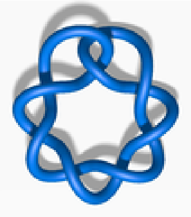
Super-A-polynomials of Twist Knots

joint work with Ramadevi and Zodinmawia
to appear soon

Satoshi Nawata

Perimeter Institute for Theoretical Physics

Aug 28 2012



Polynomial knot invariants

- Alexander polynomials $\Delta(K;q)$

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) = \left(q^{1/2} - q^{-1/2}\right)\Delta\Big($  $\Big)$

For trefoil, $\Delta\Big($  $\Big) = q - 1 + q^{-1}$

- Jones polynomials $J(K;q)$

$q^{1/2} J\Big($  $\Big) - q^{-1/2} J\Big($  $\Big) = \left(q^{1/2} - q^{-1/2}\right) J\Big($  $\Big)$

For trefoil, $J\Big($  $\Big) = q + q^3 - q^4$

- HOMFLY polynomials $P(K;a,q)$

$a^{1/2} P\Big($  $\Big) - a^{-1/2}\Delta\Big($  $\Big) = \left(q^{1/2} - q^{-1/2}\right) P\Big($  $\Big)$

For trefoil, $P\Big($  $\Big) = aq^{-1} + aq - a^2$

Categoifications

- the Poincaré polynomial of colored Khovanov homology $\mathcal{H}_{i,j}^{\mathfrak{sl}_2,R}$ [Khovanov '00]

$$Kh_R(K;q,t) = \sum_{i,j} t^j q^i \dim \mathcal{H}_{i,j}^{\mathfrak{sl}_2,R}(K) \ ,$$

- q-graded Euler characteristic gives colored Jones polynomial:

$$J_R(K;q) = Kh(q,t=-1) = \sum_{i,j} (-1)^j q^i \dim \mathcal{H}_{i,j}^{\mathfrak{sl}_2,R}(K) \ .$$

- The Poincaré polynomial of the Khovanov-Rozansky homology [Khovanov-Rozansky '04]

$$KhR_R(K;q,t) = \sum_{i,j} t^i q^j \dim \mathcal{H}_{i,j}^{\mathfrak{sl}_N,R}(K) \ .$$

is related to the colored HOMFLY polynomial via

$$KhR_R(K;q,t=-1) = P_R(K;a=q^N,q)$$

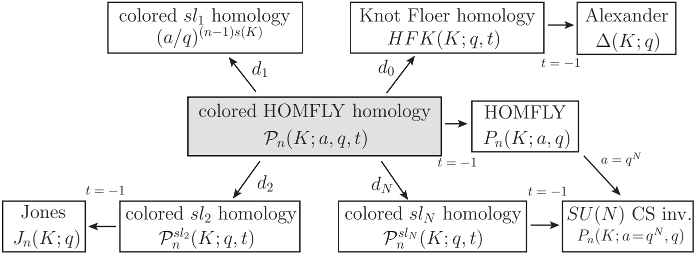
Colored superpolynomials

- the Poincaré polynomial of triply-graded homology $\mathcal{H}_{i,j}^{\mathfrak{sl}_2,R}$ [Dunfield-Gukov-Rasmussen '05]

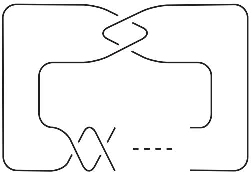
$$\mathcal{P}_R(K;a,q,t)=\sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}^R_{i,j,k}(K)\ .$$

- The (a,q)-graded Euler characteristic of the triply-graded homology theory is equivalent to the colored HOMFLY polynomial

$$P_R(K;a,q)=\sum_{i,j,k}(-1)^k a^i q^j \dim \mathcal{H}^R_{i,j,k}(K)\ .$$



Twist knot K_p



p full twist

p	-4	-3	-2	-1	0	1	2	3	4
knots	$\mathbf{10_1}$	$\mathbf{8_1}$	$\mathbf{6_1}$	$\mathbf{4_1}$	$\mathbf{0_1}$	$\mathbf{3_1}$	$\mathbf{5_2}$	$\mathbf{7_2}$	$\mathbf{9_2}$

The correspondence between the twist number and the knots in Rolfsen's table

- Colored Jones polynomials and A-polynomials are known
- Quantum A-polynomials were already computed for $p = -14 \cdots 15$.

Colored Jones polynomials of twist knots

- The double-sum expressions [Habiro '03][Masbaum '03]

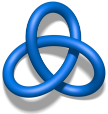
$$J_n(K_p;q) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k (q^{1-n};q)_k (q^{n+1};q)_k \times (-1)^\ell q^{\ell(\ell+1)p+\ell(\ell-1)/2}(1-q^{2\ell+1})\frac{(q;q)_k}{(q;q)_{k+\ell+1}(q;q)_{k-\ell}} .$$

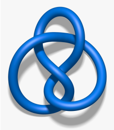
- The multi-sum expressions

$$J_n(K_{p>0};q) = \sum_{s_p\geq\cdots\geq s_1\geq 0}^{\infty} q^{s_p}\left(q^{1-n};q\right)_{s_p}\left(q^{1+n};q\right)_{s_p}\prod_{i=1}^{p-1} q^{s_i(s_i+1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

$$J_n(K_{p<0};q) = \sum_{s_{|p|}\geq\cdots\geq s_1\geq 0}^{\infty} (-1)^{s_{|p|}} q^{-\frac{s_{|p|}(s_{|p|}+1)}{2}}\left(q^{1-n};q\right)_{s_{|p|}}\left(q^{1+n};q\right)_{s_{|p|}} \times \prod_{i=1}^{|p|-1} q^{-s_i(s_{i+1}+1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

Colored superpolynomials of trefoil and figure-8





- Colored superpolynomials of trefoil and figure-8 are known [Fuji Gukov Sulkowski '12][Itoyama, Mironov, Morozov^2 '12]

$$\begin{aligned}
\mathcal{P}_n(\mathbf{3_1};a,q,t) &= (-t)^{-n+1}\sum_{k=0}^{\infty} q^k \frac{(-atq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(-at^3q^{n-1};q)_k\ , \\
\mathcal{P}_n(\mathbf{4_1};a,q,t) &= \sum_{k=0}^{\infty}(-1)^k a^{-k} t^{-2k} q^{-k(k-3)/2}\frac{(-atq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(-at^3q^{n-1};q)_k\ .
\end{aligned}$$

- Colored superpolynomials $\mathcal{P}_n(a,q,t)$ for $\mathbf{5_2}$ and $\mathbf{6_1}$ are also known up to $n=3$ [Gukov Stosic '11]

- One can do educated guess on colored superpolynomials of twist knots

Colored superpolynomials of twist knots

- The double-sum expressions

$$\begin{aligned}\mathcal{P}_n(K_p;a,q,t) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k \frac{(-atq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(-at^3q^{n-1};q)_k \\ &\qquad \times(-1)^\ell a^{p\ell}t^{2p\ell}q^{(p+1/2)\ell(\ell-1)}\frac{1-at^2q^{2\ell-1}}{(at^2q^{\ell-1};q)_{k+1}}\begin{bmatrix} k \\ \ell \end{bmatrix}_q .\end{aligned}$$

- The multi-sum expressions

$$\begin{aligned}\mathcal{P}_n(K_{p>0};a,q,t) &= (-t)^{-n+1}\sum_{s_p\geq\cdots\geq s_1\geq 0}^{\infty} q^{s_p}\frac{(-atq^{-1};q)_{s_p}}{(q;q)_{s_p}}(q^{1-n};q)_{s_p}(-at^3q^{n-1};q)_{s_p} \\ &\qquad \times\prod_{i=1}^{p-1}(at^2)^{s_i}q^{s_i(s_i-1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q\end{aligned}$$

$$\begin{aligned}&\mathcal{P}_n(K_{p<0};a,q,t) \\ &= \sum_{s_{|p|}\geq\cdots\geq s_1\geq 0}^{\infty}(-1)^k a^{-s_{|p|}}t^{-2s_{|p|}}q^{-s_{|p|}(s_{|p|}-3)/2}\frac{(-atq^{-1};q)_{s_{|p|}}}{(q;q)_{s_{|p|}}}(q^{1-n};q)_{s_{|p|}}(-at^3q^{n-1};q)_{s_{|p|}} \\ &\qquad \times\prod_{i=1}^{|p|-1}(at^2)^{-s_i}q^{-s_i(s_{i+1}-1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q .\end{aligned}$$

Colored Jones polynomials of twist knots

- The double-sum expressions [Habiro '03][Masbaum '03]

$$J_n(K_p;q) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k (q^{1-n};q)_k (q^{n+1};q)_k \times (-1)^\ell q^{\ell(\ell+1)p+\ell(\ell-1)/2}(1-q^{2\ell+1})\frac{(q;q)_k}{(q;q)_{k+\ell+1}(q;q)_{k-\ell}} .$$

- The multi-sum expressions

$$J_n(K_{p>0};q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0}^{\infty} q^{s_p} (q^{1-n};q)_{s_p} (q^{1+n};q)_{s_p} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

$$J_n(K_{p<0};q) = \sum_{s_{|p|} \geq \cdots \geq s_1 \geq 0}^{\infty} (-1)^{s_{|p|}} q^{-\frac{s_{|p|}(s_{|p|}+1)}{2}} (q^{1-n};q)_{s_{|p|}} (q^{1+n};q)_{s_{|p|}} \times \prod_{i=1}^{|p|-1} q^{-s_i(s_{i+1}+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

Colored superpolynomials of twist knots

- The double-sum expressions

$$\mathcal{P}_n(K_p; a, q, t) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k \frac{(-atq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(-at^3q^{n-1};q)_k \times(-1)^{\ell}a^{p\ell}t^{2p\ell}q^{(p+1/2)\ell(\ell-1)}\frac{1-at^2q^{2\ell-1}}{(at^2q^{\ell-1};q)_{k+1}}\begin{bmatrix} k \\ \ell \end{bmatrix}_q .$$

- The multi-sum expressions

$$\mathcal{P}_n(K_{p>0}; a, q, t) = (-t)^{-n+1}\sum_{s_p\geq\cdots\geq s_1\geq 0}^{\infty} q^{s_p}\frac{(-atq^{-1};q)_{s_p}}{(q;q)_{s_p}}(q^{1-n};q)_{s_p}(-at^3q^{n-1};q)_{s_p} \times\prod_{i=1}^{p-1}(at^2)^{s_i}q^{s_i(s_i-1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

$$\mathcal{P}_n(K_{p<0}; a, q, t) = \sum_{s_{|p|}\geq\cdots\geq s_1\geq 0}^{\infty}(-1)^k a^{-s_{|p|}}t^{-2s_{|p|}}q^{-s_{|p|}(s_{|p|}-3)/2}\frac{(-atq^{-1};q)_{s_{|p|}}}{(q;q)_{s_{|p|}}}(q^{1-n};q)_{s_{|p|}}(-at^3q^{n-1};q)_{s_{|p|}} \times\prod_{i=1}^{|p|-1}(at^2)^{-s_i}q^{-s_i(s_{i+1}-1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q .$$

Checks

- For $a = q^2$ and $t = -1$, the above formulae reduce to the colored Jones polynomials
- For $t = -1$, they reduce to the colored HOMFLY polynomials. We checked they agree with the colored HOMFLY polynomials computed by $SU(N)$ Chern-Simons theory up to 10 crossings.
- The colored HOMFLY polynomials can be reformulated into the Ooguri-Vafa polynomials. We checked that the Ooguri-Vafa polynomials have interger coefficients up to specific factors.
- We checked that the special polynomials which are the limits $q \to 1$ of the colored HOMFLY polynomials have the property,

$$\lim_{q\to 1} P_n(K_p; a, q) = \left[\lim_{q\to 1} P_2(K_p; a, q)\right]^{n-1} .$$

Cancelling differentials and Rasmussen s-invarinats

- the action of the differential d_1 on the colored superpolynomials

$$\begin{aligned}\mathcal{P}_{n+1}(K_{p>0};a,q,t) &= a^n q^{-n} + (1+a^{-1}qt^{-1})Q^{\mathfrak{sl}_1}_{n+1}(K_{p>0};a,q,t)\ ,\\ \mathcal{P}_{n+1}(K_{p<0};a,q,t) &= 1 + (1+a^{-1}qt^{-1})Q^{\mathfrak{sl}_1}_{n+1}(K_{p<0};a,q,t)\ ,\end{aligned}$$

- the action of the differential d_{-n} on the colored superpolynomials

$$\begin{aligned}\mathcal{P}_{n+1}(K_{p>0};a,q,t) &= a^n q^{n^2} t^{2n} + (1+a^{-1}q^{-n}t^{-3})Q_{n+1}(K_{p>0};a,q,t)\ ,\\ \mathcal{P}_{n+1}(K_{p<0};a,q,t) &= 1 + (1+a^{-1}q^{-n}t^{-3})Q_{n+1}(K_{p<0};a,q,t)\ .\end{aligned}$$

- The exponents of the remaining monomials are consistent with the Rassmusen's s-invariant of the twist knots, $s(K_{p>0}) = 1$ and $s(K_{p<0}) = 0$ [Rasmussen '04]

$$\begin{aligned}\deg\left(\mathcal{H}^{S^n}_{*,*,*}(K), d_1\right) &= \left(n\,s(K)\,,\,-n\,s(K)\,,\,0\right)\ ,\\ \deg\left(\mathcal{H}^{S^n}_{*,*,*}(K), d_{-n}\right) &= \left(n\,s(K)\,,\,n^2\,s(K)\,,\,2n\,s(K)\right)\ ,\end{aligned}$$

Volume conjecture and A-polynomials

- The volume conjecture relates "quantum invariants" of knots to "classical" 3d topology [Kashaev '99][Murakami^2 '00]

$$\lim_{n\to\infty} \frac{2\pi}{n} \log \left| J_n(K; q = e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \backslash K) \ .$$

- The relation b/w volume conjecture and A-polynomial [Gukov '03]

$$\log y = -x \frac{d}{dx} \lim_{\substack{n,k\to\infty \\ e^{i\pi n/k}=x}} \frac{1}{k} \log J_n(K; q = e^{\frac{2\pi i}{k}}) \ ,$$

gives the zero locus of the A-polynomial $A(K; x, y)$ of the knot K.

- A-polynomial $A(K; x, y)$ is a character variety of $SL(2, \mathbb{C})$-representation of the fundamental group of the knot complement

$$\mathcal{M}_{\mathrm{flat}}(SL(2,\mathbb{C}), T^2) \overset{\mathrm{Lag.sub.}}{\supset} \mathcal{M}_{\mathrm{flat}}(SL(2,\mathbb{C}), S^3 \backslash K) = \{(x, y) \in \mathbb{C}^\times \times \mathbb{C}^\times | A(K; x, y) = 0\}$$

AJ conjecture

- Quantum version of the volume conjecture → AJ conjecture [Gukov '03][Garoufalidis '03]

$$\widehat{A}(K;\hat{x},\hat{y},q)J_n(K;q)=0$$

where action of $\hat{x}$ and $\hat{y}$ on the set of colored Jones polynomials as

$$\hat{x}J_n(K;q^n)=q^nJ_n(K;q^n)\ ,\quad \hat{y}J_n(K;q)=J_{n+1}(K;q)\ .$$

- Find difference equations of colored Jones polynomials

$$a_kJ_{n+k}(K;q)+...+a_1J_{n+1}(K;q)+a_0J_n(K;q)=0$$

where $a_k=a_k(K;\widehat{x},q)$ and

$$\widehat{A}(K;\widehat{x},\widehat{y};q)=\sum a_i(K;\widehat{x},q)\widehat{y}^i$$

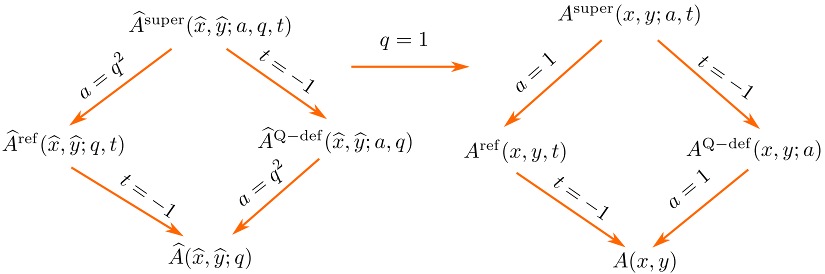
- Taking the classical limit $q=e^{\hbar}\to 1$, quantum (non-commutative) A-polynomials reduces to ordinary A-polynomials

$$\widehat{A}(K;\widehat{x},\widehat{y};q)\to A(K;x,y)\quad\text{as}\quad q\to 1$$

Super-A-polynomials

- Refinement of quantum and classical A-polynomials [Fuji Gukov Sulkowski '12]

Quantum operator	provides recursion for	classical limit
$\widehat{A}^{\rm super}(\widehat{x},\widehat{y};a,q,t)$	colored superpolynomial	$A^{\rm super}(x,y;a,t)$
$\widehat{A}^{\rm ref}(\widehat{x},\widehat{y};q,t)$	colored Khovanov-Rozansky homology	$A^{\rm ref}(x,y,t)$
$\widehat{A}^{\rm Q-def}(\widehat{x},\widehat{y};a,q)$	colored HOMFLY	$A^{\rm Q-def}(x,y;a)$
$\widehat{A}(\widehat{x},\widehat{y};q)$	colored Jones	$A(x,y)$



Classical super-A-polynomials

- Refinement of volume conjecture [Fuji Gukov Sulkowski '12]

$$\log y = -x \frac{d}{dx} \lim_{\substack{n,k\to\infty \\ e^{i\pi n/k}=x}} \frac{1}{k} \log \mathcal{P}_n(K; a, q = e^{\frac{2\pi i}{k}}, t) ,$$

gives the zero locus of the super-A-polynomial $A(K; x, y; a, q, t)$ of the knot K.

Knot	$A^{\mathrm{super}}(K; x, y; a, t)$
$\mathbf{5_2}$	$(1+at^3x)^3 y^4$ $-a(1+at^3x)^2(2-x+tx-2t^2x+3t^2x^2+at^2x^2+4at^3x^2-2at^3x^3+2at^4x^3+2at^5x^3-at^5x^4+2a^2t^5x^4+2a^2t^6x^4-a^2t^6x^5+a^2t^7x^5+a^3t^8x^6)y^3$ $-a^2(x-1)\left(1+at^3x\right)(1+tx-2t^2x+2t^2x^2-2t^3x^2+4at^3x^2+t^4x^2-3t^4x^3+at^4x^3-2at^5x^3+4at^5x^4-4at^6x^4+6a^2t^6x^4-4at^7x^4+3at^7x^5-a^2t^7x^5+2a^2t^8x^5+2a^2t^8x^6-2a^2t^9x^6+4a^3t^9x^6+a^2t^{10}x^6-a^3t^{10}x^7+2a^3t^{11}x^7+a^4t^{12}x^8)y^2$ $+a^3t^3x^2(x-1)^2(1+tx-t^2x-t^3x^2+2at^3x^2+2at^4x^2+2at^4x^3-2at^5x^3-2at^6x^3+3at^6x^4+a^2t^6x^4+4a^2t^7x^4+a^2t^7x^5-a^2t^8x^5+2a^2t^9x^5+2a^3t^{10}x^6)y$ $-a^5t^{11}x^7(x-1)^3$

Quantum super-A-polynomials

- Refinement of AJ conjecture

$$\widehat{A}^{\text{super}}(K;\hat{x},\hat{y};a,q,t)\mathcal{P}_n(K;a,q,t)=0$$

where the operators $\hat{x}$ and $\hat{y}$ acts on the set of colored superpolynomials as

$$\hat{x}\mathcal{P}_n(K;a,q,t)=q^n\mathcal{P}_n(K;a,q,t)\ ,\quad \hat{y}\mathcal{P}_n(K;q)=\mathcal{P}_{n+1}(K;a,q,t)\ .$$

- Find difference equations of colored superpolynomials

$$a_k\mathcal{P}_{n+k}(K;q)+\ldots+a_1\mathcal{P}_{n+1}(K;q)+a_0\mathcal{P}_n(K;q)=0$$

where $a_k=a_k(K;\widehat{x};a,q,t)$ and

$$\widehat{A}^{\text{super}}(K;\widehat{x},\widehat{y};a,q,t)=\sum a_i(K;\widehat{x};a,q,t)\widehat{y}^i$$

- Taking the classical limit $q=e^\hbar\to 1$, quantum (non-commutative) super-A-polynomials reduces to classical super-A-polynomials

$$\widehat{A}^{\text{super}}(K;\widehat{x},\widehat{y};a,q,t)\to A^{\text{super}}(K;x,y;a,q,t)\quad\text{as}\quad q\to 1$$

Quantum super-A-polynomials

Knot	$\widehat{A}^{\rm super}(K;\hat{x},\hat{y};a,q,t)$
$\mathbf{5_2}$	$q^6t^4\left(1+at^3\hat{x}\right)\left(1+aqt^3\hat{x}\right)\left(1+aq^2t^3\hat{x}\right)\left(1+at^3\hat{x}^2\right)\left(q+at^3\hat{x}^2\right)\left(1+aqt^3\hat{x}^2\right)\hat{y}^4$ $-aq^5t^4(1+at^3\hat{x})(1+aqt^3\hat{x})(1+at^3\hat{x}^2)(q+at^3\hat{x}^2)(1+aq^4t^3\hat{x}^2)(1+q-q^3\hat{x}+q^3t\hat{x}-q^2t^2\hat{x}-q^3t^2\hat{x}+q^4t^2\hat{x}^2+aq^4t^2\hat{x}^2+q^5t^2\hat{x}^2+q^6t^2\hat{x}^2+aqt^3\hat{x}^2+aq^2t^3\hat{x}^2+aq^5t^3\hat{x}^2+aq^6t^3\hat{x}^2-aq^4t^3\hat{x}^3-aq^8t^3\hat{x}^3+aq^4t^4\hat{x}^3+aq^8t^4\hat{x}^3+aq^5t^5\hat{x}^3+aq^6t^5\hat{x}^3+a^2q^5t^5\hat{x}^4-aq^8t^5\hat{x}^4+a^2q^9t^5\hat{x}^4+a^2q^6t^6\hat{x}^4+a^2q^7t^6\hat{x}^4-a^2q^9t^6\hat{x}^5+a^2q^9t^7\hat{x}^5+a^3q^{10}t^8\hat{x}^6)\hat{y}^3$ $-a^2q^5t^4(-1+q^2\hat{x})(1+at^3\hat{x})(q+at^3\hat{x}^2)(1+aq^2t^3\hat{x}^2)(1+aq^5t^3\hat{x}^2)(1+q^2t\hat{x}-qt^2\hat{x}-q^2t^2\hat{x}+q^3t^2\hat{x}^2+q^4t^2\hat{x}^2+at^3\hat{x}^2+aqt^3\hat{x}^2-q^3t^3\hat{x}^2+aq^3t^3\hat{x}^2-q^4t^3\hat{x}^2+aq^4t^3\hat{x}^2+q^3t^4\hat{x}^2+aq^2t^4\hat{x}^3-q^4t^4\hat{x}^3-aq^4t^4\hat{x}^3-q^5t^4\hat{x}^3-q^6t^4\hat{x}^3+aq^6t^4\hat{x}^3-aqt^5\hat{x}^3-aq^2t^5\hat{x}^3+aq^3t^5\hat{x}^3+aq^4t^5\hat{x}^3-aq^5t^5\hat{x}^3-aq^6t^5\hat{x}^3+aq^3t^5\hat{x}^4+aq^4t^5\hat{x}^4+aq^7t^5\hat{x}^4+aq^8t^5\hat{x}^4+a^2qt^6\hat{x}^4-aq^3t^6\hat{x}^4+a^2q^3t^6\hat{x}^4-aq^4t^6\hat{x}^4+2a^2q^4t^6\hat{x}^4+a^2q^5t^6\hat{x}^4-aq^7t^6\hat{x}^4+a^2q^7t^6\hat{x}^4-aq^8t^6\hat{x}^4-aq^4t^7\hat{x}^4-2aq^5t^7\hat{x}^4-aq^6t^7\hat{x}^4-a^2q^4t^7\hat{x}^5+aq^6t^7\hat{x}^5+a^2q^6t^7\hat{x}^5+aq^7t^7\hat{x}^5+aq^8t^7\hat{x}^5-a^2q^8t^7\hat{x}^5+a^2q^3t^8\hat{x}^5+a^2q^4t^8\hat{x}^5-a^2q^5t^8\hat{x}^5-a^2q^6t^8\hat{x}^5+a^2q^7t^8\hat{x}^5+a^2q^8t^8\hat{x}^5+a^2q^7t^8\hat{x}^6+a^2q^8t^8\hat{x}^6+a^3q^4t^9\hat{x}^6+a^3q^5t^9\hat{x}^6-a^2q^7t^9\hat{x}^6+a^3q^7t^9\hat{x}^6-a^2q^8t^9\hat{x}^6+a^3q^8t^9\hat{x}^6+a^2q^7t^{10}\hat{x}^6-a^3q^8t^{10}\hat{x}^7+a^3q^7t^{11}\hat{x}^7+a^3q^8t^{11}\hat{x}^7+a^4q^8t^{12}\hat{x}^8)\hat{y}^2$ $+a^3q^7t^7\hat{x}^2(-1+q\hat{x})(-1+q^2\hat{x})(1+at^3\hat{x}^2)(1+aq^4t^3\hat{x}^2)(1+aq^5t^3\hat{x}^2)(q+q^2t\hat{x}-q^2t^2\hat{x}+at^3\hat{x}^2-q^3t^3\hat{x}^2+aq^4t^3\hat{x}^2+aqt^4\hat{x}^2+aq^2t^4\hat{x}^2+aqt^4\hat{x}^3+aq^5t^4\hat{x}^3-aqt^5\hat{x}^3-aq^5t^5\hat{x}^3-aq^2t^6\hat{x}^3-aq^3t^6\hat{x}^3+aq^3t^6\hat{x}^4+a^2q^3t^6\hat{x}^4+aq^4t^6\hat{x}^4+aq^5t^6\hat{x}^4+a^2t^7\hat{x}^4+a^2qt^7\hat{x}^4+a^2q^4t^7\hat{x}^4+a^2q^5t^7\hat{x}^4+a^2q^4t^7\hat{x}^5-a^2q^4t^8\hat{x}^5+a^2q^3t^9\hat{x}^5+a^2q^4t^9\hat{x}^5+a^3q^3t^{10}\hat{x}^6+a^3q^4t^{10}\hat{x}^6)\hat{y}$ $-a^5q^8t^{15}(-1+\hat{x})\hat{x}^7(-1+q\hat{x})(-1+q^2\hat{x})(1+aq^3t^3\hat{x}^2)(1+aq^4t^3\hat{x}^2)(1+aq^5t^3\hat{x}^2)$

Augmentation polynomials of knot contact homology

- Q-deformed A-polynomial is equivalent to augmentation polynomial of knot contact homology [Ng '10][Aganagic Vafa '12]

$$A^{\rm super}\left(K_{p>0}; x=-\mu, y=\frac{1+\mu}{1+U\mu}\lambda; a=U, t=-1\right) = \frac{(-1)^{p-1}(1+\mu)^{(2p-1)}}{1+U\mu}{\rm Aug}(K_{p>0};\mu,\lambda;U,V=1)\ ,$$

$$A^{\rm super}\left(K_{p<0}; x=-\mu, y=\frac{1+\mu}{1+U\mu}\lambda; a=U, t=-1\right) = \frac{(-1)^{p}(1+\mu)^{-2p}}{1+U\mu}{\rm Aug}(K_{p<0};\mu,\lambda;U,V=1)\ ,$$

Prospects and Future Directions

- Anti-symmetric representation. Mirror symmetry
- Quantum 6j-symbolos for $U_q(\mathfrak{sl}_N)$ and their refinement
- Colored superpolynomials of other non-torus knots
- $SU(N)$ analogue of WRT invariants and one-parameter deformations of Mock modular forms.
- refinement of colored Kaufmann polynomials

Thank you