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Report on work by Oda and his school on several types of spherical functions for $\mathrm{Sp}(2, \mathrm{R})$

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Report on work by Oda and his school on several types of spherical functions for $Sp(2,\mathbb{R})$

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A lot of important work by T. Oda, T. Miyazaki, H. Narita and M. Hirano can be seen as an investigation of the following general question. Given two admissible representations π and ξ of $G = Sp(2,\mathbb{R})$, π irreducib le and ξ induced, what can be said about the space of intertwining operators T mapping the space H_π of π into the space ξ for the following special cases

– π a discrete series representation, or

– π a principal series representation induced from the minimal parabolic P of G or the Jacobi parabolic P_J, and $\xi = \mathrm{Ind}_{\mathcal{N}}^{P}\zeta$, induced from

– $\mathcal{N} = N$, the unipotent radical of P or from

– $\mathcal{N} = R_J$, the Jacobi group realized as an embedding into $P_J \subset G$.

The method to answer this question is traced and already used by Yamashita [Ya] for embeddings of the discrete series into induced representations of $SU(2,2)$. It may be very roughly described as follows.

– Fix a maximal compact K of G and distinguish in π a K–type which determines π uniquely (for the discrete series the minimal K–type will do and for the principal series "corner K–types" will be introduced).

– Relate the intertwining space

$$\mathrm{Hom}_G(H_\pi,\ \mathrm{Ind}_{\mathcal{N}}^{G}\zeta)$$

to a space

$$\mathcal{W}_{\zeta,\tau} = \mathcal{C}^\infty_{\zeta,\tau}(\mathcal{N}\backslash G/K)$$

of smooth "spherical" or "generalized ζ–Whittaker functions of type τ"

$$\text{with} \quad \begin{array}{l} F : G \rightarrow \mathcal{F}_\zeta \otimes V_\tau, \\ F(ngk) = (\zeta(n) \otimes \tau(k)^{-1})F(g) \text{ for all } n \in \mathcal{N},\ g \in G, k \in K \end{array}$$

where $\mathcal{F}_\zeta$ and V_τ are the spaces of the representations ζ of $\mathcal{N}$ resp. τ of K.

– Consider a decomposition $G = \mathcal{N}\mathcal{A}K$ such that the functions F are fixed by the restriction $\phi = F\backslash_{\mathcal{A}}$ to their "radial parts".

– Characterize the space of these radial functions ϕ by the radial parts $\mathcal{R}(D)$ of appropriate differential operators

$$D : \mathcal{W}_{\zeta,\tau} \longrightarrow \mathcal{W}_{\zeta,\tau'},$$

where τ' is a possibly not irreducible representation of K. For π in the discrete series this characterization is simply given by the condition that

the functions in the kernel of the "Schmid operator" $\mathcal{D}$ (introduced by W. Schmid [S]) and for the principal series one can use the condition that the spherical functions are eigenfunctions of the Casimir operator for a certain infinitesimal character. – Solve the system of difference–differential equations coming up by the expansion of the radial functions with respect to bases of $\mathcal{F}_\zeta \otimes V_\tau$.
Here some really ingenous work is done in each case to find substitutions which make the rather involved equations tractable and reduce the solutions to Whittaker–type classical functions (imposing eventually growth conditions on the functions F resp. ϕ).

This report has a rather strange form and the choice of the material may seem somewhat arbitrary. But it grew out of the resumés of the papers [Ya], [Na] and [Hi$_{1,2}$] which are motivated by my own interest in linking the representation theory of the Jacobi group as described in the book [BeS] by Berndt and Schmidt to the representation theory of the symplectic group. For lack of more time, the older resumés are kept as they were, and now in an introductory chapter, the method inherent in these papers –as we see it– is resumed and then illustrated by the discussion of two further examples from [O] and [MO II] not appearing in the resumés.
In this attempt to show the general line of arguments, sometimes, very subtle technical details are skipped. I hope the reader won't regret this too much and the authors will excuse it. Anyway, the best thing will be to study the original papers and I hope these lines help smoothening the way.

1. Notations

The following notations are (up to very little changes) used throughout in the papers of Oda and his collaborators.
The symplectic group is written as

$$G = Sp(2,\mathbb{R}) = \left\{ g \in M_4(\mathbb{R}),\quad {}^t g J g = J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix},\ \det g = 1 \right\}$$

and a maximal compact subgroup as

$$K = \left\{ k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G,\ A, B \in M_2(\mathbb{R} \right\} \simeq U(2) = \{u = A + iB,\ {}^t\overline{u}u = 1_2\}$$

The corresponding Lie algebras are $\mathfrak{g}$ and $\mathfrak{k}$ with

$\mathfrak{g} = \mathfrak{s}_{\mathfrak{g}}(2,\mathbb{R}) = \{X \in M_4(\mathbb{R}), JX + \ {}^tXJ = 0\}$
$\mathfrak{k} = \left\{ X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \ {}^tA = -A, \ {}^tB = B \right\}$
and the Cartan resp. Iwasawa decomposition
$\mathfrak{g} = \mathfrak{k} \otimes \mathfrak{p} = \mathfrak{k} \otimes \mathfrak{a} \otimes \mathfrak{u}$
with

$\mathfrak{k} =< T_1, T_2, Y, \overline{Y} >=< T_+, T_-, Y, \overline{Y} >, T_1 = (1/2)(T_+ + T_-), T_2 = (1/2)(T_+ - T_-)$
$\mathfrak{a} =< H_1, H_2 >,$
$\mathfrak{u} =< E_{2e_1}, E_{e_1+e_2}, E_{2e_2}, E_{e_1-e_2} >= \sum_{\alpha\in\Psi} \mathfrak{g}_\alpha.$

Here $e_1 := (1,0)$ and $e_2 := (0,1)$ and

$$\Psi = \{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$$

denotes a root system for $(\mathfrak{g}, \mathfrak{a})$.

$$\Psi_+ = \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\}$$

is taken as the positive part, the first three roots are non compact, the last one is compact.

We have

$$T_1 = \begin{pmatrix} & & 1 & \\ & & & \\ -1 & & & \\ & & & \end{pmatrix}, \qquad T_2 = \begin{pmatrix} & & & \\ & & & 1 \\ & & & \\ & -1 & & \end{pmatrix},$$

$$Y = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \quad Y' = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

$$H_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix},$$

$$E_{2e_1} = \begin{pmatrix} & & 1 & \\ & & & 0 \\ & & & \\ & & & \end{pmatrix}, \qquad E_{e_1+e_2} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & & & \\ & & & \end{pmatrix},$$

$$E_{2e_2} = \begin{pmatrix} & & 0 & \\ & & & 1 \\ & & & \\ & & & \end{pmatrix}, \qquad E_{e_1-e_2} = \begin{pmatrix} & 1 & & \\ & & & \\ & & & \\ & & -1 & \end{pmatrix}.$$

Beside the minimal parabolic $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{u}$, there will be used the "Jacobi parabolic"

$$\mathfrak{p}_J = \mathfrak{a}_J \oplus \mathfrak{m}_J \oplus \mathfrak{u}_J,$$

with

$$\mathfrak{a}_J =< H_1 >,\ \mathfrak{m}_J =< H_2, T_2, E_{2e_2} >\simeq \mathfrak{sl}_2,\ \mathfrak{n}_J =< E_{2e_1},\ E_{e_1+e_2},\ E_{e_1-e_2} >,$$

where $\mathfrak{n}_J$ is isomorphic to a Heisenberg algebra.

The corresponding subgroups are written

$$\begin{aligned} P = AN, \quad & A = \left\{ \mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1}), t_1, t_2 > 0 \right\}, \\ & N = \{ n = n(x_1, x_2, x_3, x_4), x_1, \dots, x_4 \in \mathbb{R} \} \end{aligned}$$

with

$$n(x_1,x_2,x_3,x_4) = \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_4 & & \\ & 1 & & \\ & & 1 & \\ & & -x_4 & 1 \end{pmatrix},$$

and

$$\begin{aligned} P_J = M_J\ A_J\ N_J, \quad & A_J = \{\mathrm{diag}(t,1,t^{-1},1), t>0\} \\ & N_J = \{n(x_1,x_2,0,x_4), x_1,x_2,x_4 \in \mathbb{R}\} \simeq \ \mathrm{Heis}(\mathbb{R}) \\ & M_J = SL_2(2,\mathbb{R}) \times \{\pm 1\} \end{aligned}$$

The subgroup

$$R_J = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & * & * & * \end{pmatrix} \right\} \simeq SL(2,\mathbb{R}) \ltimes N_J$$

is isomorphic to the Jacobi group $G^J(\mathbb{R})$. Unfortunately this is not the embedding as in Eichler–Zagier [EZ] and in Berndt–Schmidt [BeS].

The complexifications

We have

$$\mathfrak{g}_c := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k}_c \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_- = \mathfrak{h}_c \oplus \oplus_{\beta \in \Sigma} \mathfrak{g}_\beta$$

with $\mathfrak{g}_\beta =< X_\beta >$ and

$$\mathfrak{k}_c =< Z, H', X, \bar{X} > \supset \mathfrak{h}_c =< Z, H' >= \mathfrak{h} \underset{\mathbb{R}}{\otimes} \mathbb{C}, \quad \mathfrak{h} =< T_1, T_2 >$$

where

$$\Sigma = i\{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\}$$

is a root system for $(\mathfrak{g},\mathfrak{h})$ for the compact Cartan subgroup $\mathfrak{h}$. We take as positive non–compact roots

$$\Sigma_n^+ = \{(2,0),(1,1),(0,2)\}$$

and as positive compact root

$$\Sigma_c^+ = \{(1,-1)\}.$$

Further we use for $\beta \in \Sigma$

$$||\beta|| := \sqrt{|\beta_1|^2 + |\beta_2|^2},$$

and for an appropriate constant c

$$\{c\|\beta\|(X_\beta + X_{-\beta}), ci\|\beta\|(X_\beta - X_{-\beta})\}_{\beta\in\Sigma_n^+}$$

is an ON–basis for $\mathfrak{p} = \mathfrak{p}_{\mathbb{R}}$.
Here, the elements are given by the following matrices

$$\begin{aligned}
Z &= -i\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}, H' = -i\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}, \\
X &= (1/2)\begin{pmatrix} & 1 & & -i \\ -1 & & -i & \\ & i & & 1 \\ i & & -1 & \end{pmatrix}, \bar{X} = (1/2)\begin{pmatrix} & 1 & & i \\ -1 & & i & \\ & -i & & 1 \\ -i & & -1 & \end{pmatrix}, \\
X_{\pm(2,0)} &= \begin{pmatrix} 1 & & \pm i & \\ & 0 & & 0 \\ \pm i & & -1 & \\ & 0 & & 0 \end{pmatrix}, X_{\pm(0,2)} = \begin{pmatrix} 0 & & 0 & \\ & 1 & & \pm i \\ 0 & & 0 & \\ & \pm i & & 1 \end{pmatrix}, \\
X_{\pm(1,1)} &= \begin{pmatrix} & 1 & & \pm i \\ 1 & & \pm i & \\ & \pm i & & -1 \\ \pm i & & -1 & \end{pmatrix}, X_{\pm(1,-1)} = \begin{pmatrix} & 1 & & \pm i \\ -1 & & \pm i & \\ & \pm i & & 1 \\ \pm i & & -1 & \end{pmatrix}.
\end{aligned}$$

Here we slightly mistreat Oda's more precise notation by identifying X_β with for instance $X_{(2,0)}$, if $\beta = i(2,0)$.
The complex and real matrices are linked by the following relations
Lemma (1.1) ([O]p.7):

$$\begin{aligned}
X_{\pm(2,0)} &= \pm H_1' + H_1 \pm 2iE_{2e_1}, & H_1' &= -iT_1, \\
X_{\pm(1,1)} &= \pm 2\bar{X} + 2E_{e_1-e_2} \pm 2iE_{e_1+e_2}, & & \\
X_{\pm(0,2)} &= \pm H_2' + H_2 \pm 2iE_{2e_2}, & H_2' &= -iT_2.
\end{aligned}$$

2. Representations of K_c

For $\mathfrak{k}_c =< Z, H', X, \bar{X} >$ one has the relations

$$[Z, \mathfrak{k}_c] = 0, [H', X] = 2X, [H', \bar{X}] = -2\bar{X}, [X, \bar{X}] = H'$$

and for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ with $\lambda_1 \geq \lambda_2$ irreducible finite dimensional representations $(\tau_\lambda, V_\lambda)$ are given by

$$\begin{array}{rcl} \tau_\lambda(Z)V_k &=& (\lambda_1 + \lambda_2)V_k, \\ \tau_\lambda(H')V_k &=& (2k - d)V_k, \\ \tau_\lambda(X)V_k &=& (k+1)V_{k+1} \\ \tau_\lambda(\bar{X})V_k &=& (d+1-k)V_{k-1} \end{array}$$

with

$$V_\lambda =< V_0, \ldots, V_d >, \qquad d = \lambda_1 - \lambda_2 \geq 0.$$

For the adjoint representation of K on $\mathfrak{p}_+$, one has an isomorphism $\mathfrak{p}_+ \simeq V_{(2,0)}$ by identifying ([O],p.9)

$$(X_{(0,2)}, X_{(1,1)}, X_{(2,0)}) \mapsto (v_0, v_1, v_2)$$

and similarly $\mathfrak{p}_- \simeq V_{(0,-2)}$ by

$$(X_{(-2,0)}, X_{(-1,-1)}, X_{(0,-2)}) \mapsto (v_0, -v_1, v_2).$$

The tensor products $V_\lambda \otimes \mathfrak{p}_\pm$ decompose into irreducible components

$$V_\lambda^\pm := V_\lambda \bigotimes \mathfrak{p}_\pm = \bigoplus_{\beta \in \Sigma_n^+} V_{\lambda \pm \beta}.$$

Oda ([O])p.10) gives explicit formulas for the projections

$$P_\lambda^{\pm\beta} : V_\lambda^\pm \to V_{\lambda \pm \beta}.$$

3. The discrete series and their minimal $K-$types

Oda denotes the four positive root systems for G compatible with $\Sigma_c^+ = \{(1,-1)\}$ as follows
(I) $\Sigma_I^+ = \{(1,-1), (2,0), (1,1), (0,2)\}$
(II) $\Sigma_{II}^+ = \{(1,-1), (1,1), (2,0), (0,-2)\}$

(III) $\Sigma^+_{III} = \{(1,-1),(2,0),(0,-2),(-1,-1)\}$
(IV) $\Sigma^+_{IV} = \{(1,-1),(-2,0),(-1,-1),(0,-2)\}$
For $J \in \{I, II, III, IV\}$

$$\Sigma^+_{J,n} := \Sigma^+_J \setminus \Sigma^+_c$$

is the set of non–compact positive roots in each case.
For the compact Cartan subgroup

$$H = \exp \mathfrak{h} = \left\{ r_1(\vartheta_1) r_2(\vartheta_2) = \begin{pmatrix} \cos\vartheta_1 & & \sin\vartheta_1 & \\ & \cos\vartheta_2 & & \sin\vartheta_2 \\ -\sin\vartheta_1 & & \cos\vartheta_1 & \\ & -\sin\vartheta_2 & & \cos\vartheta_2 \end{pmatrix}, \vartheta_1, \vartheta_2 \in \mathbb{R} \right\}$$

and their characters given by

$$r_1(\vartheta_1) r_2(\vartheta_2) \mapsto e(m_1\vartheta_1 + m_2\vartheta_2) \ ,$$

one gets in each case a subset Ξ_J of dominant weights by

$$\Xi_J = \{\Lambda = (\Lambda_1, \Lambda_2) \text{ dominant w.r.t. } \Sigma^+_c, < \Lambda, \beta > 0 \text{ for all } \beta \in \Sigma^+_{J,n}\}$$

Then the set $\bigcup_{J=I}^{IV} \Xi_J$ is the set of *Harish Chandra parameters* Λ for the discrete series representations π_Λ of G. The minimal K–type of $\pi_\Lambda|K$ is given the *Blattner parameter*

$$\lambda = \lambda_{\min} = \Lambda - \rho_c + \rho_n, \rho_n = (1/2)\Sigma_{\alpha\in\Sigma^+_{J,n}} \alpha, \rho_c = (1/2, -1/2).$$

One has

$$\begin{array}{rclcrcll} \lambda & = & (\Lambda_1 + \Lambda, \Lambda_2 + 2) & \text{for} & J & = & I & \text{the "holomorphic" case,} \\ & = & (\Lambda_1 - 2, \Lambda_2 - 1) & & & = & IV & \text{the "antiholomorphic" case,} \\ & = & (\Lambda_1 + \Lambda, \Lambda_2) & & & = & II & \multirow{2}{*}{the "large" cases.} \\ & = & (\Lambda_1, \Lambda_2 - 1) & & & = & III & \end{array}$$

In particular to a holomorphic Siegel modular form of weight k belongs a representation with $J = I$ and Blattner parameter $\lambda = (k, k)$, i.e. Harish–Chandra parameter $\Lambda = (k-1, k-2)$.

4. The P_J–principal series and their corner K–types

In a similar way as the discrete series representations are characterized by their minimal K–types, the P_J–principal series representations are by their "corner K–types". This is investigated in [MOII] and moreover in [Hi2].

A P_J–principal series representation π of G is an induced representation

$$\pi = \pi(P_J, \sigma, \nu_z) = \mathrm{Ind}_{P_J}^{G}(\sigma \otimes \nu_z \otimes 1_{N_J})$$

with

$$\sigma = (\epsilon, D) \text{ a representation of } M_J = \{\pm 1\} \times SL_2(\mathbb{R}),$$

where

$$\begin{aligned} D &= \mathcal{D}_n^{\pm} (n \in \mathbb{N}) \text{ is a (limit of a) discrete series representation of } SL(2,\mathbb{R}), \\ \epsilon &= \{\pm 1\} \to C^* \text{ is a character, and} \\ \nu_z &= \text{is a quasi–character of } A_J \text{ such that} \\ & \quad \nu_z \ (\mathrm{diag}\ (a_1, 1, a_1^{-1}, 1)) = a_1^z, \quad z \in \mathbb{C}. \end{aligned}$$

This π is irreducible and admissible except for the case $(\sigma, z) = ((1, \mathcal{D}_n^{\pm}), 0)$ with $n \geq 2$. By [MOII] one knows the K–types τ of π. In particular $\tau = \tau_\lambda$ has (with $\gamma = \mathrm{diag}\ (-1, 1, -1, 1)$ multiplicity one in π in the following cases: mult $(\tau_\lambda : \pi) = 1$ for

$$\begin{aligned} \lambda &= (l,l), l \geq n, l \equiv n \bmod 2 & (1) \\ &= (n,l), l \leq n, l \equiv n \bmod 2 \text{ for } D = \mathcal{D}_n^+, \epsilon(\gamma) = (-1)^n & \\ &= (l, l-1), l \geq n, & (2) \\ &= (n, l-1), l \leq n, l \equiv n \bmod 2 \text{ for } D = \mathcal{D}_n^+, \epsilon(\gamma) = -(-1)^n & \\ &= (l,l), l \leq -n, l \equiv n \bmod 2 & (3) \\ &= (l, -n), l \geq -n, l \equiv n \bmod 2 \text{ for } D = \mathcal{D}_n^-, \epsilon(\gamma) = (-1)^n & \\ &= (l+1, l), l \leq -n & (4) \\ &= (l+1, -n), l \geq -n, l \equiv n \bmod 2 \text{ for } D = \mathcal{D}_n^-, \epsilon(\gamma) = -(-1)^n & \end{aligned}$$

The irreducible K–module τ_λ is called the *corner K–type* of π, if

$$\begin{aligned} \lambda &= (n,n) && \text{in case } (1) \\ &= (n, n-1) && \text{in case } (2) \\ &= (-n, -n) && \text{in case } (3) \\ &= (-n+1, -n) && \text{in case } (4). \end{aligned}$$

One has $\dim V_\lambda = 1$ for (1) and (3) and $= 2$ for (2) and (4).

5. Generalized Whittaker functions and intertwining spaces

Pursuing the goal to generalize the classical problem of the existence and unicity of Whittaker models, one looks at the decomposition

$$G = \mathcal{N}\mathcal{A}K,$$

where later on we take (as in [O],[MOII],[Na])

$$\mathcal{N} = N, \text{ the unipotent radical of the minimal parabolic,}$$

or as in [Hi1,2]

$$\mathcal{N} = R_J \text{ the Jacobi group, where then } R_J \cap K \cong SO(2).$$

To keep things general as long as possible, we further take (denoting by $\widehat{G}_0$ the maximal family of classes of irreducible admissible representations of G_0)

$$\begin{aligned}(\pi, H_\pi) &\in \widehat{G} \text{ uniquely fixed by} \\ (\tau^*, V_{\tau^*}) &\in \widehat{K} \text{ with } i : V_{\tau^*} \hookrightarrow H_\pi\end{aligned}$$

and

$$(\zeta, \mathcal{F}_\zeta) \in \widehat{\mathcal{N}}.$$

It is convenient to introduce the following spaces:
– the *space of* (generalized ζ–pre–) *Whittaker functions* (=: "W–functions")

$$\begin{aligned}\mathcal{W}_\zeta &:= \mathcal{C}^\infty_\zeta(\mathcal{N}\backslash G) := \{W : G \to \mathcal{F}_\zeta \text{ smooth}, \\ &\qquad W(ng) = \zeta(n)W(g) \text{ for all } n \in \mathcal{N}, g \in G\}\end{aligned}$$

– the space of *W–functions of type τ*

$$\begin{aligned}\mathcal{W}_{\zeta,\tau} &:= \mathcal{C}^\infty_{\zeta,\tau}(\mathcal{N}\backslash G/K) \\ &= \{F : G \to \mathcal{F}_\zeta \otimes V_\tau \text{ smooth}, \\ &\quad F(ngk) = (\zeta(n) \otimes \tau(k)^{-1})F(g) \text{ for all } n \in \mathcal{N}, g \in G, k \in K\}\end{aligned}$$

– the space of "*radial functions*"

$$\begin{aligned}J_{\zeta,\tau} &= \mathcal{C}^\infty(\mathcal{A}, \zeta, \tau) \\ &= \{\phi : \mathcal{A} \to \mathcal{F}_\zeta \otimes V_\tau \text{ smooth}, (\zeta(m) \otimes \tau(m))\phi(a) = \phi(a) \\ &\quad \text{for all } a \in \mathcal{A}, m \in \mathcal{N} \cap K\}\end{aligned}$$

(These spaces also make sense and will later be used, when τ is a representation of K not necessarily irreducible).
The first two spaces carry a representation of G (by right translation) and to adopt with the usual formalism of induced representations one would replace ζ by $\tilde{\zeta}$ i.e. $\zeta(n)$ multiplied by the modular function $\delta_{\mathcal{N}}$ of $\mathcal{N}$.
Apparently there is the restriction map

$$\begin{aligned} \mathrm{res} \quad : \quad & \mathcal{W}_{\zeta,\tau} \to J_{\zeta,\tau} \\ & F \to \phi := F|\mathcal{A}. \end{aligned}$$

F is uniquely fixed by its restriction to the radial part. And as to be expected, it will be necessary to restrict these spaces further by looking at spaces $\mathcal{J}^0_{\zeta,\tau}, \mathcal{W}^0_{\zeta,\tau}$ and $\mathcal{W}^0_\zeta$ consisting of functions of "moderate growth" (in a sense evident at least if – as in the second case above – $\mathcal{A} = A_J \simeq \mathbb{R}_{>0}$).
The main problem of the existence and number of different Whittaker models for $(\zeta, \mathcal{F}_\zeta)$ translates into that of the investigation of the intertwining space

$$\mathcal{I}_{\zeta,\pi} := \mathrm{Hom}_{(\mathfrak{g}_c,K)}(H_\pi, \mathcal{W}_\zeta),$$

resp. using the hypothesis that $i : V_{\tau^0} \hookrightarrow H_\tau$ uniquely fixes π and taking τ and τ^* as contragredient, the space

$$\begin{aligned} \mathcal{I}_{\zeta,\tau} \quad := \quad & \mathrm{Hom}_K(V_{\tau^*}, \mathcal{W}_\zeta) \\ \simeq \quad & \mathcal{W}_\zeta \bigotimes_K V_\tau \simeq \mathcal{W}_{\zeta,\tau}, \end{aligned}$$

thus linking the first two spaces given above. Putting all this together, one wants to investigate the space of radial functions "coming from π"

$$\mathcal{J}_{\zeta,\pi}(\tau) := \bigcup_{i \in \mathrm{Hom}_K(\tau^*, \pi|_K)} \{\mathrm{res}\, Ti, T \in \mathcal{I}_{\zeta,\pi}\}$$

resp. its subspace $\mathcal{J}^0_{\zeta,\pi}(\tau)$ consisting of functions of moderate growth.

6. The Schmid operator and the shift operators

In [S] W. Schmid proposed a geometric construction of discrete series representations using a differential operator $\mathcal{D}$ which generalizes the Cauchy Riemann operator $\bar{\partial}$. In [Y] H. Yamashita refined this to the following general setting allowing for the determination of the spaces of interest here.

For an ON–basis $(X_1, \ldots, X_6)$ of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$, we define an operator $\nabla_{\zeta,\tau}$ acting on functions $F \in \mathcal{W}_{\zeta,\tau}$ by

$$\nabla_{\zeta,\tau}F(g) = \sum_{i=1}^{6} \mathcal{L}_{X_i}F(g) \otimes X_i \in \mathcal{F}_\zeta \otimes (V_\tau \otimes \mathfrak{p})$$

where

$$\mathcal{L}_X F(g) := \frac{d}{dt}F(g \exp tX)|_{t=0}.$$

As stated in 2. the adjoint representation of K on $\mathfrak{p}_c$ may be decomposed into two parts

$$V_\tau \otimes \mathfrak{p}_c = V_{\tau^+} \oplus V_{\tau^-}$$

with

$$V_{\tau^-} := \bigoplus_{\beta \in \Sigma^+} V_{\tau_{\lambda \pm \beta}}, \quad \text{where we use} \quad V_\tau = V_\lambda \text{ for } \tau = \tau_\lambda.$$

P_τ^- denotes the operator projecting on V_{τ^-}. Now, the Schmid operator

$$\mathcal{D}_{\zeta,\tau} : \mathcal{W}_{\zeta,\tau} \to \mathcal{W}_{\zeta,\tau^-}$$

is defined by the composition $\mathcal{D}_{\zeta,\tau} := P_\tau^- \nabla_{\zeta,\tau}$.
Similarly, one defines the “shift operators”

$$\mathcal{D}^\beta_{\zeta,\tau} : \mathcal{W}_{\zeta,\tau_\lambda} \to \mathcal{W}_{\zeta,\tau_{\lambda-\beta}}$$

by the composition of $\nabla_{\zeta,\tau}$ with the operators $P^{-\beta}$ projecting on $V_{\tau_{\lambda_\beta}}$. The $\mathcal{D}^\beta_{\zeta,\tau}$ are kind of constituents of $\mathcal{D}_{\zeta,\tau}$. The central fact to be used here is the following (Prop.2.1) in [Ya]:
Let $\pi = \pi_{\Lambda^*}$ be a representation of discrete series with Harish–Chandra parameter Λ^* (and Blattner parameter $\lambda^* = \Lambda^* - \rho_c + \rho_n$) then each $F \in \mathcal{W}_{\zeta,\tau_\lambda}$ is characterized by

$$\mathcal{D}_{\zeta,\tau}F = 0.$$

if Λ^* is “far from the walls of the Weyl chambers”, i.e. if $\lambda^* - \Sigma_{\beta \in Q}\beta$ is Σ_c^+ dominant for any $Q \subset \Sigma_n^+$.
Moreover the condition $\mathcal{D}_{\zeta,\tau}F = 0$ is equivalent to F being an eigenfunction for the Casimir operator Ω namely

$$\Omega F = (\|\Lambda\| - \|\rho\|)F \quad \text{for} \quad \rho = \rho_c + \rho_n.$$

Here the Casimir operator $\Omega \in Z(\mathfrak{g}_c) \subset U(\mathfrak{g}_c)$ is

$$\Omega = H_1^2 + H_2^2 - 4H_1 - 2H_2 - 4E_{2e_1}E_{-2e_1} - 2E_{e_1+e_2}E_{-e_1-e_2} - 4E_{2e_2}E_{-2e_2} - 2E_{e_1-e_2}E_{-e_1+e_2}.$$

When π is a principal series representation the Schmid operator doesn’t work, but the functions F may even then be characterized by being eigenfunctions of Ω.

7. The differential equations for the radial parts

The method traced by Yamashita and extensively used by Oda and his school now proceeds as follows.
As the functions $F \in \mathcal{W}_{\zeta,\tau}$ are fixed by their radial parts $\phi \in J_{\zeta,\tau}$, we look for certain differential operators D and their “radial parts” $\mathcal{R}(D)$, defined by the appropriate restrictions

$$\begin{array}{ccccc} & \mathcal{W}_{\zeta,\tau} & \stackrel{D}{\rightarrow} & \mathcal{W}_{\zeta,\tau'} & \\ \mathrm{res} & \downarrow & & \downarrow & \mathrm{res} \\ & \mathcal{J}_{\zeta,\tau} & \underset{\mathcal{R}(D)}{\rightarrow} & \mathcal{J}_{\zeta,\tau'} & \end{array}$$

Then, the functions $\phi \in \mathcal{J}_{\zeta,\tau}$, being annihilated by the radial part $\mathcal{R}(\mathcal{D}_{\zeta,\tau})$ of the Schmid operator or being eigenvalues of the Casimir $\mathcal{R}(\Omega)$, are fixed by a system of difference–differential equations which comes up by the expansion

$$\phi(a) = \Sigma_{(i)} c_{(i)}(a) w_{(i)}$$

where $(w_{(i)})$ symbolizes a basis of $\mathcal{F}_\zeta \otimes V_\tau$ which has to be filled in for each case of representations π of G and ζ of $\mathcal{N}$ under investigation.
The application of the (at most) second order differential operators produce recurrent relations of the type

$$D_{(i+2)} C_{(i+2)} + D_{(i+1)} C_{(i+1)} + D_{(i)} C_{(i)} = 0$$

where here $D_{(i)}$ are differential operators (in two variables if $\mathcal{A} = A$ and in one if $\mathcal{A} = A_J$) and the changes $(i) \longrightarrow (i+1)$ or $(i+2)$ indicate that in the tupels discribing the bases $w_{(i)}$ each component i_j is changed at most from i_j to $i_j + 1$ or $i_j + 2$. Then in each case these systems are solved by finding ingenious substitutions and reducing finally to a Whittaker–type differential equation, where one solution is singled out by a growth condition.

8. The case of the large discrete series

To illustrate the procedure by an example different from these in the résumes given later, we reproduce parts of Oda [O].

Here we take the usual Iwasawa decomposition

$$G = NAK$$

and take for the representations $(\rho, \mathcal{F}_\zeta)$ of $\mathcal{N} = N$ here simply the characters η fixed by the purely imaginary numbers

$$\eta_0 = d\eta(E_{l_1 - l_2}), \quad \eta_3 := d\eta(E_{2l_2}).$$

(the general case is considered in Narita [Na]) and for π the cases of the "large" discrete series $\pi = \pi_\Lambda, \Lambda \in \Xi_{II} \cup \Xi_{III}$. To describe the radial part of the Schmid operator and the shift operators Oda introduces "macro symbols" which appear again in several papers from his school. These are linear differential operators $\mathcal{L}_j^\pm$ and $\mathcal{S}^\pm$ defined for smooth functions φ living on $A = \{\,\mathrm{diag}\,(a_1, a_2, a_1^{-1}, a_2^{-1}), a_1 a_2 > 0\}$ by

$$\begin{array}{rcl} \mathcal{L}_j^\pm \varphi &=& (\partial_j \pm 2ia_j^2 \eta(E_{2j}))\varphi, \quad \partial_j = \mathcal{L}_{H_j},\ j = 1, 2, \\ \mathcal{S}^\pm \varphi &=& (a, a_2^{-1} \eta(E_{l_1 - l_2}) \pm i a_1 a_2 \eta(E_{l_1 + l_2}))\varphi. \end{array}$$

η being a character, we have in this case $\mathcal{F}_\zeta = \mathbb{C}$ and $\phi \in J_{\zeta,\tau} = \mathcal{C}^\infty(A, V_\lambda)$ may be written as

$$\phi(a) = \sum_{k=0}^{d} c_k(a) v_k \quad \text{with} \quad c_k : A \to \mathbb{C}.$$

Then the condition $\mathcal{R}(\mathcal{D}_{\zeta,\tau})\phi = 0$ produces a system of equations for (c_k) which may be reduced to the system ([O]p.17):

$$\begin{array}{rcl} (\mathcal{L}_1 - \lambda_1 - 1)c_d(a) + \mathcal{S}^+ c_{d-1}(a) &=& 0, \\ (\mathcal{L}_1 - \lambda_1 - 1)c_{d-1}(a) + \mathcal{S}^+ c_{d-2}(a) &=& 0, \\ \mathcal{S} c_d(a) + (\mathcal{L}_2^+ - \lambda_1 + d - 1)c_{d-1}(a) &=& 0, \\ (\mathcal{L}_1 \pm \lambda_2 - d)c_{d-2}(a) - 2\mathcal{S}_{d-1}^+(a) + (\mathcal{L}_2^- + \lambda_1 - d)c_d(a) &=& 0. \end{array}$$

By the substitution

$$c_k(a) = a_1^{\lambda_1 + 1 - d} a_2^{\lambda_1} \left(\frac{a_1}{a_2}\right)^k e^{-i\eta(E_{(0,2)})a_2^2} h_k(a)$$

this system is for *generic* η (i.e. $\eta_0 \cdot \eta_3 \neq 0$) equivalent to

$$\begin{array}{rcl} (\partial_1 \partial_2 - \mathcal{S}^2) h_d(a) &=& 0 \\ \{(\partial_1 + \partial_2)^2 + (2\lambda_2 - 2)(\partial_1 + \partial_2) - (2\lambda_2 + 1) - 2\mathcal{S}' \partial_2\} h_d(a) &=& 0 \end{array}$$

where $\mathcal{S} = \mathcal{S}^+ = \mathcal{S}^-$ and $\mathcal{S}' = (1/2)(\mathcal{L}_2^+ - \mathcal{L}_2^-)$. This system is recognized as a holonomic system of rank 4 and thus one has $\dim_c \ker(\mathcal{D}_{\eta,\tau_\lambda}) = 4$.
Using results by Kostant on the relation between $(\mathfrak{g}_c, K)$– and G–intertwining operators and relating the h_d to classical Whittaker functions, Oda finally gets the result ([O],p.19)
For $\Lambda \in \Xi_{II}$ one has

$$\begin{array}{rcll} \mathrm{Hom}_G^\infty(\pi_\Lambda^*, \mathcal{C}_\eta^\infty(N\backslash G)) & \simeq & \mathbb{C} \quad \text{if} & \mathrm{Im}\ \eta_3 < 0 \\ & = & 0 & \mathrm{Im}\ \eta_3 > 0. \end{array}$$

With the notation introduced in section 5 this may also be written as

$$\begin{array}{rcll} \dim_{\mathbb{C}} J^0_{\eta,\pi}(\tau_\lambda) & = & 1 \quad \text{if} & \mathrm{Im}\ \eta_3 < 0 \\ & = & 0 & \mathrm{Im}\ \eta_3 > 0. \end{array}$$

(For the holomorphic and antiholomorphic cases π_Λ, $\Lambda \in \Xi_I \cup \Xi_{IV}$, it was known before, that there are no Whittaker models for these characters η).

9. The case of the principal series induced from the Jacobi parabolic

As shown in the paper [MOII], if $\pi = \pi(P_J, \sigma, \nu_z)$ is a principal series representation induced from the Jacobi parabolic group, things go very similar to the case discussed above, as π is fixed by a "corner K–type" τ_λ^* which does the same as the minimal K–type did for the discrete series and the condition that a radial function ϕ is annihilated by the Schmid operator is replaced by the condition that ϕ is an eigenfunction for the Casimir operator Ω, which in turn is expressed using again the shift operators. Here the main point (before doing again the delicate job of discussing the differential equations!) is to determine the eigenvalue of Ω characterizing π. This is done by seeing ([MOII],Prop.7.1) π as a quotient $(\mathfrak{g}_c, K)$–module of

$$\pi'' = \mathrm{Ind}_P^G(\sigma_M \otimes \mu'' \otimes 1_N),$$

where $\sigma_M \in \widehat{M}$ is fixed by

$$\sigma_M(\gamma_{2e_1}) = \epsilon(\gamma_{2e_1}) \quad \text{and} \quad \sigma_M(\gamma_{2e_2}) = (-1)^k$$

($\gamma_{2e_1} = \mathrm{diag}\ (-1,1,-1,1), \gamma_{2e_2} = \mathrm{diag}\ (1,-1,1,-1)$ and μ'' is the quasi–character of A given by

$$\mu''(\mathrm{diag}\ (a_1, a_2, a_1^{-1}, a_2^{-1})) = a_1^z a_2^{1-k}.$$

Hence the radial part $r(\Omega) := a^{-\rho}R(\Omega)a^{\rho} \quad (\rho = \rho_n + \rho_c)$ has the eigenvalue ([MOII],Prop.7.2)

$$\chi_\Omega = z^2 + (k-1)^2 - 5.$$

A Resumé of (a part) of H. Yamashita: Embeddings of discrete series into induced representations of semisimple Lie groups I – General theory and the case of $SU(2,2)$ – Japan. J. Math. 16 (1990) 31–95

A.1 The Problem

Embedding of any, not necessarily holomorphic, discrete series representation π into various important induced G–modules such as principal series and (generalized) Gelfand–Graev representations, by using an appropriate differential operator $\mathcal{D} = \mathcal{D}_\tau$. Here, τ indicates the lowest K–type occuring in π (which then characterizes π).

A.2 Discrete Series

$\triangle$ root system of $\mathfrak{g}_c$ with respect to $\mathfrak{t}_c$, $\mathfrak{k} \supset \mathfrak{t}$ Cartan subalgebra, $\triangle_c$, $\triangle_n$ compact resp. non compact roots, W, W_c Weyl groups of $\triangle$ resp. $\triangle_c$

$$\begin{array}{rl} \Xi := \{\Lambda \in \mathfrak{t}_c^*, \ \ (\Lambda,\alpha) \neq & 0 \text{ for any } \alpha \in \triangle \ (\Lambda \text{ is } \triangle\text{–}\underline{\text{regular}}), \\ K \supset T := & \exp \mathfrak{t} \to \mathbb{C}^* \\ & \exp H \mapsto \exp < \Lambda + \rho, H > \ \text{ is a unitary character} \\ & \hfill (\Lambda + \rho \text{ is } K \ \underline{\text{integral}})\} \end{array}$$

$\rho :=$ half sum of positive roots in $\triangle$.

Remark: Ξ is W–stable.

The W_c– orbits in Ξ parametrize the discrete series for G

$$\Xi_c^+ := \{\Lambda \in \Xi, \ < \Lambda, \alpha > \geq 0 \text{ for every } \alpha \in \triangle_c^+ \ (\Lambda \text{ is } \triangle_c^+\text{– } \underline{\text{dominant}})\}$$

Remark: $\Xi_c^+ \simeq \Xi / W_c$ the <u>"H–C parameters"</u> of π_Λ.

$q : Z(\mathfrak{g}_c) \xrightarrow{\sim} U(\mathfrak{t}_c)^W$ the H–C isomorphism
$\mathfrak{t}_c^*/W \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{alg}}(Z(\mathfrak{g}_c), \mathbb{C})$, the infinitesimal characters,
$\mu \longmapsto \chi_\mu$ with $\chi_\mu(D) = \mu(q(D))$ for $D \in \mathbb{Z}(\mathfrak{g}_c)$

$$\pi_\Lambda \text{ (discrete series)} \longmapsto \chi_\Lambda \ : \ \begin{array}{l} Z(\mathfrak{g}_c) \text{ acts through differentiation} \\ \text{on the space of } C^\infty\text{–vectors for } \pi_\Lambda \\ \text{by scalars } \chi_\Lambda^{(D)} \text{ for } D \in Z(\mathfrak{g}_c) \end{array}$$

Remark: There are $|W/W_c|$ inequivalent π_Λ's with the same χ_Λ.

(τ_μ, V_μ) finite dimensional irreducible representations of K with $\triangle_c^+$–highest weight $\mu \in \mathfrak{A}_c^*$, $\triangle_c^+$–dominant and K–integral.

$m_\Lambda(\mu) :=$ multiplicity of τ_μ in $\pi_\Lambda|_K$
$\lambda := \Lambda - \rho_c + \rho_n = \Lambda + 2\rho_n - \rho$ $\quad$ ($\triangle_c^+$– dominant, K– integral).

Blattner formula:

$$m_\lambda(\mu) = \sum_{w \in W_c} \det w \; Q(w(\mu + \rho_c) - (\lambda + \rho_c)) \text{ for } \mu \text{ as above,}$$

where $Q(\beta) =$ number of distinct ways that $\beta \in A_c^*$ can be written as a sum of non–negative integral multiples of elements of $\triangle_n^+ = \triangle^+ \cap \triangle_n$.

Proposition: $m_\Lambda(\lambda) = 1$. λ the "Blattner parameter" of π_Λ, τ_λ lowest K–type of π_Λ

A.3 Differential operators

$\mathcal{C}_\tau^\infty(G) := \{\phi : G \to V \text{ smooth, } \phi(kg) = \tau(k)\phi(g) \text{ for } k \in K\}$ where (τ, V) finite dimensional continuous K–representation.

$$\begin{array}{rcl} R_Y\phi(g) & = & \frac{d}{dt}\phi(g \exp tY)|_{t=0} \text{ for } Y \in \mathfrak{g}, \\ L_Y\phi(g) & = & \frac{d}{dt}\phi(\exp(-tY)g)|_{t=0}, \end{array}$$

$$\begin{array}{rcl} \nabla_\tau\phi : G & \to & V \otimes \mathfrak{p}_c \\ x & \to & \sum\limits_{i=1}^{2n} L_{X_i}\phi(x) \otimes X_i, \end{array}$$

where $(X_i)_{i=1,\dots,2n}$ ON–basis of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ ($n = |\triangle_n^+|$).
Then

$$\nabla_\tau : C_\tau^\infty(G) \to C_{\tau \otimes Ad_C}^\infty(G),$$

Ad_c the adjoint representation of K on $\mathfrak{p}_c$, commutes with the G– and $\mathfrak{g}_c$– actions from the right.

Now $\tau = \tau_\lambda$

$$(\tau_\lambda \otimes Ad_{\mathfrak{p}_c},\ v_\lambda \otimes \mathfrak{p}_c) \simeq (\tau_\lambda^+,\ V_\lambda^+) \otimes (\tau_\lambda^-,\ V_\lambda^-)$$

where $\tau_\lambda^\pm = \bigotimes_{\beta \in \triangle_n^+} \tau_{\lambda \pm \beta}$.

$P_\lambda : V_\lambda \otimes \mathfrak{p}_c \to V_\lambda^-$ the projection

$$\begin{array}{rcl} \mathcal{D}_\lambda : \mathcal{C}_{\tau_\lambda}^\infty(G) & \to & \mathcal{C}_{\tau_\lambda^-}^\infty(G) \text{ the "Schmid operator"} \\ \phi & \mapsto & P_\lambda(\nabla_\lambda\phi(g)) \end{array}$$

Ω, the Casimir operator on G, induces an elliptic differential operator on $\mathcal{C}^\infty_{\tau_\lambda}(G)$, and one has

$$\Omega\phi = (\|\Lambda\|^2 - \|\mathfrak{p}\|^2)\phi \text{ with } \Lambda = \lambda + \mathfrak{p}_c - \mathfrak{p}_n,\ \mathfrak{p} = \mathfrak{p}_c + \mathfrak{p}_n$$
$$\text{for } \phi \in \text{Ker } \mathcal{D}_\lambda.$$

There are two Hilbert spaces

$\tilde{\mathcal{H}}_\lambda := \{\phi \in \mathcal{C}^\infty_{\tau_\lambda}(G),\ \Omega\phi = (\|\Lambda\|^2 - \|\mathfrak{p}\|^2)\phi,\ |\phi\|_{L^2} < \infty\}$,

$\tilde{\mathcal{H}}_\lambda := \{\phi \in \mathcal{C}^\infty_{\tau_\lambda}(G),\ \mathcal{D}_\lambda\phi = 0,\ \|\phi\|_{L^2} < \infty\}$,

and one has (Th. 1.5)

$$\mathcal{H}_\lambda = \tilde{\mathcal{H}}_\lambda$$

and π_Λ is realized by right translation on $\mathcal{H}_\lambda$.

λ lowest highest weight of π_Λ "far from the walls"

$$\Leftrightarrow \lambda - \sum_{\beta\in Q}\beta \text{ is } \triangle_c^+ \text{ dominant for any } Q \subset \triangle_n^+$$
$$\Rightarrow \mathcal{H}_\lambda^\circ = \text{ Ker } (\mathcal{D}_\lambda)^\circ,$$

that is, any right K–finite function in $\mathcal{C}^\infty_{\tau_\lambda}(G)$ annihilated by $\mathcal{D}_\lambda$ is square–integrable on G.

$N \subset G$ closed subgroup

$(\eta, \mathcal{F})$ continuous representation of N

$\mathcal{C}^\infty(G;\eta) := \{\phi : G \to \mathcal{F} \text{ smooth};\ \phi(gn) = \tilde{\eta}(n)^{-1}\phi(g), (n,g) \in N \times G\}$

with left G–action, and $\tilde{\eta}(n) = \delta_N(n)^{-1/2}\eta(n),\ \delta_N$ modular function of N for the left Haar measure.

Abbreviation for this

$$\pi(\eta) = \mathcal{C}^\infty_\eta \text{ Ind}_N^G(\eta).$$

(τ, V) finite–dimensional representation of K

$\mathcal{C}^\infty_\tau(G;\eta) = \{F : G \to V \otimes \mathcal{F} \text{ smooth};\ F(kgn) = (\tau(k) \otimes \tilde{\eta}(n)^{-1})F(g),\ k \in K, n \in N\}$

For the space of K–finite vectors for $\pi(\eta)$, there is a decomposition into a direct sum of irreducible K–modules (where μ runs over $\triangle_c^+$–dominant K integral elements in $\mathfrak{t}_c^*$)

$$\begin{array}{rcl} \bigotimes_\mu (V_\mu^* \otimes C^\infty_{\tau_\mu^*}(G;\eta)) & \simeq & \mathcal{C}^\infty(G;\eta)^\circ \\ v^* \otimes F & \longrightarrow & F_{v^*} \text{ with } F_{v^*}(g) = < v^*, F(g) > \end{array}$$

The in this context central operator

$$\mathcal{D}_{\lambda,\eta} : \mathcal{C}^\infty_{\tau_\lambda}(G;\eta) \to \mathcal{C}^\infty_{\tau_\lambda^-}(G;\eta)$$

is induced from

$$\mathcal{D}_\lambda \otimes \mathrm{Id}_\mathcal{F} : \underbrace{(\mathcal{C}^\infty_{\tau_\lambda}(G) \otimes \mathcal{F})^N}_{=\mathcal{C}^\infty_{\tau_\lambda}(G;\eta)} \to \underbrace{(\mathcal{C}^\infty_{\tau_\lambda^-}(G) \otimes \mathcal{F})^N}_{\mathcal{C}^\infty_{\tau_\lambda^-}(G,\eta)} \ .$$

A.4 $\mathcal{D}_{\lambda,\eta}$ and the embedding of discrete series

Remark: w_0 longest element of the compact Weyl group

$$\Rightarrow \begin{array}{ccc} \Xi_c^+ & \xrightarrow{\sim} & \Xi_c^+ \\ \Lambda & \longmapsto & -w_o\Lambda \end{array}$$

is a bijection and π_λ^* is unitarily equivalent to $\pi_{-w_o\Lambda}$. Hence, one may deal with π_λ^* instead of π_Λ.

$$\mathcal{J}_{\Lambda,\eta} := \ \mathrm{Hom}_{(\mathfrak{g}_c,K)}(\pi_\lambda^*, \pi(\eta))$$

There is (Prop. 2.1) an injective linear map

$$\mathcal{J}_{\lambda,\eta} \to \mathrm{Ker}(\mathcal{D}_{\lambda,\eta}) \subset \mathcal{C}^\infty_{\tau_\lambda}(G,\eta), \ \lambda = \Lambda - \mathfrak{p}_c + \mathfrak{p}_n$$

given by

$$i \mapsto F^{[i]} \text{ with } < v^*, F^{[i]}(\cdot) >= F^{[i]}_{v^*}(\cdot) = i(v^*) \text{ for all } v^* \in V_\lambda^*.$$

Moreover, one has an isomorphism (Th. 2.4)

$$\mathcal{J}_{\Lambda,\eta} = \mathrm{Hom}_{(\mathfrak{g}_c,K)}(\pi_\Lambda^*, \pi(\eta)) \simeq \mathrm{Ker}\ (\mathcal{D}_{\lambda,\eta})$$

if the Blattner parameter λ is far from the walls and $(\eta, \mathcal{F})$ is a weakly cyclic representation of N, i.e. there exists a non–zero continuous linear functional T on $\mathcal{F}$ such that $< T, \eta(n)v >= 0$ for all $n \in N$ implies $v = 0$. (Any irreducible unitary representation is weakly cyclic).

Now, the problem of describing the embeddings

$$\pi_\lambda^* \to \pi(\eta)$$

as $(\mathfrak{g}_c, K)$–Modules is reduced to solving the differential equation

$$\mathcal{D}_{\lambda,\eta}F = 0 \text{ for } F \in \mathcal{C}^\infty_{\tau_\lambda}(G;\eta).$$

A.5 General theory of embeddings into the principal series

Then, the preceding general theory is specialized to the important cases where the G–modules $\pi(\nu) = \mathcal{C}^\infty-\ \mathrm{Ind}_N^G(\nu)$ are generalized principal series representations induced from various parabolic subgroups. We take:

$P = MAN$ Langlands decomposition of a parabolic,

$$L = MA = M \times A \text{ the Levi subgroup,}$$

ξ an admissible irreducible representation of L,
$\xi = \sigma \otimes e^\psi$, σ irreducible admissible M–module, $\psi \in \mathfrak{a}_c^*$,
$\pi(\xi \otimes 1_N) = C^\infty - \mathrm{Ind}_P^G(\xi \otimes 1_N)$,
$\pi_P[\xi]$ its extension to a continuous representation on the Hilbert space $\mathcal{H}_\xi$, is admissible and unitary if so is $\xi = \sigma \otimes e^\psi$.
$\pi_P[\xi]$ is called a principal P–series representation of G.

By a theorem of Casselman each irreducible admissible representation of G is embedded, as a $(\mathfrak{g}_c, K)$–module, into some member of the principal P–series representation. The main part of the paper makes this explicit for the discrete series of $G = SU(2,2)$.

B Resumé of H. Narita: Fourier expansion of holomorphic Siegel modular forms with respect to the minimal parabolic subgroup. Math. Z. 231 557–588 (1999)

B.1 Problem:

Given a holomorphic Siegel modular form F of degree 2, lift F to a function f on $G = Sp(2,\mathbb{R})$ and expand $f(xg)$ for fixed $g \in G$ as a function of $x \in N$, the unipotent radical of the standard Borel B, as a series indexed by the representations $\eta \in \widehat{N}$.

B.2 Determination of the unitary dual $\widehat{N}$

$$N = \left\{ \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_4 & & \\ & 1 & & \\ & & 1 & \\ & & -x_4 & 1 \end{pmatrix}, x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

$$\mathfrak{n} = < X_1 = E_{2e_1}, X_2 = E_{e_1+e_2}, X_3 = E_{2e_2}, X_4 = E_{e_1-e_2} >_{\mathbb{R}}$$

By Kirillov theory we have

$$\widehat{N} \simeq \mathfrak{n}^* \,/\, \mathrm{Ad}^* N.$$

There are exactly 3 types of representations $\eta = \eta_l \in \widehat{N}$, where $l = \sum_{j=1}^4 \xi_j l_j \in \mathfrak{n}^*$, $< l_1, \ldots, l_4 >$ dual basis of $< X_i >$,
namely

(1) η_l is the character given by $(l = (0,0,\xi_3,\xi_4))$
$\eta_l(x) = e(x_3\xi_3 + x_4\xi_4)$

(2) η_l realized on $L^2(\mathbb{R})$ by $(l = (0, \xi_2 \neq 0, 0, 0))$
$\eta_l(x)\phi(t) = e((x_2 + x_3 t)\xi_2)\phi(t + x_4)$ for $\phi \in L^2(\mathbb{R})$

(3) η_l realized on $L^2(\mathbb{R})$ by $(l = (\xi_1 \neq 0, 0, \xi_3, 0))$
$\eta_l(x)\phi(t) = e((x_1 + 2x_2 t + x_3 t^2)\xi_1 + x_3\xi_3)\phi(t + x_4)$ for $\phi \in L^2(\mathbb{R})$.

All three types are got as induced representations

$$\eta_l = \mathrm{Ind}_{M_l}^N \chi_l$$

where $M_l = N$ in case (1) and $M_l = \{(x_1, x_2, x_3, 0)\}$ in cases (2) and (3). (2) is here called *Schrdinger* and (3) *generic representation.*

B.3 Generalized Whittaker functions for the holomorphic discrete series

To the holomorphic Siegel form F belongs a holomorphic discrete series representation of G. So, let be

$\pi_\wedge$ a holomorphic discrete series representation with HC–parameter $\wedge$ and Blattner parameter λ and minimal K–type τ_λ.

$\mathcal{C}^\infty_\eta\ (N\backslash G) := \{f : G \rightharpoondown V_\eta \text{ smooth; } f(xg) = \eta(x)f(g), x \in N\}$, where $V_\eta \subset H_\eta$ the subspace of $\mathcal{C}^\infty$–vectors.

$\mathcal{C}^\infty_{\eta,\tau^*_\lambda}\ (N\backslash G/K) := \{W : G \rightharpoondown V_\eta \otimes V^*_\lambda \text{ smooth;}$ $W(xgk) = \eta(x) \otimes \tau^*_\lambda(k)^{-1}W(g), x \in N, k \in K\}$ where V^*_η is the representation space of the contragredient τ^* of τ.

One has by restriction to the minimal K–type τ_λ of $\pi_\wedge$

$$\mathrm{Hom}_{(\mathfrak{g}_c,K)}(\pi_\wedge, \mathcal{C}^\infty_\eta(N\backslash G)) \overset{\mathrm{res}|_{\tau_\lambda}}{\rightarrow} \mathrm{Hom}_K(\tau_\lambda, \mathcal{C}^\infty_\eta(N\backslash G)) \simeq \mathcal{C}^\infty_{\eta,\tau_\lambda}(N\backslash G/K)$$

and more sharply, the "well known fact"

$$\mathrm{Hom}_{(\mathfrak{g}_c,K)}(\pi_\wedge, \mathcal{C}^\infty\eta_\eta(N\backslash G)) \simeq \{W \in \mathcal{C}^\infty_{\eta,\tau^*_\lambda}(N\backslash G/K); dR_X W = 0 \text{ for } X \in \mathfrak{p}_-\},$$

where R is the respresentation of G by right translation.

Determination of this space by solving a system of partial differential equations for the "radial part" W_A of W:
Obviously $W \in \mathcal{C}^\infty_{\eta,\tau^*}(N\backslash G/K)$ is determined by its restriction $W_A := W|_A$ to

$$A = \exp\,\mathfrak{a} = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}), a_1, a_2 \in \mathbb{R}_{>0}\}.$$

Compute the A–radial part $\rho_A(R_{X_{-\beta}})$ for $X_{-\beta} \in \mathfrak{p}_-$. (using the "macro" symbols introduced in a former paper by Oda).
Example (s. Lemma 6.1)

$$\rho_A(R_{X_{(-1,-1)}})W_A(a) = 2\sum_{k=0}^{d}(kw_{k-1}(a) + (\underbrace{a_1a_2^{-1}d\eta(E_{l_1-l_2}) - ia_1a_2d\eta(E_{l_1+l_2})}_{=:\varphi^-})w_k(a))v^*_k,$$

where

$$W_A(a) = \sum_{k=0}^{d} w_k(a)v^*_k,$$

$$\{v_k^*\}_{0\le k\le d} \quad \text{the standard basis of} \quad V_\lambda^* = V_{\lambda'}, \lambda' = (-\lambda_2, -\lambda_1).$$

From the equations $\rho_A(R_{X_{-\beta}})W_A = 0$, $X_{-\beta} \in \mathfrak{p}_-$, one gets a system of differential equations for the w_k which can be solved (Prop. 7.1) for the different cases of $\eta \in \widehat{N}$:

Case (1) (the one–dimensional representation)

$$\begin{aligned} w_k(a) &= 0,\ 0 \le k \le d \qquad \text{if } \eta \text{ is a unitary character with } \xi_4 \ne 0, \\ &= 0,\ 0 \le k \le d-1, \\ &= C a_1^{\lambda_1} a_2^{\lambda_2} \exp(-2\pi a_2^2 \xi_3), \qquad \text{if } \eta \text{ as above with } \xi_4 = 0. \end{aligned}$$

Case (2) (the "Schrdinger representation" on $L^2(\mathbb{R})$)

$$w_k(a) = C a_1^{\lambda_1 - k} a_2^{\lambda_2 + k} t^k \exp(2\pi a_2^2 \xi_2 t), \quad 0 \le k \le d.$$

Case (3) (the "generic representation" on $L^2(\mathbb{R})$)

$$w_k(a) = C a_1^{\lambda_1 - k} a_2^{\lambda_2 + k} t^k \exp(-2\pi(a_1^2 \xi_1 + a_2^2 \xi_3 + a_2^2 \xi_1 t^2)),\ 0 \le k \le d.$$

Here C a constant and t the variable in $H_\eta = L^2(\mathbb{R})$.
An immediate consequence is (Th. 7.2)

$$\begin{aligned} \mathrm{Hom}_{(\mathfrak{g}_c, K)}(\pi_\wedge, \mathcal{C}_\eta^\infty(N\backslash G)) &= \{0\} \quad \text{Case (1) }, \xi_4 \ne 0 \\ &= \mathbb{C} \quad \text{Case (1) }, \xi_4 = 0 \\ &= \{0\} \quad \text{Case (2) }, \\ &= \mathbb{C} \quad \text{Case (3) }, \xi_1 > 0 \\ &= \{0\} \quad \text{Case (3) }, \text{otherwise.} \end{aligned}$$

Another formulation of this Theorem on the *existence and unicity of the W–model* for discrete series representations is (Th 7.3)

$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_c, K)}(\pi_\wedge, A_\eta(N\backslash G)) = 1$
$\Longleftrightarrow$
η a unitary character with parameter $\xi_3 \ge 0, \xi_4 = 0$
or
η a generic representation with $\xi_1 > 0, \xi_3 \ge 0$.

Here

$$\begin{aligned} A_\eta(N\backslash G) \;=\; & \{\varphi \in \mathcal{C}_\eta^\infty(N\backslash G); C_{\varphi,v}(\cdot) = <v, \varphi(\cdot)> \\ & K\text{–finite for all } v \in H_\eta \text{ and } \varphi_{\varphi,v}|_A \text{ of moderate growth}\} \end{aligned}$$

B.4 The "Fourier expansion with respect to N"

Notation:
$\Gamma = \mathrm{Sp}\,(2,\mathbb{Z}), N_Z = N \cap \Gamma$,
$\pi_\wedge$ holomorphic discrete series representation as in 2.
f a holomorphic Siegel modular form of weight τ_λ with respect to Γ, i.e.
$f : G \to V_\lambda^*$ smooth with

i) $f(\gamma g k) = \tau_\lambda^*(k)^{-1} f(g)$ for $\gamma \in \Gamma, g \in G, k \in K$,

ii) $\rho \simeq \pi_\wedge$ for ρ the representation of G by right translation on
$V_f =< c_{f,v}^{g_0}, v \in V_\lambda, g_0 \in G >$ and
$dR_X c_{\delta,v} = 0$ for all $X \in \mathfrak{p}_-$,

where $C_{f,v}(g) :=< f(g), v >$ and $\varphi^{g_0}(g) = \varphi(gg_0)$.
$L^2(N_\mathbb{Z}\backslash N) \simeq \oplus_{\eta\in\widehat{N}} m(\eta) H_\eta \simeq \oplus_{\eta\in\widehat{N}} \mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N)) \otimes_\mathbb{C} N_\eta$
$\{\phi_1^{(\eta)}, \ldots, \phi_{m(\eta)}^{(\eta)}\}$ basis of $\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$.
Then we have a *first decomposition* of f (for $x \in N, g \in G$)

$$f(xg) = \sum_{K=0}^{d} \sum_{\eta\in\widehat{N}} \sum_{M=1}^{M(\eta)} (\phi_M^{(\eta)} \otimes W_f^{(\eta,k)}(g))(x) \otimes v_k^*.$$

Here it comes out, that
$$W_f^{(\eta,k)} \in \mathcal{C}_\eta^\infty(N\backslash G)$$
and
$$W_f^{(\eta)} \in \mathcal{C}_{\eta,\tau_\lambda^*}^\infty(N\backslash G/K) \quad \text{for} \quad W_f^{(\eta)}(g) := \sum W_f^{(\eta,k)}(g) \otimes v_k^*.$$

Remark: $W_f^{(\eta)}$ is a generalized W–function with K–type τ_λ. By the results of section 2, one has now only to look at the cases (1) and (3).
There is some work to be done to determine explicitely the bases $\{\phi_1^{(\eta)}, \ldots, \phi_{m(\eta)}^{(\eta)}\}$ of $\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$. For the unitary character η with $\xi_4 = 0$ one has rather easily
$$\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N)) = \mathbb{C}\phi^{\xi_3}$$
with $\phi^{\xi_3} : \mathbb{C} \to L^2(N_\mathbb{Z}\backslash N))$, where $\phi^{\xi_3}(1)(x) = e(\xi_3 x_3)$ (and ergo only non-trivial for $\xi_3 \in \mathbb{Z}$!). For the generic representation η from case (3) with parameters ξ_1, ξ_3 by using differential equations one comes up with intertwining operators
$$\phi_M^{\xi_1,\xi_3} : H_\eta = L^2(\mathbb{R}) \to L^2(N_\mathbb{Z}\backslash N)$$
specified by
$$\phi_M^{\xi_1,\xi_3}(h_i)(x) := \sum_{m\in\mathbb{Z}} h_i\Big(x_4 + \frac{2\xi_1 m + M}{2\xi_1}\Big) \cdot e\Big(\xi_1 x_1 + (2\xi_1 m + M)x_2 + \Big(\frac{(2\xi_1 m + M)^2}{4\xi_1} + \xi_3\Big)x_3\Big).$$

Here are
$h_i (i \in \mathbb{N}_0)$ the Hermite polynomials in $L^2(\mathbb{R})$

$$\begin{aligned} M \in \mathfrak{M}(\xi_1, \xi_3) &= \{M \in \mathbb{Z}; (M^2/(4\xi_1)) + \xi_3 \in \mathbb{Z}\} /\!\sim \quad \text{for} \quad \xi_1 \in \mathbb{Z} \\ &= \emptyset \qquad \text{for} \quad \xi_1 \notin \mathbb{Z} \end{aligned}$$

with $M \sim M' \iff M \equiv M' \bmod 2\xi_1$.
All this can be put together to give the expansion (for $x \in N, a \in A$)

$$\begin{aligned} f(xa) &= \sum_{\xi_3 \in \mathbb{N}_0} C^f_{\xi_3} w_{\xi_3}(a) e(\xi_3 x_3) v_d^* \\ &\quad + \sum_{k=0}^{d} \sum_{\xi_1 \in \mathbb{N}} \sum_{\xi_3 \in \mathbb{N}_0} \sum_{M \in \mathfrak{M}(\xi_1,\xi_3)} C^f_{\xi_1,\xi_3,M} \cdot \sum_{m \in \mathbb{Z}} \\ &\quad \Big\{ w^k_{\xi_1,\xi_3,M}(a,m) e\Big(\xi_1 x_1 + (2\xi_1 m + M) x_2 + \Big(\frac{(2\xi_1 m + M)^2}{4\xi_1} + \xi_3\Big) x_3\Big) \Big\} v_k^* \end{aligned}$$

where

$$\begin{aligned} w_{\xi_3}(a) &= \exp(-2\pi a_2 \xi_3) \\ w^k_{\xi_1,\xi_3,M}(a,m) &= a_1^{\lambda_1 - k} a_2^{\lambda_2 + k} \Big(x_4 + \frac{2\xi_1 m + M}{2\xi_1}\Big)^k \\ &\quad \cdot \exp\Big\{ -2\pi \Big(a_1^2 \xi_1 + a_2^2 \xi_3 + a_2^2 \xi_1 \Big(x_4 + \frac{2\xi_1 m + M}{2\xi_1}\Big)^2 \Big) \Big\} \end{aligned}$$

B.5 Comparison to the classical cases

F a $\mathbb{C}$–valued holomorphic Siegel modular form of weight l has the classical *Fourier expansion*

$$F(Z) = \sum_{T\geq 0} C_T^F e(t_1z_1 + t_2z_2 + t_3z_3)$$

for $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathfrak{H}_2, T = \begin{pmatrix} t_1 & t_2/2 \\ t_2/2 & t_3 \end{pmatrix}$ positive semi–definite.
Remark (Lemma 10.1):

$$\begin{aligned} T &= T_{\xi_3} = \begin{pmatrix} 0 & 0 \\ 0 & \xi_3 \end{pmatrix} \quad \text{for} \quad t_1 = \det T = 0, & (5) \\ &= T^m_{\xi_1,\xi_3,M} = \begin{pmatrix} \xi_1 & \frac{2\xi_1 m+M}{2} \\ \frac{2\xi_1 m+M}{2} & \xi_3 + \frac{(2\xi_1 m+M)^2}{4\xi_1} \end{pmatrix} \quad \text{otherwise}, & (6) \\ & \text{where } \xi_1 \in \mathbb{N}, \xi_3 \in \mathbb{Q}_{\geq 0} \text{ with } \mathfrak{M}(\xi_1,\xi_3) \neq 0 \quad \text{and} \quad M \in \mathfrak{M}(\xi_1,\xi_3). \end{aligned}$$

Let f be the usual lift from f to a function on G, i.e.

$$f(g) = J(g,I)^{-l}F(g < I >), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $J(g,I) = \det(CI + D), I = iE_2$ and with the coordinates used here

$$g < I >= \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}, \quad z_1 = x_1 + i(a_1^2 + a_2^2x_4^2),\ z_2 = x_2 + ia_2^2x_4, z_3 = x_3 + ia_2^2.$$

Then comparing the Fourier expansion of $f(xa)$ with the last result from section 3., one has (Th. 10.2)

$$\begin{aligned} C_T^F &= C^f_{\xi_3} \quad \text{for } T = T_{\xi_3} & (7) \\ C^F_{T^m_{\xi_1,\xi_3,M}} &= C^f_{\xi_1,\xi_3,M} \quad \text{for } T = T^m_{\xi_1,\xi_3,M} \quad \text{for any } m \in \mathbb{Z}. & (8) \end{aligned}$$

The Siegel form F has a *Fourier–Jacobi expansion*

$$F(Z) = \sum_{t_1=0}^{\infty} \phi_{t_1}(z_2,z_3)e(t_1z_1)$$

where

$$\phi_{t_1}(z_2,z_3) = \sum_{T\geq 0} C_T^F e(t_2z_2 + t_3z_3)$$

is a Jacobi form of weight l and index t_1. The Jacobi group $G^J = SL_2(\mathbb{R}) \ltimes N_J$ is here embedded into G by

$$g^J = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \lambda & \kappa & \mu \\ & 1 & \mu & \\ & & 1 & \\ & & -\lambda & 1 \end{pmatrix} \right) \stackrel{j}{\mapsto} \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix} \begin{pmatrix} 1 & \lambda' & \kappa & \mu' \\ & 1 & \mu' & \\ & & 1 & \\ & & -\lambda' & 1 \end{pmatrix}$$

where

$$(\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

ϕ_{t_1} is lifted to a function Φ_{t_1} on G^J using the slash operator from the Jacobi forms

$$\Phi_{t_1}(g^J) := \phi_{t_1} \mid_{t_1,l}(g^J)(0, i).$$

Then the N–expansion for the lift f of the Siegel form F from the end of section 2 can be read as (Th. 10.4)

$$f(xa) = \sum_{\xi_1=0}^{\infty} \Phi_{\xi_1}(a_2, 0, x_2, x_3, x_4) a_1^l e(\xi_1(x_1 + i a_1^2)),$$

where Φ_{ξ_1} comes from the Fourier–Jacobi coefficient ϕ_{ξ_1} of F by defining

$$\Phi_{\xi_1}(a_2, x_1, x_2, x_3, x_4) := \Phi_{\xi_1}(g_J)$$

for g_J the AN–part of $j(g^J)$

$$g_j = \begin{pmatrix} 1 & & & \\ & a_2 & & a_2^{-1} x_3 \\ & & 1 & \\ & & & a_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_2 x_4 & x_1 - x_2 x_4 & a_2^{-1} x_2 \\ & 1 & a_2^{-1} x_2 & \\ & & 1 & \\ & & -a_2 x_4 & 1 \end{pmatrix}.$$

C Resumé of M. Hirano: Fourier–Jacobi type spherical functions for discrete series representations of $Sp(2,\mathbb{R})$ preprint Dec. 1999

C.1 Problem:

Correspondence between Siegel modular forms, ordinary modular forms and Jacobi modular forms for the non–holomorphic case.

As a first step consider the space of intertwining operators between discrete series representations π of $G = Sp(2,\mathbb{R})$ and representations of G which are induced from representations ρ of the Jacobi group, called here R_J.

By relating this space to a certain space of Fourier–Jacobi type spherical functions of type (ϱ, π, τ) (where (τ^*, V_{τ^*}) is a K–type of π), one can get a condition such that the intertwining space is at most one dimensional.

C.2 Description of the representations treated here

C.2.1 The usual notations

$G = Sp(2,\mathbb{R}) \supset K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right\} \longmapsto A + iB \in U(2)$

$\mathfrak{g} = \mathrm{Lie}G = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition (for θ)
$\mathfrak{h} =< T_1, T_2 >_{\mathbb{R}}$ compact Cartan subalgebra
$\triangle = \triangle(\mathfrak{h}_c, \mathfrak{g}_c) = \{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\}$ $\triangle^+ = \triangle_c^+ \cup \triangle_n^+$ compact and non–compact positive roots $\mathfrak{g}_\beta$ root space for $\beta \in \triangle$, $\|\beta\| := \sqrt{\beta_1^2 + \beta_2^2}$
$\mathfrak{p} \supset \mathfrak{a}_\mathfrak{p} =< H_1, H_2 >_{\mathbb{R}}$ maximal abelian subalgebra $\{e_i\}_{i=1,2}$ basis of $\mathfrak{a}_\mathfrak{p}^*$
$\sum = \sum(\mathfrak{a}_\mathfrak{p}, \mathfrak{g}) \supset \sum^+ = \{2e_1, 2e_2, e_1 \pm e_2\}$
$\mathfrak{g} = \mathfrak{n}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{k}$ Iwasawa decomposition, with $\mathfrak{n}_\mathfrak{p} =< E_{2e_1}, E_{2e_2}, E_{e_1 \pm e_2} >$

C.2.2 "Jacobi type" notations

$\mathfrak{a}_J =< H_1 >$, $\mathfrak{n}_J =< E_{2e_1}, E_{e_1+e_2}, E_{e_1-e_2} >\simeq$ heis $(\mathbb{R})$
$\mathfrak{m}_J =< H_2, E_{2e_2}, E_{-2e_2} >\simeq \mathfrak{sl}(2\mathbb{R})$, $\mathfrak{g}_{2e_1}$ center of $\mathfrak{n}_J$.
$A_J = \exp \mathfrak{a}_J, N_J = \exp \mathfrak{n}_J \simeq$ Heis $(\mathbb{R})$
$M_J = Z_K(\mathfrak{a}_J) \exp \mathfrak{m}_J \simeq \{\pm 1\} \times SL(2,\mathbb{R})$
$P_J := M_J A_J N_J$ <u>Jacobi maximal parabolic subgroup</u>
<u>(corresponding to the long root $2e_2$)</u>
$\mathfrak{g}_n := \{X \in \mathfrak{g} : [H_1, X] = nX\}$ for $n \in \mathbb{Z}$, that is
$\mathfrak{g}_0 = \mathfrak{m}_J \oplus \mathfrak{a}_J$, $\mathfrak{g}_1 = \mathfrak{g}_{e_1+e_2} \oplus \mathfrak{g}_{e_1-e_2}$, $\mathfrak{g}_2 = \mathfrak{g}_{2e_1}$, $\mathfrak{g}_{-i} = \theta \mathfrak{g}_i$
$\mathfrak{g}_n = \{0\}$ for $n \neq 0, \pm 1, \pm 2$.
$R_J := M_J^0 \times N_J \simeq SL(2,\mathbb{R}) \ltimes$ Heis $(\mathbb{R})$ <u>Jacobi group</u>
$\mathfrak{r}_J =$ Lie $R_J = \mathfrak{m}_J \oplus \mathfrak{n}_J$
$G = R_J A_J K$ and $\mathfrak{g} = \mathfrak{r}_J \oplus \mathfrak{a}_J \oplus \mathfrak{k}$ <u>Iwasawa decompositions</u>

C.2.3 Parametrization of the representations $(\tau_\lambda, V_\lambda)$ of the compact K

$V_\lambda = \bigoplus_{k=0}^{d_\lambda} \mathbb{C} v_k^\lambda$ for $d_\lambda = \lambda_1 - \lambda_2$, $\lambda \in \wedge = \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2, \ \lambda_1 \geq \lambda_2\}$
$\{v_k^\lambda\}_{k=0,\ldots,d_\lambda}$ standard basis of V_λ
$\tau_\lambda(Z) v_k^\lambda = (\lambda_1 + \lambda_2) v_k^\lambda \quad , \quad \tau_\lambda(H') v_k^\lambda = (2k - d\lambda) v_k^\lambda$
$\tau_\lambda(X) v_k^\lambda = (k+1) v_{k+1}^\lambda \quad , \quad \tau_\lambda(\overline{X}) v_k^\lambda = (d_\lambda + 1 - k) v_{k-1}^\lambda$
for $k = 0, \ldots, d_\lambda \quad , \quad \mathfrak{k}_c =< Z, H', X, \overline{X} >$

C.2.4 Representations $(\rho, \mathcal{F}_\rho)$ of the Jacobi group R_J

There is a reference to the book of Berndt–Schmidt, but the author prefers to use another notation:

$\rho = \rho_{\pi_1,m}$ non–trivial on the center of R_J, "of type $m \neq 0$" where
$\rho_{\pi_1,m}(g^J) = \pi_1(g) \otimes \tilde{\rho}_m(g^J)$ for $g^J = g \cdot n \in M_J^0 \times N_J \simeq R_J$,
$(\tilde{\nu}_m, U_m)$ corresponds to the Schrödinger–Weil representation π_{SW}^m in [BeS] and
$(\pi_1, \mathcal{W}_{\pi_1})$ is a metaplectic representation;

$\rho = \rho_{\pi_1,0}$ trivial on the center, where

(1) $\rho_{\pi_1,0}(g^J) = \pi_1(g) \otimes \tilde{\nu}_0(g^J)$, with
$(\pi_1, \mathcal{W}_{\pi_1})$ an irreducible unitary representation of M_J^0
$(\tilde{\nu}_0, \mathcal{U}_0)$ the trivial representation of R_J

(2) $\rho = \rho_r = \mathrm{Ind}_{N_0}^{R_J} \eta_r$, $r \in \mathbb{R}$, $\eta_r(n) = e(n_0 + r n_3)$

for

$$N_0 = \left\{ n = n(n_0, n_1, n_2, n_3) = \begin{pmatrix} 1 & n_0 & & \\ & 1 & & \\ & & 1 & \\ & & -n_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ & & 1 & \\ & & & 1 \end{pmatrix}, n_i \in \mathbb{R} \right\}$$

Bases for the Heisenberg – resp. the $SL(2)$–part

$$(\mathfrak{n}_J)_c =< X_\pm, Z >_\mathbb{C} \quad (\mathfrak{m}_J)_c =< V_\pm, U >_\mathbb{C}$$

The infinite simal formulae for different $(\rho, \mathcal{F}_\rho)$ are reproduced in terms of a basis $\{w_l \otimes u_j^m\}_{l \in L, j \in J}$ of $\mathcal{W}_{\pi_1} \otimes U_m$.

C.2.5 Parametrization of discrete series representations of G

Here we find a reference to Knapp's book [Kn] on Representation Theory of Semisimple Groups, Ch. IX.
Positive root systems compatible to $\triangle_c^+ = \{(1, -1)\}$

(I) $\triangle_I^+ = \{(1, -1), (2, 0), (1, 1), (0, 2)\}$

(II) $\triangle_{II}^+ = \{(1, -1), (2, 0), (1, 1)(0, -2)\}$

(III) $\triangle_{III}^+ = \{(1, -1), (2, 0), (0, -2), (-1, -1)\}$

(IV) $\triangle_{IV}^+ = \{(1, -1), (0. - 2), (-1, -1), (-2, 0)\}$.

Use these to define (with respect to the compact Cartan $\mathfrak{h} =< T_1, T_2 >$ in $\mathfrak{g}$) a subset Ξ_L ($L \in \{I, II, III, IV\}$) of $\triangle_c^+$–dominant weights by

$$\Xi_L = \{\wedge = (\wedge_1, \wedge_2), \triangle_c^+ \text{ –dominant weight; } < \wedge, \beta >< 0, \ \forall \beta \in \triangle_{L,n}^+\}$$

where $\triangle_{L,n}^+ := \triangle_L^+ - \triangle_c^+$. Then

$$\bigcup_{L=I}^{IV} \Xi_L$$

gives the H-C parametrization of the discrete series representations $\pi = \pi_\wedge$ of G. The minimal K–type $(\tau_\lambda, V_\lambda)$ of $\pi_\wedge$ is given by the <u>"Blattner parameter"</u>

$$\lambda = \wedge - \rho_c + \rho_n \quad (\rho_c, \rho_n \text{ half sums of positive } c-\text{ resp. } nc-\text{ roots in } \triangle_L^+)$$

i.e. one has

$$\begin{array}{lllll}\lambda & = (\wedge_1+1,\wedge_2+2) & \text{if} & \wedge\in\Xi_I & \text{(the “holomorphic case”)}\\ & = (\wedge_1-2,\wedge_2-1) & & \wedge\in\Xi_{IV} & \text{(the “anti–holomorphic case”)}\\ & = (\wedge_1+1,\wedge_2) & & \wedge\in\Xi_{II} & \text{(the “large” case (Vogan))}\\ & = (\wedge_1,\wedge_2-1) & & \wedge\in\Xi_{III} & \text{(the “large” case (Vogan))}\end{array}$$

C.3 Fourier–Jacobi type spherical functions

For π resp. ρ an irreducible admissible representation of G resp. the Jacobi group R_J we have the intertwining space

$$\mathcal{I}_{\rho,\pi} := \mathrm{Hom}_{(\mathfrak{g}_c,K)}(\pi, \mathcal{C}^\infty\ \mathrm{Ind}^G_{R^J}(\rho))$$

For finite dimensional K–module $(\tau_\lambda, V_\lambda)$ and a K–equivariant map $i:\tau^*\to\pi|_K$ with pullback i^* one gets by restriction

$$\mathcal{I}_{\rho,\pi} \xrightarrow{i^*} \mathrm{Hom}_K(\tau^*, \mathcal{C}^\infty\mathrm{Ind}^G_{R^J}(\rho)) =: \mathcal{I}_{\rho,\tau}$$

Now this space may be interpreted as

$$\begin{array}{rcl}\mathcal{I}_{\rho,\tau} & \simeq & \mathcal{C}^\infty_{\rho,\tau}(R_J\backslash G/K)\\ & := & \{F: G\to\mathfrak{F}_\rho\otimes V_\tau \text{ smooth;}\\ & & F(rgk) = (\rho(r)\otimes\tau(k)^{-1})F(g),\ r\in R_J, g\in G, k\in K\}\end{array}$$

And by the Iwasawa decomposition $G = R_J A_J K$, one further has an injective linear map given by restriction to A_J

$$\mathcal{C}^\infty_{\rho,\tau}(R_J\backslash G/K) \xrightarrow{\mathrm{res}|_{A_J}} \mathcal{C}^\infty(A^J,\rho,\tau),$$

where

$$\begin{array}{l}\mathcal{C}^\infty(A^J,\rho,\tau) := \{\varphi: A_J\to\mathfrak{F}_\rho\otimes V_\tau;\\ (\rho(m)\otimes\tau(m))\varphi(a) = \varphi(a))\ \ a\in A_J, m\in R_J\cap K\simeq SO(2)\},\end{array}$$

Now for each intertwining operator $T\in\mathcal{I}_{\rho,\pi}$ and each $i:\tau^*\to\pi|_K$ denote by T_i the image of T in $\mathcal{C}^\infty_{\rho,\tau}(R_J\backslash G/K)$ and take these elements to define the <u>space of Fourier–Jacobi type spherical functions</u> of type (ρ,π,τ)

$$J_{\rho,\pi}(\tau) := \bigcup_{i\in\mathrm{Hom}_K(\tau^*,\pi|_K)} \{T_i;\ T\in\mathcal{I}_{\rho,\pi}\}.$$

and its subspace

$$J^\circ_{\rho,\pi}(\tau)$$

consisting of those functions f, where $f|_{A_J}$ is of moderate growth, when

$$a = a_r = \exp rH_1 = \mathrm{diag}\,(e^r, 1, e^{-r}, 1) \text{ and } r \to \infty$$

The main goal of the paper is to determine this space $J^{\circ}_{\rho,\pi}(\tau)$ for π a discrete series representation. This is done by translating the condition characterizing π via the Cauchy–Riemann type differential operators into the space $\mathcal{C}^{\infty}_{\rho,\tau}(R_J\backslash G/K)$ resp. its "radial part" $\mathcal{C}^{\infty}(A_J, \rho, \tau)$.

C.4 Differential Operators

As usual, for a function f on G and $X \in \mathfrak{g}$ $R_X f$ is defined by

$$R_X f(g) = \frac{d}{dt} f(g \text{ expt } tX)|_{t=0}.$$

Now, here "shift operators" (for $\beta \in \sum_n^{\pm}$)

$$\nabla^{\pm\beta}_{\rho,\tau_\lambda} : \mathcal{C}^{\infty}_{\rho,\tau_\lambda}(R_J\backslash G/K) \to \mathcal{C}_{\rho,\tau_{\lambda\pm\beta}}(R_J\backslash G/K)$$

play a central role. They are defined by composition of the operators $\nabla_{\rho,\tau_\lambda}$ defined by

$$\nabla^{\pm}_{\rho,\tau_\lambda} f = (1/4) \sum_{\beta \in \sum_n^+} \|\beta\|^2 R_{X_{F\beta}} f \otimes X_{\pm\beta} \text{ for } f \in \mathcal{C}^{\infty}_{\rho,\tau_\lambda}(R_J\backslash G/K)$$

and the projectors on the irreducible factors

$$P^{\pm\beta} : V_\lambda \otimes \mathfrak{p}_{\pm} = \bigoplus_{\beta \in \triangle_n^{\pm}} V_{\lambda\pm\beta} \longrightarrow V_{\lambda\pm\beta}$$

(remembering that via the adjoint representation one has $\mathfrak{p}_+ = V_{(2,0)}$ and $\mathfrak{p}_- \simeq V_{(0,-2)}$). As these functions f are completely determined by their radial part $f|_{A^J}$, one looks for a (differential) operator u only at the "radial part"

$$R(u) : \mathcal{C}^{\infty}(A_J; \rho, \tau) \longrightarrow \mathcal{C}^{\infty}(A_J; \rho, \tau')$$

with

$$(uf)|_{A_J} = \mathcal{R}(u)(f|_{A_J}).$$

In Prop 4.3 the operators $\mathcal{R}(\nabla^{\pm}_{\rho,\tau})$ are determined. And then in Prop 4.4 the operators $\mathcal{R}(\nabla^{\pm\beta}_{\rho,\tau})$ are applied to functions

$$\varphi \in \mathcal{C}^{\infty}(A_J, \rho_{\pi_{1,m}}, \tau_\lambda)$$

using the expansion

$$\varphi(a) = \sum_{k=0}^{d_\lambda} \sum_{l\in L} \sum_{j\in J} c_{l,j,k}(a)(w_l \otimes u_j^m \otimes v_k^\lambda)$$

coming from the bases of $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes U_m)$ and $(\tau_\lambda, V_\lambda)$ given above in section 3.

C.5 Characterization of Spherical Functions

Starting from the Cauchy–Riemann differential equations characterizing holomorphic functions, one can identify for a discrete series representation $\pi = \pi_1$ the intertwining space $\mathcal{I}_{\rho,\pi}$ with a solution space of differential equations:

$$\mathcal{I}_{\rho,\pi} \simeq \bigcap_{\beta\in\triangle^+_{L^*,n}} \ker\ \nabla^{-\beta}_{\rho,\tau_\lambda} \subset \mathcal{C}^\infty_{\rho,\tau_\lambda}(R_J\backslash G/K)$$

for $\pi_\wedge$ with $\wedge \in \Xi_L$ and $L^* = IV, III, II$ and I for $L = I, II, III$ and IV respectively. And thus also

$$J_{\rho,\pi}(\tau_\lambda) = \left\{F \in \mathcal{C}^\infty_{\rho,\tau_\lambda}(R_J\backslash G/K);\ \nabla^{-\beta}_{\rho,\tau_\lambda} F = 0 \text{ for all } \beta \in \triangle^+_{L^*,n}\right\}$$

The main part of the paper consists in solving these differential equations for the different cases, using the expressions of the differential operators and the expansions of the radial functions from the previous section.
Before giving a summary of the general results, we will single out the (most interesting) special case:

$\pi = \pi_\wedge$	with	$\wedge = (\wedge_1, \wedge_2) \in \Xi_I$ the "holomorphic" case
		τ_λ^* the minimal k–type, where
		$d_\lambda = 0$ (i.e. $\wedge_1 - \wedge_2 = 1$)
$\rho = \rho_{\pi_1,m}$		of type $m > 0$.

Then one has

$$J_{\rho,\pi}(\tau_\lambda) = \{0\} \text{ except for the case } \pi_1 = \mathcal{D}^+_{\wedge_2+3/2}$$

For $\pi_1 = \mathcal{D}^+_{\wedge_2+3/2}$ this space is one–dimensional. Namely $\varphi \in J_{\rho,\pi}(\tau_\lambda)|_{A_J}$ is written as

$$\varphi(a_r) = c_{1/2,0}(a_r)(w_{\wedge_2+3/2} \otimes u^m_{1/2} \otimes v_0^\lambda),$$

where $c_{1/2,0}$ is a solution of the first order differential equation

$$\left\{2x\frac{d}{dx} - (\wedge_1 + 1) + x\right\} c_{1/2,0}(a_r) = 0,\ x = 4\pi m\xi^2,\ a_r = \text{diag}\,(\xi, 1, \xi^{-1}, 1)$$

and so

$$c_{1/2,0}(a_r) = cx^{(\wedge_1+1)/2}e^{-x/2}.$$

There are very explicit formulae also for the cases $d_\lambda \geq 1$. The most general result is for $\rho \neq \rho_r$

$$\dim\, J^0_{\rho,\pi}(\tau_\lambda) \leq 1$$

This is given precisely for the holomorphic and the antiholomorphic case in the Theorems 6.3 – 6.9 .

In most cases $J_{\rho,\pi}(\tau_\lambda)$ and thus $J^0_{\rho,\pi}(\pi_\lambda)$ are $\{0\}$ exceptions are

$$\dim\, J_{\rho,\pi}(\tau_\lambda) = \dim\, J^0_{\rho,\pi}(\tau_\lambda) = 1$$

for

$$\begin{array}{llll} \wedge \in \Xi_I, & m > 0, & \pi_1 = \mathcal{D}^+_{n_1}, & \wedge_2 + 3/2 \leq n_1 \leq \wedge_1 + 1/2 \\ \wedge \in \Xi_{IV}, & m < 0, & \pi_1 = \mathcal{D}^-_{n_1}, & -\wedge_1 + 3/2 \leq n_1 \leq -\wedge_2 + 1/2 \\ \wedge \in \Xi_I, & m = 0, & \pi_1 = \mathcal{D}^+_{\wedge_1+1} & \pi_1 \neq id \\ \wedge \in \Xi_{IV}, & m = 0, & \pi_1 = \mathcal{D}^-_{-\wedge_2+1} & \end{array}$$

Here the spaces are fixed by an "elementary" function of the type $c_{1/2,0}$ given above. Theorems 7.5 – 7.8 and 7.10 – 7.12 treat the "large" cases $\wedge \in \Xi_{II} \cup \Xi_{III}$. Here again "most" spaces $J_{\rho,\pi}(\tau_\lambda)$ are $\{0\}$, but there are also cases with

$$\dim\, J_{\rho,\pi}(\tau_\lambda) = l, \quad l = 1, 2, 3\ ,$$

for example one has $l = 3$ (and $\dim\, J^0_{\rho,\pi}(\tau_\lambda) = 1$) for

$$\begin{array}{rll} m > 0, \ \ \pi_1 = & \mathcal{P}^\tau_s & \text{with } \tau = \pm 1/2, \ \ s \in i\mathbb{R}, \ \text{ or} \\ = & \mathcal{C}^\tau_s & \text{with } \tau = \pm 1/2, \ \ 0 < s < 1/2 \end{array}$$

Here the spaces are spanned by non elementary functions, where Meijer's G–function

$$G^{3,0}_{2,3}\left(x \;\middle|\; \begin{matrix} a_1 \ , \ a_2 \\ \\ b_1 \ , \ b_2 \ , \ b_3 \end{matrix} \right)$$

enters (which however may degenerate considerably to a classical Whittaker function $W_{\kappa,\mu}(x)$ of the form $x^\alpha e^{\beta x}$ and is discussed more thouroughly in the appendix). Theorems 8.1 and 8.2 treat the case of

$$\rho = \mathrm{Ind}^{R_J}_{N_0}\mu_r$$

excepted up to now. Here

$$\mathcal{I}_{r,\pi} := \mathrm{Hom}_{(\mathfrak{g}_c,K)}\left(\pi, \mathcal{C}^\infty\ \mathrm{Ind}^G_{R_j}\ \left(\mathcal{C}^\infty\ \mathrm{Ind}^{R_J}_{N_0}\mu_r\right)\right)$$

is again $\{0\}$ for the holomorphic and antiholomorphic cases, but one has more complicated results for the "large cases" (already known for $r \neq 0$), for example

$$\dim \mathcal{I}_{0,\pi} = 4 \ \text{ and even } \ \dim \mathcal{I}^0_{0,\pi} = 2.$$

D Resumé of M. Hirano: Fourier–Jacobi type sherical functions for P_J–principal series representations of $Sp(2,\mathbb{R})$ preprint Dec.1999

This paper is a continuation of the paper with the analogue title where 'P_J–principal' is replaced by 'discrete'.

D.1 Problem:

Determine the space of intertwining operators from the space of the principal series representations π of $G = Sp(2,\mathbb{R})$ induced from the Jacobi parabolic subgroup P_J (in other sources also called Klingen parabolic) to the space of representations of G which are induced from representations ρ of the Jacobi group R_J inside G. The space of these operators is translated to a space of generalized Whittaker functions of type (ρ,π,τ) , where (τ^*, V_{τ^*}) is a representation of K associated to π, and the dimension of this space (resp. a sub–space of functions of moderate growth) is shown to be at most one–dimensional.

D.2 The P_J–principal Series and their corner K–types

The notations of the discrete series paper are taken over. In addition, we have the "Jacobi maximal parabolic subgroup (resp. subalgebra)

$$P_J = M_J A_J N_J \quad , \mathfrak{p}_J = \mathfrak{m}_J \oplus \mathfrak{a}_J \oplus \mathfrak{n}_J,$$

where

$$\begin{array}{ll} N_J = \exp\, \mathfrak{n}_J \simeq \text{ Heis } (\mathbb{R}), & \mathfrak{n}_J =< E_{2e_1}, E_{e_1+e_2}, E_{e_1-e_2} >_{\mathbb{R}} \simeq \text{ heis } (\mathbb{R}) \\ A_J = \exp\, \mathfrak{a}_J \simeq \mathbb{R}^*, & \mathfrak{a}_J =< H_1 >_{\mathbb{R}} \\ M_J = Z_K(\mathfrak{a}_J)\, \exp\, \mathfrak{m}_J \simeq \{\pm 1\} \times SL_2(\mathbb{R}), & \mathfrak{m}_J =< H_2, E_{2e_2}, E_{-2e_2} > \simeq \mathfrak{sl}(2,\mathbb{R}). \end{array}$$

A P_J–principal series representation π of G is an induced representation

$$\pi = \mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J}),$$

where $\sigma = (\varepsilon, D)$ a representation of M_J with a character $\varepsilon : \{\pm 1\} \to \mathbb{C}^*$ and a (limit of) discrete series representation $D = \mathcal{D}_n^{\pm}$ $(n \in \mathbb{N})$ of $SL(2,\mathbb{R})$

ν_z a quasi–character of A_J such that $\nu_z(\mathrm{diag}(a,1,a^{-1},1)) = a^z$ with $z \in \mathbb{C}$.

This π is irreducible and admissible except for the case $(\sigma, z) \neq ((1,\mathcal{D}_n^{\pm}), 0)$ with $n \geq 2$. It is unitarizable for

(1) $z \in i\mathbb{R}^*$,

(2) $z = 0$ and $\sigma = (-1, \mathcal{D}_n^{\pm})$ or $(1, \mathcal{D}_1^{\pm})$,

(3) $0 < z < 1$ and $\sigma = (-1, \mathcal{D}_n^{\pm})$ or $(1, \mathcal{D}_1^{\pm})$.

One has a unitary equivalence $\mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J}) \simeq \mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_{-z} \otimes 1_{N_J})$.

By older work (Miyazaki–Oda) one knows the K–types τ of $\pi = \mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$. In particular, one has (with $\gamma = \mathrm{diag}\,(-1, 1, -1, 1)$)

I) $[\pi : \tau_\lambda] = 0$ for τ_λ with $\lambda = (\lambda_1, \lambda_2), \lambda_1 < n$ (resp. $\lambda_2 < -n$) and $D = \mathcal{D}_n^+$ (resp. $\mathcal{D}_n^-$).

II) $[\pi : \tau_\lambda] = 1$ for τ_λ with

(1) $\lambda = (l, l), l \geq n, l \equiv n \bmod 2$, and
$\lambda = (n, l), l \leq n, l \equiv n \bmod 2$ for
$D = \mathcal{D}_n^+$ and $\varepsilon(\gamma) = (-1)^n$

(2) $\lambda = (l, l-1), l \geq n$ and $\lambda = (n, l-1) l \leq n, l \equiv n \bmod 2$ for $D = \mathcal{D}_n^+, \varepsilon(\gamma) = (-1)^{n+1}$.

(3) $\lambda = (l, l), l \leq -n, l \equiv n \bmod 2$, and
$\lambda = (l, -n) l \geq -n, l \equiv n \bmod 2$ for
$D = \mathcal{D}_n^-, \varepsilon(\gamma) = (-1)^n$

(4) $\lambda = (l+1, l), l \leq -n$, and $\lambda = (l+1, -n), l \geq -n, l \equiv n \bmod 2$, for $D = \mathcal{D}_n^-, \varepsilon(\gamma) = (-1)^{n+1}$

The irreducible K–module τ_λ with $\lambda = (n, n)$ is called the <u>corner K–type</u> of π for case (1) (i.e. $D = \mathcal{D}_n^+, \varepsilon(\gamma) = (-1)^n$) and with

$$\begin{aligned} \lambda &= (n, n-1) && \text{for case (2)} \\ &= (-n, -n) && \text{for case (3)} \\ &= (-n+1, -n) && \text{for case (4)} \end{aligned}$$

One has dim $V_\lambda = 1$ for (1) and (3) and $= 2$ for (2) and (4).

Now, the same way as in the case of π discrete series with minimal K–type, these corner K–types are used to define the space of spherical functions $\mathcal{C}^\infty_{\rho,\tau}(R_J \backslash G/K)$ and of the radial parts $\mathcal{C}^\infty(A_J; \rho, \tau)$.

D.3 The Casimir operator

The Casimir element Ω in the center $Z(\mathfrak{g}_c)$ of $U(\mathfrak{g}_c)$ is given by

$$\begin{aligned}\Omega = H_1^2 \quad & +H_2^2 - 4H_1 - 2H_2 \\ & -4E_{2e_2}E_{-2e_1} - 2E_{e_1+e_2}E_{-e_1-e_2} - 4E_{2e_2}E_{-2e_2} - 2E_{e_1-e_2}E_{-e_1+e_2}.\end{aligned}$$

Prop. 4.3 states the radial part $\mathcal{R}(\Omega)$ as an operator on $\mathcal{C}^\infty(A_J;\rho,\tau)$.
As in the discrete series case $\varphi \in \mathcal{C}^\infty(A_J;\rho,\tau)$ is expanded

$$\varphi(a_r) = \sum_{k=0}^{d_\lambda}\sum_{l\in L}\sum_{j\in J} c_{l,j,k}(a)(w_l \otimes u_j^m \otimes v_k^\lambda)$$

resp.

$$= \sum_{\substack{k=0 \\ l=l(j,k)\in L}}^{d_\lambda} \sum_{j\in J} c_{j,k}(a)(w_l \otimes u_j^m \otimes v_k^\lambda)$$

with $l(j,k) = -j + k - \lambda_1$ for $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,\mathfrak{m}}, \mathcal{W}_{\pi_1} \otimes U_\mathfrak{m})$ and $(\tau_\lambda, V_\lambda)$. Prop. 4.4 expresses $\mathcal{R}(\Omega)\varphi(a_r)$ in terms of this expansion.
It is known that Ω acts on π, hence on $J_{\rho,\pi}(\tau_\lambda)|_{A_J}$, as a scalar operator χ_Ω. The main point is that this is used to characterize the Fourier–Jacobi type spherical functions belonging to π. This is done in a case by case discussion.

D.4 The 1–dimensional case

For $\pi = \mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ with 1–dimensional corner type τ^* i.e. the cases (1) and (3) in section 2: Take at first case (1) with the data

$$\sigma = (\varepsilon, \mathcal{D}_n^+), \varepsilon(\gamma) = (-1)^n, \nu_z, z \in \mathbb{C}, \text{ with } (z,n) \neq (0, 2N),$$

τ_λ^* the corner K–type of π, i.e. $\tau_\lambda = \tau_{(-n,-n)}$.
Here the expansion reduces to

$$\varphi(a_r) = \sum_{\substack{j\in J \\ l=l(j)=-j+n}} c_{j,0}(a_r)(w_l \otimes u_j^m \otimes v_o^\lambda)$$

and we have

$$\chi_\Omega = z^2 + (n-1)^2 - 5,$$

thus the differential equation

$$(\mathcal{R}(\Omega) - \chi_\Omega)\varphi(a_r) = 0.$$

And moreover also

$$\mathcal{R}\left(\nabla^{(0,2)}_{\rho,\tau_\lambda+(2,0)}\right)\cdot\mathcal{R}\left(\nabla^{(2,0)}_{\rho,\tau_\lambda}\right)\varphi(a_r)=0$$

Here in the first equation the shift operators from section 3 of the first part appear, i.e. operators

$$\nabla^{\beta}_{\rho,\tau}:\mathcal{C}^\infty_{\rho,\tau}(R_J\backslash G/K)\to\mathcal{C}^\infty_{\rho,\tau+\beta}(R_J)\backslash G/K),$$

and the fact is used, that $\tau^\infty_{\lambda+(2,2)}=\tau_{(n-2,n-2)}$ does not appear in the K–types of π and so the composition $\nabla^{(0,2)}_{\rho,\tau_\lambda+(2,0)}\cdot\nabla^{(2,0)}_{\rho,\tau_\lambda}$ has to be zero.

Lemma 5.1 expresses this operator in terms of the expansion of $\varphi(a_r)$ given above. Together with the expression for $\mathcal{R}(\Omega)$ this leads to a recursive system of (second order) differential equations for the $c_{j,0}$ (in Prop. 5.2).

By a careful and lengthy analysis, one arrives for $m>0$ at the result (Th. 5.4):

$$\dim\ J_{\rho,\pi}(\tau_\lambda)=:d,\ \dim\ J^0_{\rho,\pi}(\tau_\lambda)=:d^0$$

where

(1)	$d\le 3, d^0\le 1$	for $\pi_1=\mathcal{P}^\tau_s,\ \tau=\pm1/2,\ s\in\mathbb{R},$ or $\mathcal{C}^\tau_s,\tau=\pm1/2,\ 0<s<1/2,$
(2)	$d=d^0=0$	for $\pi=\mathcal{D}^+_{n_1},\ n_1+1/2>n,\ n_1\in(1/2)\mathbb{Z}_{\ge3}\backslash\mathbb{Z},$
(3)	$d\le2,\ d^0\le1$	for $\pi_1=\mathcal{D}^+_{n_1},\ n_1+(1/2)\le n,\ n_1\in1/2\mathbb{Z}_{\ge3}\backslash\mathbb{Z}$
(4)	$d\le1,\ d^0=0$	for $\pi_1=\mathcal{D}^-_{n_1},\ n_1\in(1/2)\mathbb{Z}_{\ge3}\backslash\mathbb{Z}$

Here in the non–zero cases, we have fundamental coefficient functions $c_{j_0,0}(a)$ of the type

$$e^{x/2}G^{3,0}_{2,3}(x|\dots),\ x=4\pi m\xi^2,$$

where the G's are Meijer's functions.

For $m<0$ Th. 5.5 gives an analogous result. Both can be carried over by a symmetry argument to the case (3) and are given in Th. 5.6 and 5.7 .

For $m=0$ a similar but new discussion leads to the Theorems 5.8 – 5.10. For instance, one has in Th. 5.8 the case

(4) $\quad d=d^0\le2$ for $\rho=\rho_{\pi_1,0},\ \pi_1=\mathcal{D}^+_n.$

Here the fundamental coefficient function $c_{0,0}(a)$ is of the form

$$c_1\xi^{z+2}+c_2\xi^{-z+2}\ (\text{resp. } c_1\xi^2+c_2\xi^2\ \log\ \xi)$$

with some constants c_1 and c_2 if $z\neq0$ (resp. $z=0$) and $a=\mathrm{diag}(\xi,1,\xi^{-1},1)$.

D.5 The 2–dimensional case

For irreducible P_J–principal series representations π with 2–dimensional corner types τ^* (i.e. the cases (2) and (4)) one has again a differential equation coming from the Casimir operator and this time another one coming from the shift operator

$$\nabla^{(1,1)}_{\rho,\tau_\lambda} : \mathcal{C}^\infty_{\rho,\tau_\lambda}(R_J\backslash G/K) \to \mathcal{C}^\infty_{\rho,\tau_{\lambda+(1,1)}}(R_J\backslash G/K)$$

using the fact that $\tau^*_{\lambda+(1,1)} = \tau_{(n-2,n-1)}$ does not occur in the K–types of the π looked at here. I.e. we have the system of the differential equations for $\varphi \in J_{\rho\pi}(\tau_\lambda)|_{A_J}$

$$(\mathcal{R}(\Omega) - \chi_\Omega)\varphi(a_r) = 0$$

$$\mathcal{R}\left(\nabla^{(1,1)}_{\rho,\tau_{\lambda+(1,1)}}\right)\varphi(a_r) = 0$$

In Prop. 6.1 the corresponding system of differential equations for the expansion

$$\varphi(a_r) = \sum_{\substack{k=0,1 \\ l=l(j,k)=-j+k+n-1}} \sum_{j\in J} c_{j,k}(a_r)(w_e \otimes u^{\mathfrak{m}}_j \otimes v^\lambda_k)$$

is established and in Th. 6.3 – 6.8 the results are assembled which qualitatively don't differ from those described above in the 1–dimensional case, though they demand some careful and lengthy considerations.