \in D_n(K)} \frac{\omega \wedge \bar{\omega}}{(\int\limits_R \omega \wedge \bar{\omega})^{1+t/2}}, \qquad \mathrm{Re}\, t > g \cdot p_n$$

($p_n = \dim D_n(K)$). Integration gives a birationally invariant function

$$\int\limits_{R} \Omega(t) = \sum_{\omega \in D_n(K)} \Big(\int\limits_{R} \omega \wedge \bar{\omega} \Big)^{-t/2}.$$

Moreover, if $\Omega(t)$ is written as

$$\Omega(t) = \Phi(t)\ dx_1 \wedge d\bar{x}_1 \wedge \cdots \wedge dx_n \wedge d\bar{x}_n$$

for the coordinates $(x_1, \ldots, x_n)$ of a chart of one of the manifolds R_ν , the expression

$$i \sum_{j,l=1}^{n} \frac{\partial^2 \log \Phi}{\partial x_j \partial \bar{x}_l} \, dx_j \; d\bar{x}_l$$

defines a metric on R_ν, which is Kählerian in the now customary notation.

— The **"zeta calculus":** In generalization of the classical zeta functions of algebraic number fields Kähler associates to each subvariety C of the variety V – at least formally – a zeta function $\zeta(C,t)$ in the following way. If a subvariety B of V is given in the local ring $s \in V$ by the ideal $\mathfrak{b}(s)$, one defines as a "norm of B"

$$N\mathfrak{b} \;:=\; \prod_{s \in V} \#(s/\mathfrak{b}(s)).$$

Then $\zeta(C,t)$ is given by the sum over all subvarieties B contained in C

$$\zeta(C,t) \;:=\; \sum (N\mathfrak{b})^{-t}.$$

By a result of Lustig from 1956 this converges for the case of arithmetic dimensions $r \leq 2$ but by a result of Witt from 1968 it diverges for $r > 2$. So Kähler's hope that the logarithmic derivative of his zeta function could be an arithmetic integral (GA 485.) is only partially fulfilled. Nevertheless the resemblance of the right hand side of Planck's radiation formula in the form

$$Ed\Big(-\frac{1}{\kappa T}\Big) = d \;\log\; \frac{1}{1 - e^{\frac{-h\nu}{\kappa T}}}$$

with the local factor of a classical zeta function was an intriguing theme leading Kähler to postulate a correspondence between Galois fields and harmonic oscillators (in his paper "osservazioni a proposito della dinamica" from 1954).

As years went by, it became more and more important to Kähler to use the mathematical system of his Geometria Aritmetica as a universal language to describe the world around us and even in us as a whole. The notions already mentioned in Weil's review indicate that Kähler was thinking of mathematical objects in philosophical terms. Thus, in his Geometria Aritmetica and in particular in his book "Algebra und Differentialrechnung" from 1958 he used local algebra to build a mathematical model for Leibniz's "Monadologie". This model, he continuously extended using Siegel's theory of theta functions, the theory of modular and Jacobi functions, and he proposed a model for the periodic system of the elements, a model of the system of elementary particles and even of phenomenae in theology – this as an outcome of his studies in theology and philosophy, in particular Platon, Schopenhauer and Nietzsche.

Here I just show you briefly two examples of Kähler's attempts to find convincing models using his Monadologie and automorphic functions. I have not the time to comment on these but hope that some of you get a bit curious and perhaps interested. Anyway, Riemenschneider and I, we plan to edit Kähler's Works in the Verlag Walter de Gruyter and you will be able to find easily more of Kähler's mathematics there. Let me finish my panorama with the last theme:

5. **The "New Poincaré Group"**

Though all these models mentioned above use primarily fields and their structure, Kähler returned also to the description of space–time. Starting in 1983, he proposed in several articles a model describing the Minkowski space of special relativity as a limit case of the hyperbolic four–space with the metric

$$ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}$$

and a 10–parameter quaternionic invariance group $SL(2, \mathbb{H})) \simeq SO(1,4)$ which he called the "New Poincaré Group". This is a fertile point of view for the theory of automorphic functions, as we will see in Krieg's lecture. But beyond this, in his last paper "Raum Zeit Individuum" from 1992 he discussed the Dirac equation for the metric above showing that Maxwell's equations appear here as a natural consequence. Moreover, he expressed his hope that this should be the frame to unify quantum theory and relativity – and even biology.

We are in the fortunate situation that at this meeting we have experts who will inform us about the state of affairs in some of the topics Kähler was so much interested in. And even if it should come out, that his attempts to give answers prove to be insufficient, we all – I hope – can be sure, that the mathematics he proposed is the right tool and we can accept and admire the audacity to insist on asking his questions about the predominance of the absolute over the relative in the tradition of Platon, Leibniz, Planck and Einstein.

Kähler took as a theme for his article "Osservazioni a proposito della dinamica" from 1954 the following words of Galois

> "Tout voir, tout entendre, ne perdre aucune idée".

I hope, this may guide us here, today and tomorrow.