Exercises on the course 'Supersymmetric field theories'
Antoine Van Proeyen, Hamburg Sep. 12-14, 2012

The course is based on chapters of the book 'Supergravity', D. Freedman, and A. Van Proeyen, Cambridge Univ. Press, 2012

A link for a discount prize of 20% is
http://www.cambridge.org/knowledge/discountpromotion?code=L2SUPE

1 Scalar field theory and its symmetries

Ex. 1.5 Show that the action

$$S = \int \mathrm{d}^D x\, \mathcal{L}(x) = -\tfrac{1}{2}\int \mathrm{d}^D x \left[\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i + m^2\phi^i\phi^i\right]. \tag{1}$$

is invariant under the transformation

$$\phi^i(x) \xrightarrow{\Lambda} \phi'^i(x) \equiv \phi^i(\Lambda x). \tag{2}$$

Remember: $(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu$ and

$$\Lambda^\mu{}_\rho \eta_{\mu\nu}\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}. \tag{3}$$

Only fields transform, not spacetime coordinates
Ex. 1.6 Compute the commutators $[L_{[\mu\nu]}, L_{[\rho\sigma]}]$ and show that they agree with that of

$$[m_{[\mu\nu]}, m_{[\rho\sigma]}] = \eta_{\nu\rho}m_{[\mu\sigma]} - \eta_{\mu\rho}m_{[\nu\sigma]} - \eta_{\nu\sigma}m_{[\mu\rho]} + \eta_{\mu\sigma}m_{[\nu\rho]} \tag{4}$$

for matrix generators. Show that to first order in $\lambda^{\rho\sigma}$

$$\phi^i(x^\mu) - \tfrac{1}{2}\lambda^{\rho\sigma}L_{[\rho\sigma]}\phi^i(x^\mu) = \phi^i(x^\mu + \lambda^{\mu\nu}x_\nu). \tag{5}$$

Remember:

$$L_{[\rho\sigma]} \equiv x_\rho\partial_\sigma - x_\sigma\partial_\rho. \tag{6}$$

2 The Dirac field

Ex. 2.9 Show, using only

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}, \tag{7}$$

that $[\Sigma^{\mu\nu}, \gamma^\rho] = 2\gamma^{[\mu}\eta^{\nu]\rho} = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}$.

Prove the consistency of

$$\delta\Psi = -\tfrac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\Psi, \qquad \delta\bar\Psi = \tfrac{1}{2}\lambda^{\mu\nu}\bar\Psi\Sigma_{\mu\nu}$$

Prove then the invariance of the action

$$S[\bar\Psi, \Psi] = -\int \mathrm{d}^D x \bar\Psi[\gamma^\mu\partial_\mu - m]\Psi(x)$$

3 Clifford algebras and spinors

Ex 3.40 Rewrite

$$S[\Psi] = -\tfrac{1}{2}\int \mathrm{d}^D x\, \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

as

$$\begin{aligned} S[\psi] &= -\tfrac{1}{2}\int \mathrm{d}^4 x\, \left[\bar{\Psi}\gamma^\mu \partial_\mu - m\right](P_L + P_R)\Psi \\ &= -\int \mathrm{d}^4 x\, \left[\bar{\Psi}\gamma^\mu \partial_\mu P_L \Psi - \tfrac{1}{2}m\bar{\Psi}P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_R\Psi\right]. \end{aligned}$$

and prove that the Euler-Lagrange equations are

$$\not{\partial} P_L \Psi = m P_R \Psi\,, \qquad \not{\partial} P_R \Psi = m P_L \Psi\,. \tag{8}$$

Derive $\Box P_{L,R}\Psi = m^2 P_{L,R}\Psi$ from the equations above.

Answers to questions asked

- Give an explicit construction of γ-matrices
- To construct a C that satisfies

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)}\,, \qquad t_r = \pm 1\,, \tag{9}$$

where $\Gamma^{(r)}$ is a matrix in the set

$$\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1\cdots\mu_D}\} \tag{10}$$

of rank r, the following two properties are relevant

$$C^T = -t_0 C\,, \qquad (C\gamma^\mu)^T = t_1\, C\gamma^\mu\,. \tag{11}$$

or the last one is also

$$\gamma^{\mu T} = t_0 t_1\, C\gamma^\mu C^{-1}$$

Indeed for all indices different

$$\begin{aligned} (C\gamma^{\mu_1\mu_2\cdots\mu_r})^T &= \left(C\gamma^{\mu_1}C^{-1}C\gamma^{\mu_2}C^{-1}\ldots C\gamma^{\mu_r}C^{-1}C\right)^T \\ &= -t_0(t_0t_1)^r C\gamma^{\mu_r\cdots\mu_1} \end{aligned} \tag{12}$$

See that this gives a number for r mod 4.

- See that there are two solutions C_+ and C_- for even dimension,
- To go to another representation

$$\gamma'^\mu = S\gamma^\mu S^{-1}\,, \qquad C' = S^{-1\,T} C\, S^{-1}\,.$$