Complex Functions
for students of engineering sciences

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Contents.

„Function Theory“or „Complex Analysis“:

- Functions of one complex variable
- Möbius transforms
- Complex differentiation and integration
- Potential problems in the plane
- Conformal mappings
- Cauchy’s integration formula and applications
- Taylor and Laurent Series
- Isolated singularities and residues
- Integral transforms: Fourier and Laplace transforms
- Shannon’s sampling theorem

1 Complex Numbers

Starting point: We want to solve all equations of the form

$$x^2 = a \qquad \text{for } a \in \mathbb{R}.$$

Good news:
For non-negative $a \in [0, \infty)$ there is (at least) one $x \in \mathbb{R}$ satisfying $x^2 = a$.

Bad news: For negative $a \in (-\infty, 0)$ there is no $x \in \mathbb{R}$ satisfying $x^2 = a$.

Example: For $a = -1$ there is no real number x satisfying

$$x^2 + 1 = 0.$$

What now? To solve all equations of the form $x^2 = a$, we need to extend the real numbers. This extension leads to the `field of the complex numbers`, $\mathbb{C}$.

In the following we discuss the algebraic and geometric structure of $\mathbb{C}$.

First ideas to introduce the complex numbers.

Starting point: Use *symbolic solution* i for equation $x^2 + 1 = 0$, so that

$$i^2 = -1.$$

This number i is called `imaginary unit`.

Next step: Using the imaginary unit, we define the number set

$$\mathbb{C} := \{a + ib \,|\, a, b \in \mathbb{R}\}.$$

Then, we introduce operations on $\mathbb{C}$ as follows.

- **Summation:**

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}.$$

- **Multiplication:**

$$(a_1+ib_1)(a_2+ib_2) = (a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}.$$

So then $\mathbb{C}$ has an algebraic structure.

General questions concerning complex numbers.

- What is $\mathfrak{i}$?
- Can we use the operations $+$ and $\cdot$ without contradictions?
- Are the operations $+$ and $\cdot$ consistent with those on $\mathbb{R}$?
- Can we order the complex numbers?
- Are there alternative representations for complex numbers?
- Are there geometric interpretations for $+$ and $\cdot$?
- ...
- Why do we deal with complex nunmbers?
- ...and later with complex functions?
- Are there relevant applications in engineering?

Reminder: The real numbers $\mathbb{R}$ are — in combination with summation $'+'$ and multiplication $'\cdot'$ — a field, i.e., the following axioms hold:

- Axioms for summation.

associative law	$\forall x, y, z \in \mathbb{R}:$	$x + (y + z)$	$=$	$(x + y) + z$
commutative law	$\forall x, y \in \mathbb{R}:$	$x + y$	$=$	$y + x$
the zero element	$\forall\, x \in \mathbb{R}\ \exists\, 0 \in \mathbb{R}:$	$x + 0$	$=$	x
the inverse	$\forall\, x \in \mathbb{R}\ \exists -x \in \mathbb{R}:$	$x + (-x)$	$=$	0

- Axioms for multiplication.

associative law	$\forall x, y, z \in \mathbb{R}:$	$(xy)z$	$=$	$x(yz)$
commutative law	$\forall x, y \in \mathbb{R}:$	xy	$=$	yx
the one element	$\forall\, x \in \mathbb{R}\ \exists\, 1 \in \mathbb{R}:$	$x \cdot 1$	$=$	x
the inverse	$\forall\, x \in \mathbb{R} \setminus \{0\}\ \exists x^{-1} \in \mathbb{R}:$	xx^{-1}	$=$	$1.$

- distributive law $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{R}$.

On the construction of the complex numbers.

Starting point: Regard the set $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$ with the **summation**

$$(a_1,b_1) + (a_2,b_2) := (a_1 + a_2, b_1 + b_2) \qquad \text{for } (a_1,b_1),(a_2,b_2) \in \mathbb{R}^2$$

and the **multiplication**

$$(a_1,b_1)\cdot(a_2,b_2) := (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) \qquad \text{for } (a_1,b_1),(a_2,b_2) \in \mathbb{R}^2.$$

Observation: The multiplication is associative and commutative; moreover, we have

$$(a,b)\cdot(1,0) = (a,b) \qquad \text{for } (a,b) \in \mathbb{R}^2,$$

i.e., $(1,0) \in \mathbb{C}$ is the `neutral element of multiplication`. The equation

$$(a,b)\cdot(x,y) = (1,0) \qquad \text{for } (a,b) \neq (0,0)$$

has a unique solution, the `multiplicative inverse` of (a,b),

$$(x,y) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right).$$

On the structure of the complex numbers.

Remark: The set $\mathbb{R}^2$ is — with the summation $'+'$ and the multiplication $'\cdot'$ — a *field*, the field of the complex numbers, denoted as $(\mathbb{C}, +, \cdot)$, or $\mathbb{C}$.

Exercise:

Verify all axioms of summation and multiplication and the distribution rule. $\square$

Observation: The mapping $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$, defined as $\varphi(a) = (a, 0)$ is injective. For all $a_1, a_2 \in \mathbb{R}$, we have

$$\begin{aligned} \varphi(a_1 + a_2) &= (a_1 + a_2, 0) = (a_1, 0) + (a_2, 0) = \varphi(a_1) + \varphi(a_2) \\ \varphi(a_1 a_2) &= (a_1 a_2, 0) = (a_1, 0) \cdot (a_2, 0) = \varphi(a_1) \cdot \varphi(a_2) \end{aligned}$$

Conclusion:

- We can identify the real numbers with the complex numbers of the form $(a, 0)$;
- The real numbers are a subfield of $\mathbb{C}$;
- The operations $+$ and $\cdot$ in $\mathbb{C}$ are consistent with those in $\mathbb{R}$.

The field of the real numbers is ordered.

Remark: The real numbers are an ordered field, i.e., the following ordering axioms hold.

- For any $x \in \mathbb{R}$ we have $x > 0$ or $x = 0$ or $x < 0$;
- For $x > 0$ and $y > 0$ we have $x + y > 0$;
- For $x > 0$ and $y > 0$ we have $xy > 0$.

Question: Are the complex numbers $\mathbb{C}$ ordered?

Answer: **NO!**

Reason: In any ordered field all square numbers (except for zero) are positive. Now suppose $\mathbb{C}$ is ordered. Then, we have

$$0 < 1^2 = 1 \qquad \text{and} \qquad 0 < i^2 = -1$$

giving the contradiction $0 < 1 + (-1) = 0$. $\blacksquare$

A simpler notation for complex numbers.

Simplification of notations:

- For $a \in \mathbb{R}$ we write just 'a' rather than $(a, 0)$;
- The complex unit $(0, 1)$ is denoted as i;
- Thereby, any complex number (a, b) can be written as

$$(a, b) = (a, 0) + (b, 0) \cdot (0, 1) = a + b \cdot i = a + ib.$$

and we have

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0) = -1.$$

Conclusion: We have constructed a field, $\mathbb{C}$, which comprises $\mathbb{R}$. The equation

$$x^2 + 1 = 0$$

has a solution in $\mathbb{C}$. The only solutions for that equation are $\pm i$.

Real part and imaginary part.

From now we denote complex numbers as z or w. For

$$z = x + iy \in \mathbb{C} \qquad \text{for } x, y \in \mathbb{R}$$

we call x the real part and y the imaginary part of z, in short:

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z)$$

The following properties hold.

$$\begin{aligned}
\mathrm{Re}(z+w) &= \mathrm{Re}(z) + \mathrm{Re}(w) && \text{for } z, w \in \mathbb{C} \\
\mathrm{Im}(z+w) &= \mathrm{Im}(z) + \mathrm{Im}(w) && \text{for } z, w \in \mathbb{C} \\
\mathrm{Re}(az) &= a\mathrm{Re}(z) && \text{for } z \in \mathbb{C}, a \in \mathbb{R} \\
\mathrm{Im}(az) &= a\mathrm{Im}(z) && \text{for } z \in \mathbb{C}, a \in \mathbb{R}
\end{aligned}$$

and moreover

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \qquad \text{for } z \neq 0.$$

The complex plane.

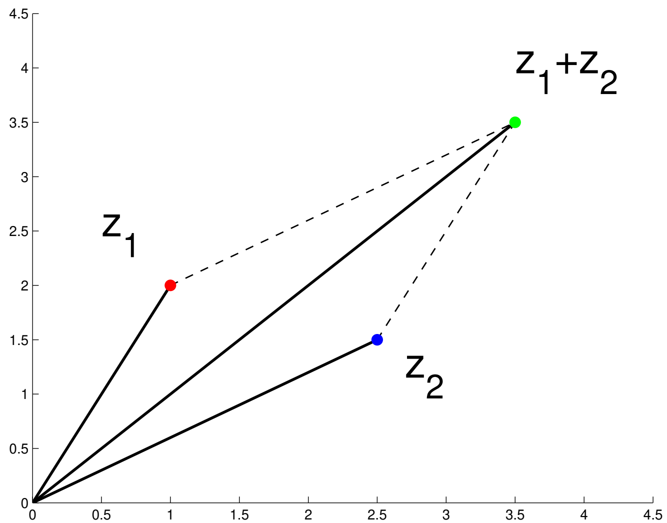
Geometric illustration: We represent $z = (x, y) \in \mathbb{C}$ as a `point` in the

`complex plane` (`Gaussian plane of complex numbers`)

by the Cartesian coordinate system of $\mathbb{R}^2$, with one `real axis`, $\mathbb{R}$, and one `imaginary axis`, $i \cdot \mathbb{R}$.

Geometric illustration of summation:

By summation of vectors according to parallelogram formula.



Summation of two complex numbers.

Conjugation of complex numbers.

By reflection of the complex number $z = x + iy$ with respect to the real axis we obtain the complex number

$$\bar{z} := x - iy \in \mathbb{C}$$

called the `complex conjugate` of z.

The following properties hold.

$$\begin{array}{rcll}
\overline{w+z} & = & \overline{w} + \bar{z} & \text{for } w, z \in \mathbb{C} \\
\overline{wz} & = & \overline{w} \cdot \bar{z} & \text{for } w, z \in \mathbb{C} \\
\overline{(\bar{z})} & = & z & \text{for } z \in \mathbb{C} \\
z\bar{z} & = & x^2 + y^2 & \text{for } z = x + iy \in \mathbb{C} \\
\mathrm{Re}(z) & = & (z + \bar{z})/2 & \text{for } z \in \mathbb{C} \\
\mathrm{Im}(z) & = & (z - \bar{z})/(2i) & \text{for } z \in \mathbb{C}
\end{array}$$

In particular, we have $z = \bar{z}$ if and only if $z \in \mathbb{R}$.

The modulus function.

By
$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} \qquad \text{for } z = x + iy \in \mathbb{C}$$
we denote the modulus of z and by $|w - z|$ we denote the distance between the two numbers $w, z \in \mathbb{C}$ in the complex plane.

- Therefore, $|z| = |z - 0|$ is the Euclidean distance of z to the origin.
- For $z \in \mathbb{R}$, its usual modulus (i.e., absolute value) of real numbers is $|z|$.
- The following inequalities hold.

$$-|z| \leq \mathrm{Re}(z) \leq |z| \qquad \text{and} \qquad -|z| \leq \mathrm{Im}(z) \leq |z| \qquad \text{for } z \in \mathbb{C}.$$

Theorem: The modulus function $|\cdot|$ is a norm on $\mathbb{C}$, i.e., we have

- $|z| \geq 0$ for all $z \in \mathbb{C}$ and $|z| = 0$, if and only if $z = 0$;
- $|w + z| \leq |w| + |z|$ for all $w, z \in \mathbb{C}$ (triangular inequality);
- $|wz| = |w| \cdot |z|$ for all $w, z \in \mathbb{C}$.

The Euler formula.

In the complex plane, we have for

$$z = x + iy$$

with polar coordinates

$$(x, y) = |z|(\cos(\varphi), \sin(\varphi))$$

the Euler formula

$$z = |z| \exp(i\varphi) = |z|(\cos(\varphi) + i \sin(\varphi)),$$

where $\varphi \in (-\pi, \pi]$ is, for $z \neq 0$, the (unique) angle between the positive real axis and the straight line from 0 to $z = (x, y)$.

The angle $\varphi \in (-\pi, \pi]$ is also called the argument of $z \neq 0$, in short

$$\varphi = \arg(z) \in (-\pi, \pi].$$

Example: $i = (0, 1) = \exp(i\pi/2)$, $-1 = i^2 = \exp(i\pi)$, whereby $\boxed{e^{i\pi} + 1 = 0}$.

The geometry of multiplication and division.

Using polar coordinates we can interpret the multiplication of two complex numbers $w, z \in \mathbb{C}$ as a stretching rotation in the complex plane, since by

$$w = |w|(\cos(\psi), \sin(\psi)) \qquad \text{and} \qquad z = |z|(\cos(\varphi), \sin(\varphi))$$

we have

$$\begin{aligned} wz &= |w| \cdot |z|(\cos(\psi) + \mathrm{i}\sin(\psi))(\cos(\varphi) + \mathrm{i}\sin(\varphi)) \\ &= |w| \cdot |z|(\cos(\psi + \varphi) + \mathrm{i}\sin(\psi + \varphi)) = |w| \cdot |z| \exp(\mathrm{i}(\psi + \varphi)) \end{aligned}$$

or, by the Euler formula

$$wz = |w| \cdot |z| \exp(\mathrm{i}\psi) \cdot \exp(\mathrm{i}\varphi) = |w| \cdot |z| \exp(\mathrm{i}(\psi + \varphi)).$$

Likewise, for the division of complex numbers $w, z \in \mathbb{C}$, where $z \neq 0$, we have

$$\frac{w}{z} = \frac{|w|}{|z|} \exp(\mathrm{i}(\psi - \varphi)) = \frac{|w|}{|z|}(\cos(\psi - \varphi) + \mathrm{i}\sin(\psi - \varphi)).$$

Powers and unit roots.

For the n-th power z^n, $n \in \mathbb{N}$, of $z \in \mathbb{C}$ we have the representation

$$z^n = (|z|\exp(i\varphi))^n = |z|^n \exp(in\varphi) = |z|^n(\cos(n\varphi) + i\sin(n\varphi)).$$

The equation

$$z^n = 1$$

has the n pairwise distinct solutions

$$z_k = \exp\left(i\frac{2\pi k}{n}\right) \qquad \text{for } k = 0, \ldots, n-1.$$

These solutions are called the n-th unit roots.

2 Complex Functions

We regard complex-valued functions f of one complex variable.

2.1 Notation and geometric interpretation

Definition: *A* complex function *is a function, whose domain and range are, respectively, point sets in the complex plane.* $\square$

Remark: A complex function $f : A \longrightarrow B$ with domain $A \subset \mathbb{C}$ and image $B = f(A) \subset \mathbb{C}$ maps each $z \in A$ to one unique $w = f(z) \in B$, i.e., $z \longmapsto f(z)$.

This unique assignment (by the map $f : A \longrightarrow B$)

$$z \longmapsto f(z) \qquad \text{for } z \in A$$

is usually determined by an *explicit* formula for $f(z)$, for $z \in A$.

However, complex functions may also be determined *implicitly*.

Examples for complex functions.

- $f(z) = (3z+1)^2$ for $z \in \mathbb{C}$;
- $f(z) = \exp(ix) + y$ for $z = x + iy \in \mathbb{C}$;
- $f(z) = 1/z$ for $z \in \mathbb{C} \setminus \{0\}$.

We usually use the symbol $z \in \mathbb{C}$ for the argument and $w \in \mathbb{C}$ for the (function) value of f at z, i.e., $w = f(z)$. Moreover, we write $z = x + iy$ and

$$w = u + iv \qquad \text{i.e., } u = \mathrm{Re}(w) \text{ and } v = \mathrm{Im}(w)$$

or

$$u(z) = \mathrm{Re}(f(z)) \qquad \text{and } v(z) = \mathrm{Im}(f(z)).$$

Question: How can we display f graphically?

Answer: We sketch the domain and the image in two different complex planes, namely in the z-plane (for the domain) and in the w-plane (for the image).

Complex functions in one real variable.

We also regard *complex-valued* functions $f : I \longrightarrow \mathbb{C}$ for one *real* variable, i.e., for arguments in $I \subset \mathbb{R}$,

$$f : t \longmapsto f(t) \in \mathbb{C} \qquad \text{for } t \in I.$$

Examples.

- $f(t) = a + bt$ for $a, b \in \mathbb{C}$, where $b \neq 0$;
- $f(t) = \exp(i\omega t)$ for $\omega \in (0, \infty) \subset \mathbb{R}$;

2.2 Lineare functions

Definition: *A complex function* f *is said to be* affine-linear *(or just* linear*), iff* f *has the form*

$$f(z) = az + b \qquad \text{for } z \in \mathbb{C}$$

for fixed complex constants $a, b \in \mathbb{C}$*, where* $a \neq 0$*.* □

Question: How can we interpret linear functions geometrically?

Special case 1: The choice $a = 1$ leads us to a translation about b,

$$f(z) = z + b \qquad \text{for } z \in \mathbb{C}$$

Special case 2: The choice $a \in (0, \infty)$ and $b = 0$ leads us to a dilation

$$f(z) = az \qquad \text{for } z \in \mathbb{C}$$

i.e., the argument z is elongated (for $a > 1$) or shrunk ($0 < a < 1$).
But in general, we just use the term dilation by scaling factor $a > 0$.

Special case 3: The choice $a \in \mathbb{C}$ with $|a| = 1$ and $b = 0$ leads us to a rotation

$$f(z) = az \qquad \text{for } z \in \mathbb{C}$$

More precisely, a rotation about one angle $\alpha \in [0, 2\pi)$, where $\alpha = \arg(a)$, or, $a = \exp(i\alpha)$.

Special case 4: The choice $a \in \mathbb{C}$, with $a \neq 0$ and $b = 0$ leads us to a stretching rotation

$$f(z) = az \qquad \text{for } z \in \mathbb{C}$$

which we can interpret as a composition between one rotation and one dilation.

More precisely: For

$$a = |a| \exp(i\alpha) \qquad \text{with } \alpha = \arg(a)$$

we have one rotation about angle $\alpha \in [0, 2\pi)$ and one dilation by scaling $|a|$.

General case: For $a, b \in \mathbb{C}$, $a \neq 0$, any linear function

$$f(z) = az + b = |a| \exp(i\alpha)z + b$$

can be written as composition

$$f = f_3 \circ f_2 \circ f_1$$

of three functions:

- $f_1(z) = \exp(i\alpha)z$, i.e., rotation about angle $\alpha \in [0, 2\pi)$;
- $f_2(z) = |a|z$, i.e., dilation by scaling factor $|a| > 0$;
- $f_3(z) = z + b$, i.e., translation about b.

Remark: Rotation f_1 and dilation f_2 are commutative, i.e., we have

$$f_2 \circ f_1 = f_1 \circ f_2$$

and so

$$f = f_3 \circ f_2 \circ f_1 = f_3 \circ f_1 \circ f_2.$$

□

2.3 Quadratic functions

Definition: *A complex function* f *is said to be* quadratic, *if* f *has the form*

$$f(z) = az^2 + bz + c \qquad \textit{for } z \in \mathbb{C}$$

for fixed constants $a, b, c \in \mathbb{C}$, *where* $a \neq 0$. □

Let us first regard the geometric behaviour of the quadratic function

$$f(z) = z^2 \qquad \text{for } z \in \mathbb{C}.$$

To this end, we regard the images of straight lines in the complex plane that are parallel to the (real and imaginary) axes.

Let $w = z^2$. Then, we have for $z = x + iy$ and $w = u + iv$ the representation

$$w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

and so

$$u = x^2 - y^2 \qquad \text{and} \qquad v = 2xy.$$

Images of axes parallel lines under $z \longmapsto z^2$.

For the image of a straight lines parallel to the real axis, i.e., $y \equiv y_0$, we obtain

$$\begin{aligned} u &= x^2 - y_0^2 \\ v &= 2xy_0 \end{aligned}$$

For $y_0 = 0$ (the real axis) we obtain $u = x^2$ and $v = 0$.

For $y_0 \neq 0$ we can eliminate x by letting $x = v/(2y_0)$, whereby we have

$$u = \frac{v^2}{4y_0^2} - y_0^2$$

a (right-open) parabola which is symmetric to the u-axis, with focal point zero, and with intersections $u = -y_0^2$ (with u-axis) and $v = \pm 2y_0^2$ (with v-axis).

Conclusion: The family of parallel lines to the x-axis is mapped under the quadratic function $f(z) = z^2$ onto a family of `co-focal` (i.e., having the same symmetry axis, and the same focal point) (right-open) parabolas.

The straight lines $y \equiv y_0$ and $y \equiv -y_0$ are mapped onto the same parabola.

Images of axes parallel lines under $z \longmapsto z^2$.

For the image of a straight lines parallel to the imaginary axis, i.e., $x \equiv x_0$, we obtain

$$\begin{aligned} u &= x_0^2 - y^2 \\ v &= 2x_0 y \end{aligned}$$

For $x_0 = 0$ (the imaginary axis) we obtain $u = -y^2$ and $v = 0$.

For $x_0 \neq 0$ we can eliminate y by letting $y = v/(2x_0)$, whereby we have

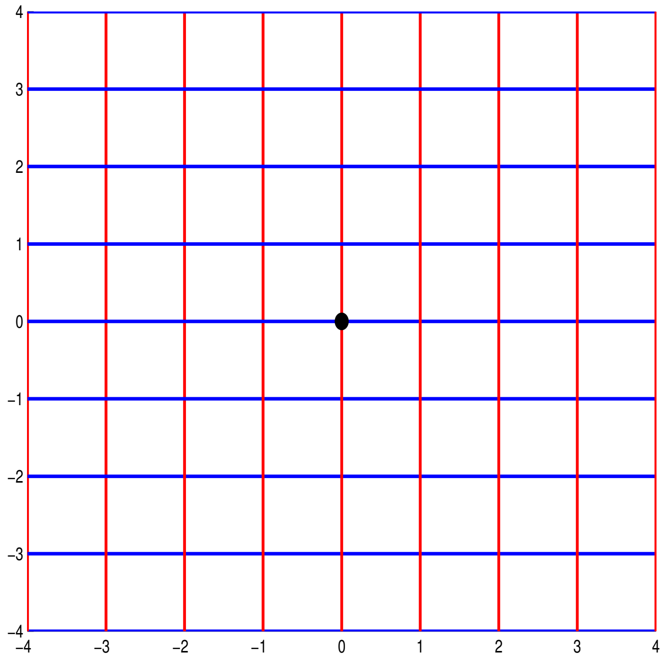
$$u = x_0^2 - \frac{v^2}{4x_0^2}$$

a (left-open) parabola which is symmetric to the u-axis, with focal point zero, and with intersections $u = -x_0^2$ (with u-axis) and $v = \pm 2x_0^2$ (with v-axis).

Conclusion: The family of parallel lines to the y-axis is mapped under the quadratic function $f(z) = z^2$ onto a family of `co-focal` (left-open) parabolas.

The straight lines $x \equiv x_0$ and $x \equiv -x_0$ are mapped onto the same parabola.

Images of axes parallel lines under $z \longmapsto z^2$.



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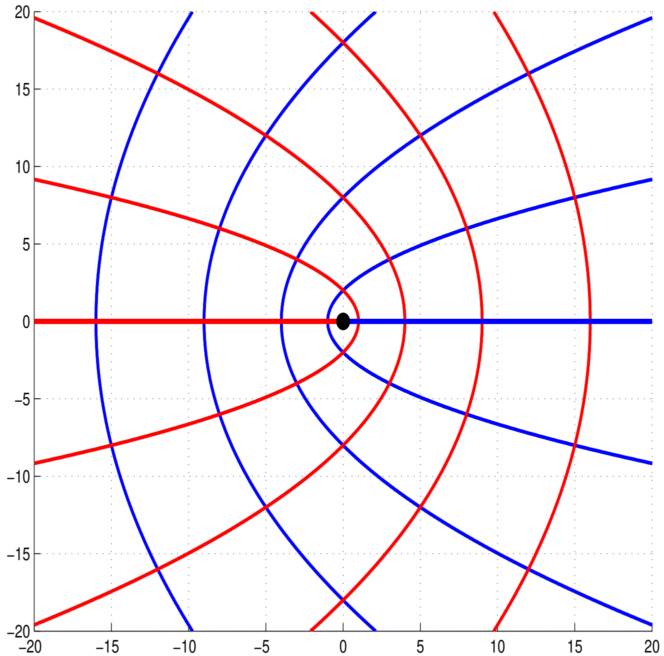


image of $f(z) = z^2$.

General quadratic functions.

Starting from the representation

$$f(z) = az^2 + bz + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

for $a, b, c \in \mathbb{C}$, $a \neq 0$, we can write any quadratic function as a composition

$$f = f_4 \circ f_3 \circ f_2 \circ f_1$$

of four mappings:

- the translation $f_1(z) = z + \frac{b}{2a}$;
- the quadratic function $f_2(z) = z^2$;
- the dilation $f_3(z) = az$;
- the translation $f_4(z) = z - \frac{b^2}{4a} + c$.

$\square$

2.4 The exponential function

Definition: *The* complex exponential function $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ *is defined as*

$$\exp(z) \equiv e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)) \qquad \text{for } z = x + iy.$$

$\square$

Observe: We have the functional equation

$$e^{z_1+z_2} = e^{z_1} e^{z_2} \qquad \text{for } z_1, z_2 \in \mathbb{C}.$$

Question: How can we sketch the complex exponential function $z \longmapsto \exp(z)$?

For $w = \exp(z)$, $z = x + iy$ and $w = u + iv$ we obtain

$$w = u + iv = e^z = e^x(\cos(y) + i\sin(y))$$

and so

$$u = e^x \cos(y) \qquad \text{and} \qquad v = e^x \sin(y).$$

Images of axes parallel lines under $z \longmapsto \exp(z)$.

For the image of a straight lines parallel to the real axis, i.e., $y \equiv y_0$, we obtain

$$\begin{aligned} u &= e^x \cos(y_0) \\ v &= e^x \sin(y_0) \end{aligned}$$

- For fixed y_0 this yields a straight line from the origin through the positive quadrant, whose angle with the positive real line is y_0.
- For angles y_0 and y_1, that are different by an integer multiple of 2π, i.e.,

  $$y_1 = y_0 + 2\pi k \qquad \text{for one } k \in \mathbb{Z}$$

  we obtain the same straight line.
- More precisely: by the periodicity of $\exp(z)$ we have

  $$e^{z+2\pi i k} = e^z e^{2\pi i k} = e^z(\cos(2\pi k) + i \sin(2\pi k)) = e^z \cdot 1 = e^z.$$

  i.e., two complex numbers with coincident real parts, but whose imaginary parts differ about one integer multiple of 2π, are mapped onto the same value.

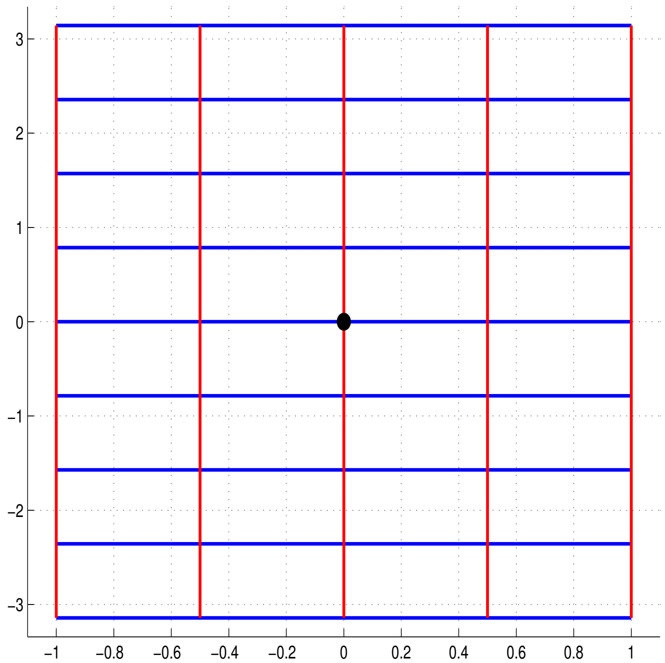
Images of axes parallel lines under $z \longmapsto \exp(z)$.

For images of straight lines parallel to the imaginary axis, i.e., $x \equiv x_0$, we obtain

$$u = e^{x_0}\cos(y) \qquad \text{and} \qquad v = e^{x_0}\sin(y)$$

- For fixed x_0 we obtain a circle with origin zero and radius e^{x_0}.
- **Observation:** The origin is not contained in the image of the exponential function, i.e., there is no argument $z \in \mathbb{C}$ satisfying $\exp(z) = 0$. Therefore, we have $e^z \neq 0$ for all $z \in \mathbb{C}$.
- **Observation:** The function exp maps rectangular grids in the Cartesian coordinate system onto families of curves with orthogonal intersections.
- **More precisely:** Curves with orthogonal intersections in the Cartesian coordinate system are being mapped by onto curves, whose intersections in the Cartesian coordinate system are also orthogonal.
- **Terminology:** We say that the exponential function `preserves angles`, or, $\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$ is said to be a `conformal` map. More details later.

Images of axes parallel lines under $z \longmapsto \exp(z)$.



domain.

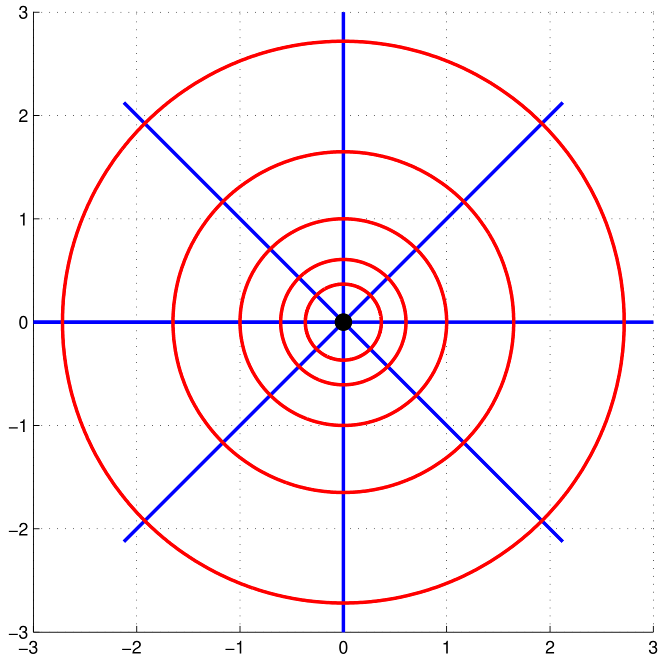


image of $f(z) = \exp(z)$.

2.5 The inverse function

Definition: *A complex function $f(z)$ is said to be* injective *(*one-to-one*), if for every point $w \in \mathbb{C}$ in its image there is one and only one point $z \in \mathbb{C}$ of its domain satisfying $f(z) = w$.* □

Remark: Injective functions attain each value of its image only once.

Examples.

- any linear function $f(z) = az + b$, where $a \neq 0$, is injective.
- the quadratic function $f(z) = z^2$, is *not* injective, since we have $f(z) = f(-z)$ for all $z \in \mathbb{C}$.
- the complex exponential function $\exp(z)$ is *not* injective, since we have $\exp(z) = \exp(z + 2\pi i k)$ for all $k \in \mathbb{Z}$ and all $z \in \mathbb{C}$.

Restriction of the domain.

Remark: A non-injective function may be turned into an injective function by a suitable restriction of its domain.

Example: The quadratic function

$$f(z) = z^2 \qquad \text{for } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0$$

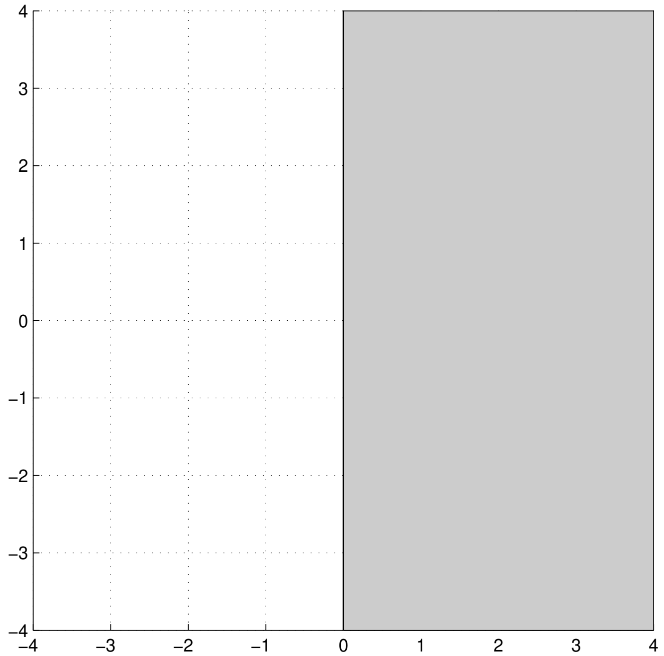
is injective on the right half plane $A := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}$.

Moreover, in this case the image of f is $f(A) = \mathbb{C}^-$, where we let

$$\begin{aligned} \mathbb{C}^- &= \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \neq 0 \text{ or } \mathrm{Re}(z) > 0\} \\ &= \mathbb{C} \setminus \{z \in \mathbb{R} \,|\, z \leq 0\} \end{aligned}$$

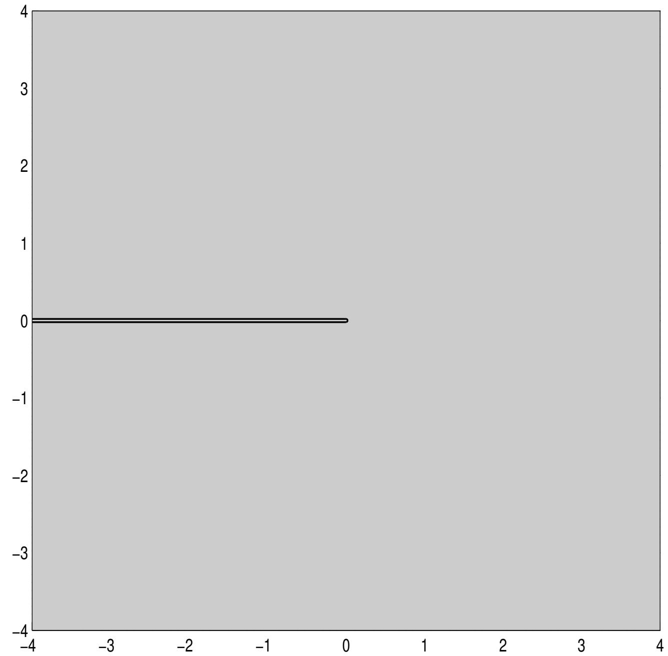
denote the cut complex plane. □

Function values of $z \longmapsto z^2$ on the right half plane.



the right half plane

(domain)



the cut complex plane $\mathbb{C}^-$

(image)

The inverse function.

Definition: *Let* f *be an injective function with domain* $D(f)$ *and range* $W(f)$. *Then, the (unique)* inverse function $f^{-1} : W(f) \longrightarrow D(f)$ *of* f *is that function, which assigns each value* $w \in W(f)$ *the (unique) argument* $z \in D(f)$ *satisfying* $f(z) = w$, *i.e.,* $f^{-1}(w) = z$, *or,*

$$
\begin{aligned}
(f^{-1} \circ f)(z) &= z \qquad \textit{for all } z \in D(f) \\
(f \circ f^{-1})(w) &= w \qquad \textit{for all } w \in W(f)
\end{aligned}
$$

$\square$

Example: For the domain

$$D(f) = \{z = re^{i\varphi} \in \mathbb{C} \,|\, r > 0 \text{ and } -\pi/2 < \varphi < \pi/2\}$$

there is a (unique) inverse function f^{-1} of $f(z) = z^2$ with range $W(f) = \mathbb{C}^-$.

For the principal value of the square root $f^{-1} : W(f) \longrightarrow D(f)$ we get

$$w = f^{-1}(z) = \sqrt{r}e^{i\varphi/2} \qquad \text{for } z = re^{i\varphi} \text{ with } \varphi = \arg(z) \in (-\pi/2, \pi/2).$$

The inverse function of the n-th power.

Example: For $n \geq 2$, the power function

$$f(z) = z^n \qquad \text{for } z \in \mathbb{C}$$

is injective on the domain

$$D(f) = \left\{ z \in \mathbb{C} \;\middle|\; -\frac{\pi}{n} < \arg(z) < \frac{\pi}{n} \right\}.$$

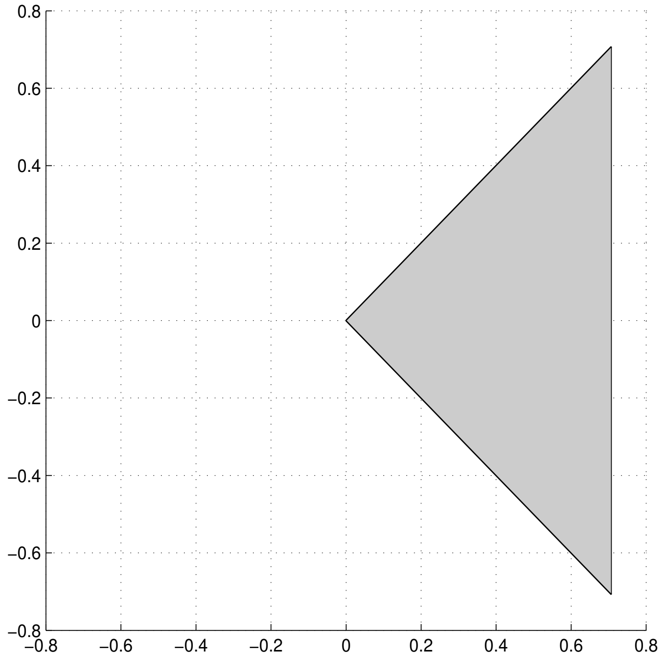
In this case, the range of f on $D(f)$ is $W(f) = \mathbb{C}^-$.

For the inverse function $f^{-1} : W(f) \longrightarrow D(f)$ we get

$$w = f^{-1}(z) = \sqrt[n]{r} e^{i\varphi/n} \qquad \text{for } z = re^{i\varphi} \text{ with } \varphi = \arg(z) \in \left(-\frac{\pi}{n}, \frac{\pi}{n}\right).$$

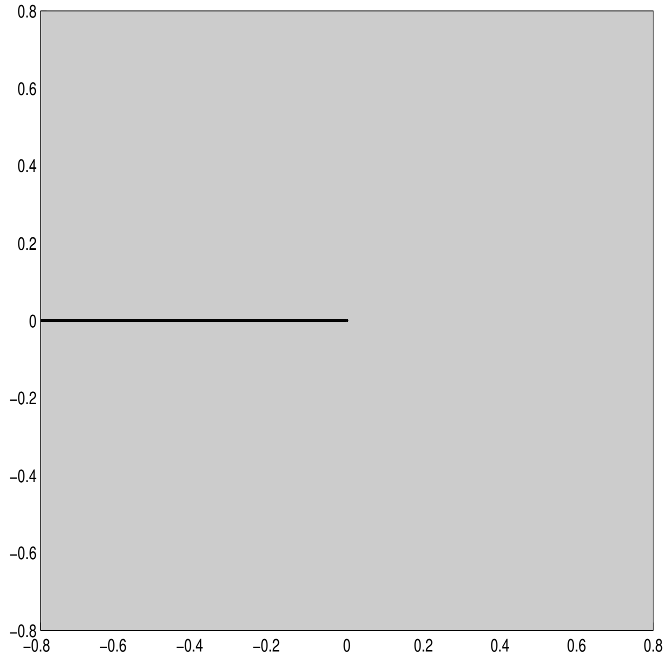
$\square$

Example $n = 4$: Regard the function $z \longmapsto z^4$.



$\{z \in \mathbb{C} \mid \arg(z) \in (-\pi/4, \pi/4)\}$

(domain)



the cut complex plane $\mathbb{C}^-$

(image)

2.6 The complex logarithm and general power functions

Aim: Inversion of the complex exponential function

$$f(z) = \exp(z).$$

Observe: The exponential function $\exp(z)$ is for *all* $z \in \mathbb{C}$ well-defined, and we have

$$D(\exp) = \mathbb{C} \qquad \text{and} \qquad W(f) = \mathbb{C} \setminus \{0\}$$

for its domain and range.

But: The exponential function is not injective on $\mathbb{C}$.

Therefore: For the construction of an inverse function $\exp^{-1}$ of exp we need to restrict the domain of exp suitably.

Question: Let $z = x + iy \in W(\exp)$. Which values $w = u + iv$ are valid, so that

$$e^w = z$$

holds?

Construction of the complex logarithm.

Starting point: For $z = x + iy \in W(\exp)$ we wish to have

$$e^w = z \qquad \text{for one } w = u + iv \in \mathbb{C}.$$

Then, we have

$$|e^w| = |e^u| = |z|$$

and therefore $u = \log(|z|)$, where $\log : (0, \infty) \longrightarrow \mathbb{R}$ is the *real* logarithm.

Moreover, we have

$$\arg(e^w) = \arg(e^{u+iv}) = \arg(e^u e^{iv}) = v$$

and so $v = \arg(z)$. Therefore, the set of solutions of $e^w = z$ consists of the complex numbers

$$w = \log(|z|) + i(\arg(z) + 2\pi k) \qquad \text{for } k \in \mathbb{Z}$$

and every $w \in \mathbb{C}$ satisfying $e^w = z$ is called `logarithm` of z. For $z \in \mathbb{C}$, the set $\{\mathrm{Log}(z)\} := \{w \in \mathbb{C} \mid e^w = z\}$ is called `set-valued complex logarithm` of z.

The set-valued complex logarithm.

Example 1: How does the complex logarithm $\{\mathrm{Log}(-1)\}$ of -1 look like?

First we have $\log(|-1|) = \log(1) = 0$.

The numbers $\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ are the arguments of -1. Therefore, we have

$$\{\mathrm{Log}(-1)\} = \{i(2k+1)\pi \mid k \in \mathbb{Z}\}$$

for the values of the logarithm of -1.

Example 2: How does the complex logarithm $\{\mathrm{Log}(-1+i)\}$ of $-1+i$ look like?

First we have $|-1+i| = \sqrt{2}$ and, moreover, $\arg(-1+i) = \frac{3\pi}{4}$ is one argument of $-1+i$. Therefore, we have

$$\{\mathrm{Log}(-1+i)\} = \left\{ \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right) \,\middle|\, k \in \mathbb{Z} \right\}$$

for the values of the logarithm of $-1+i$.

Example 3: For $x > 0$ we have $\{\mathrm{Log}(x)\} = \{\log(x) + 2\pi i k \mid k \in \mathbb{Z}\}$.

The principal value of the logarithm.

Our previous discussion on the solutions of the equation

$$z = e^w$$

showed that the exponential function is injective on the strip

$$S = \{w \in \mathbb{C} \mid -\pi < \operatorname{Im}(w) < \pi\},$$

with image $\mathbb{C}^-$. The only value of $\{\operatorname{Log}(z)\}$ belonging to the strip S is

$$w = \log(|z|) + \mathrm{i}\arg(z) \qquad \text{with } -\pi < \arg(z) < \pi.$$

This value is the `principle value of the logarithm` of z, in short: $\operatorname{Log}(z)$.

Remark: The principle value of the logarithm is only defined in the cut complex plane $\mathbb{C}^-$. The logarithm $\operatorname{Log}(z)$ is not defined on the negative real axis and at $z = 0$. On the positive real axis, $\operatorname{Log}(z)$ coincides with the real logarithm $\log(x)$.

The general power function.

Definition: *For* $a, b \in \mathbb{C}$ *we denote by* $\{a^b\}$ *the set of complex numbers*

$$e^{b\{\mathrm{Log}(a)\}} \qquad \text{for } a \neq 0$$

where $\{\mathrm{Log}(a)\} = \{\log(|a|) + i(\arg(a) + 2\pi k) \,|\, k \in \mathbb{Z}\}$*. Therefore, we have*

$$\{a^b\} = \left\{ e^{b\,[\log(|a|)+i(\alpha+2\pi k)]} \,|\, k \in \mathbb{Z} \right\}$$

where $\alpha = \arg(a)$*. If* a *lies in the cut complex plane,* $a \in \mathbb{C}^-$*, then the set* $\{a^b\}$ *contains the value*

$$e^{b\mathrm{Log}(a)} = e^{b(\log(|a|)+i\alpha)} \qquad \text{with } \alpha = \arg(a) \in (-\pi, \pi).$$

This value is called the `principle value` *of* $\{a^b\}$*.* □

Examples.

1.) Let $a = re^{i\alpha} \in \mathbb{C} \setminus \{0\}$ and $b = n \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
\{a^b\} &= \left\{ e^{n(\log(r)+i\alpha+2\pi ik)} \,\middle|\, k \in \mathbb{Z} \right\} = \left\{ e^{n\log(r)+in\alpha+2\pi ikn} \,\middle|\, k \in \mathbb{Z} \right\} \\
&= \left\{ r^n e^{in\alpha} e^{2\pi ikn} \,\middle|\, k \in \mathbb{Z} \right\} = \left(re^{i\alpha} \right)^n = r^n e^{in\alpha} = \underbrace{a \cdot \ldots \cdot a}_{n\text{-fold}}
\end{aligned}
$$

2.) For $x > 0$, $e^{i\operatorname{Log}(x)} = \cos(\log(x)) + i\sin(\log(x))$ is the principle value of $\{x^i\}$.

3.) Let $a = re^{i\alpha} \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
\{a^{1/n}\} &= \left\{ e^{(1/n)(\log(r)+i\alpha+2\pi ik)} \,\middle|\, k \in \mathbb{Z} \right\} = \left\{ r^{1/n} e^{i\alpha/n} e^{2\pi ik/n} \,\middle|\, k \in \mathbb{Z} \right\} \\
&= \left\{ r^{1/n} e^{i\alpha/n} e^{2\pi ik/n} \,\middle|\, 0 \le k < n \right\}
\end{aligned}
$$

i.e., the values z of $\{a^{1/n}\}$ are the n-th roots of a, so that $z^n = a$, in short

$$z = \sqrt[n]{a}$$

with principle value $r^{1/n} e^{i\alpha/n}$ for $\alpha/n = \arg(a)/n \in (-\pi/n, \pi/n)$. $\square$

Remark.

The functional equation (from real analysis)

$$\log(ab) = \log(a) + \log(b) \qquad \text{for all } a, b > 0$$

do in the general case *not* hold for the principle values of the complex logarithm, i.e., there are $a, b \in \mathbb{C}^-$ satisfying

$$\operatorname{Log}(ab) \neq \operatorname{Log}(a) + \operatorname{Log}(b).$$

Example: For $a = i$ and $b = -1 + i$ we have

$$\begin{aligned} \operatorname{Log}(i) + \operatorname{Log}(-1+i) &= i\frac{\pi}{2} + \log(\sqrt{2}) + i\frac{3}{4}\pi = \log(\sqrt{2}) + i\frac{5}{4}\pi \\ &\neq \log(\sqrt{2}) - i\frac{3}{4}\pi = \operatorname{Log}(-1-i) = \operatorname{Log}(i(-1+i)). \end{aligned}$$

Remark. We have the equation

$$\text{principle value of } \{a^b\} \cdot \text{principle value of } \{a^c\} = \text{principle value of } \{a^{b+c}\}.$$

Proof: For $\alpha := \arg(a) \in (-\pi, \pi)$,

$$A := e^{b\,[\log(|a|)+i\alpha]}$$

is the principle value of $\{a^b\} = \left\{e^{b\,[\log(|a|)+i(\alpha+2\pi k)]}\right\}$. Likewise,

$$B := e^{c\,[\log(|a|)+i\alpha]}$$

is the principle value of $\{a^c\}$ and

$$C := e^{(b+c)\,[\log(|a|)+i\alpha]}$$

is the principle value of $\{a^{b+c}\}$.

Finally, we have

$$A \cdot B = e^{b\,[\log(|a|)+i\alpha]} \cdot e^{c\,[\log(|a|)+i\alpha]} = e^{(b+c)\,[\log(|a|)+i\alpha]} = C.$$

$\blacksquare$

2.7 The Joukowski function

The Joukowski function, defined as

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{for } z \neq 0,$$

is relevant for fluid flow problems (details later).

Observation: We have the symmetry

$$f(z) = f(1/z) \qquad \text{for } z \neq 0.$$

Aim: Analyze the geometric behaviour of the Joukowski function.

To this end, determine for

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

the images of the circles $|z| \equiv \text{const}$ and the straight lines $\arg(z) \equiv \text{const}$.

Geometric behaviour of the Joukowski function.

For

$$z = re^{i\varphi} \qquad \text{and} \qquad w = u + iv$$

we obtain

$$u + iv = \frac{1}{2}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)$$

and therefore

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\varphi) \qquad \text{and} \qquad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\varphi).$$

For the image of the circle $r \equiv r_0 > 0$, we get the parameter representation

$$\left.\begin{array}{rcl} u & = & \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right)\cos(\varphi) \\ v & = & \frac{1}{2}\left(r_0 - \frac{1}{r_0}\right)\sin(\varphi) \end{array}\right\} \qquad 0 \leq \varphi < 2\pi,$$

for the unit circle $r_0 \equiv 1$. Therefore, $u = \cos(\varphi)$, for $0 \leq \varphi < 2\pi$ and $v \equiv 0$, i.e., the straight line between the points -1 and 1, being traversed *twice*. $\square$

Geometric behaviour of the Joukowski function.

For $r_0 \neq 1$ we can eliminate φ whereby we get the ellipse

$$\frac{u^2}{\frac{1}{4}\left(r_0 + \frac{1}{r_0}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r_0 - \frac{1}{r_0}\right)^2} = 1$$

with half axis

$$a = \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right) \qquad \text{and} \qquad b = \frac{1}{2}\left|r_0 - \frac{1}{r_0}\right|$$

and focal points ± 1.

Conclusion: The Joukowski function maps a family of circles $r \equiv \text{const}$ onto a family of `co-focal ellipses`. The two circles $r \equiv r_0$ and $r \equiv 1/r_0$ are mapped onto the same ellipse. $\square$

Geometric behaviour of the Joukowski function.

For the image of a straight line $\varphi \equiv \varphi_0$ we get

$$\left.\begin{aligned} u &= \tfrac{1}{2}\left(r + \tfrac{1}{r}\right)\cos(\varphi_0) \\ v &= \tfrac{1}{2}\left(r - \tfrac{1}{r}\right)\sin(\varphi_0) \end{aligned}\right\} \qquad 0 < r < \infty.$$

For the positive x-axis $\varphi_0 = 0$, we get

$$\left.\begin{aligned} u &= \tfrac{1}{2}\left(r + \tfrac{1}{r}\right) \\ v &= 0 \end{aligned}\right\} \qquad 0 < r < \infty,$$

the piece $\{(u, 0) \,|\, 1 \leq u < \infty\}$ of the u-axis.

Likewise, for the negative x-axis $\varphi_0 = \pi$ we get the piece $-\infty < u < -1$.

The straight lines $\varphi_0 = \pi/2$ (positive y-axis) and $\varphi_0 = 3\pi/2$ (negative y-axis) are being mapped onto the entire v-axis.

Geometric behaviour of the Joukowski function.

For $\varphi_0 \notin \{0, \pi/2, \pi, 3\pi/2\}$, we can eliminate r, whereby we get the hyperbola

$$\frac{u^2}{\cos^2(\varphi_0)} - \frac{v^2}{\sin^2(\varphi_0)} = 1$$

with half axis

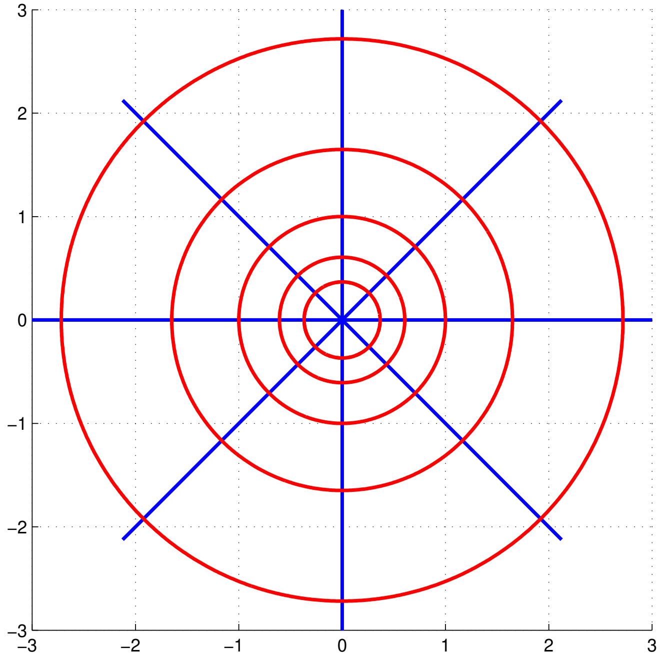
$$a = |\cos(\varphi_0)| \qquad \text{and} \qquad b = |\sin(\varphi_0)|.$$

The distance between the focal points and the origin is

$$\sqrt{a^2 + b^2} = \sqrt{\cos^2(\varphi_0) + \sin^2(\varphi_0)} = 1.$$

Therefore, the two focal points are ± 1.

Image of the Joukowski function.



domain.

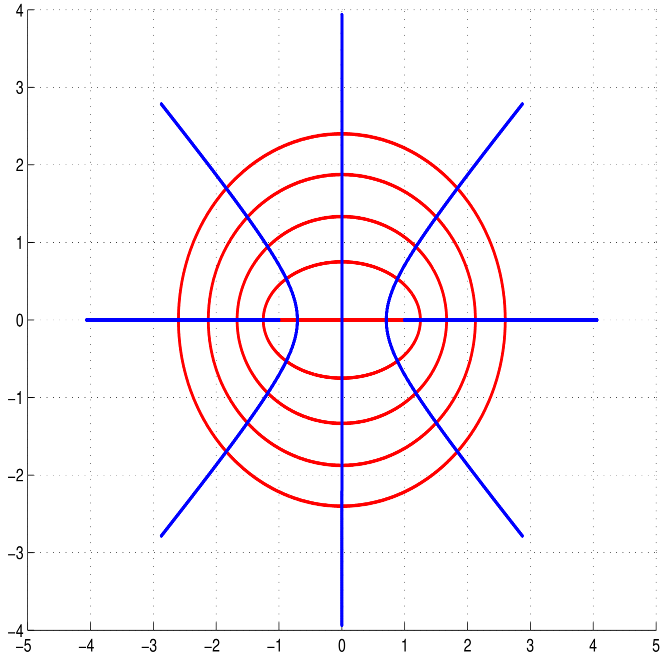


image of the Joukowski function.

Further remarks on the Joukowski function.

Remark and Conclusion: The Joukowski function maps the polar coordinate grid onto a net of ellipses and hyperbolas, at orthogonal intersections, respectively. In fact, the Joukowski function *preserves angles* (cf. our previous discussion on the exponential function).

Remark: The Joukowski function is *not* injective on its domain $\mathbb{C} \setminus \{0\}$, since for any $z \in \mathbb{C} \setminus \{\pm 1, 0\}$ we have $z \neq 1/z$, but $f(z) = f(1/z)$.

Remark: The Joukowski function is injective for the following two subdomains.

(a) on the complement of the unit disk $D(f) = \{z \in \mathbb{C} \mid |z| > 1\}$.

(b) on the upper half plane $D(f) = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

Remark: The inverse function $w = f^{-1}(z)$ of the Joukowski function $f(w)$ is obtained by the solution of the resulting quadratic equation

$$w^2 - 2zw + 1 = 0$$

for w in the corresponding domain $D(f)$, and so $w = z + \sqrt{z^2 - 1}$. □

2.8 Complex trigonometric functions

Recall that by the relation of the Eulerian formula for $x \in \mathbb{R}$, i.e.,

$$\begin{aligned} e^{ix} &= \cos(x) + i\sin(x) \\ e^{-ix} &= \cos(x) - i\sin(x) \end{aligned}$$

we obtain by summation and subtraction the two formulas

$$\begin{aligned} \cos x &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \text{for } x \in \mathbb{R} \\ \sin x &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \qquad \text{for } x \in \mathbb{R} \end{aligned}$$

But the right hand sides are also defined for arbitrary complex arguments. This motivates us to let

$$\begin{aligned} \cos z &:= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \text{for } z \in \mathbb{C} \\ \sin z &:= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \qquad \text{for } z \in \mathbb{C} \end{aligned}$$

Calculations with complex trigonometric functions.

We have

$$\begin{aligned}\cos(z+2\pi) &= \frac{1}{2}\left(e^{i(z+2\pi)} + e^{-i(z+2\pi)}\right) \\ &= \frac{1}{2}\left(e^{iz}e^{2\pi i} + e^{-iz}e^{-2\pi i}\right) \\ &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ &= \cos(z)\end{aligned}$$

for all $z \in \mathbb{C}$. Likewise, we can show

$$\sin(z+2\pi) = \sin(z) \qquad \text{for all } z \in \mathbb{C}.$$

Conclusion: The complex trigonometric functions sin and cos are (like the real trigonometric functions) periodic with period 2π. $\square$

Calculations with complex trigonometric functions.

Symmetry.

$$\cos(z) = \cos(-z) \qquad \text{for all } z \in \mathbb{C}$$
$$\sin(z) = -\sin(-z) \qquad \text{for all } z \in \mathbb{C}$$

Phase shifts.

$$\begin{aligned}\sin\left(z+\frac{\pi}{2}\right) &= \frac{1}{2i}\left(e^{i(z+\pi/2)} - e^{-i(z+\pi/2)}\right) = \frac{1}{2i}\left(e^{iz}e^{i\pi/2} - e^{-iz}e^{-i\pi/2}\right)\\ &= \frac{1}{2i}\left(ie^{iz} - (-i)e^{-iz}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos(z)\end{aligned}$$

Partition of the unity.

$$\cos^2(z) + \sin^2(z) = 1 \qquad \text{for all } z \in \mathbb{C}.$$

Summation rules.

$$\cos(z_1+z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) \qquad \text{for all } z_1, z_2 \in \mathbb{C}$$
$$\sin(z_1+z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) \qquad \text{for all } z_1, z_2 \in \mathbb{C}.$$

3 Möbius Transformations

3.1 The stereographic projection

Preliminary remark: In the investigation of rational functions

$$R(z) = \frac{p(z)}{q(z)} \qquad \text{with polynomials } p, q : \mathbb{C} \longrightarrow \mathbb{C}$$

it makes sense to close the *gaps* of the domain, (i.e., the zeroes z_0 of the polynomial $q(z)$) by assigning the $R(z)$ the "value" ∞ at the zeroes z_0 of $q(z)$, respectively, unless the nominator polynomial $p(z)$ also vanishes at z_0.

Notation: If q has a zero at $z^* \in \mathbb{C}$, i.e., $q(z^*) = 0$, and $p(z^*) \neq 0$, then we let $R(z^*) := \infty$, i.e., the image of R (containing all values of R) will be extended by the "number" ∞.

Definition: *In the extension $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ of the complex plane, we call ∞ the* `infinite point`*.* □

Extension of calculations on $\mathbb{C}^*$.

For the extended complex plane C^* we introduce the following rules (in addition to the usual ones on $\mathbb{C}$):

$$
\begin{aligned}
a + \infty &:= \infty \qquad \text{for } a \in \mathbb{C} \\
a \cdot \infty &:= \infty \qquad \text{for } a \in \mathbb{C} \setminus \{0\} \\
a/\infty &:= 0 \qquad \text{for } a \in \mathbb{C}
\end{aligned}
$$

Warning: There is no *useful* definition for the values $0 \cdot \infty$ and $\infty \pm \infty$, i.e., these values cannot be defined without contradiction.

Topological description: The extended complex plane $\mathbb{C}^*$ is a topological space. For a sequence of complex numbers $\{z_n\}_n$, $z_n \neq 0$, we have

$$z_n \longrightarrow \infty \quad \text{for } n \to \infty \qquad \Longleftrightarrow \qquad 1/z_n \longrightarrow 0 \quad \text{for } n \to \infty.$$

$\mathbb{C}^*$ is compact, i.e., *every* sequence in $\mathbb{C}^*$ has (at least) one accumulation point. Therefore, $\mathbb{C}^*$ is called compactification of $\mathbb{C}$. $\square$

The stereographic projection.

Definition: *The* stereographic projection $P : \mathbb{S}^2 \longrightarrow \mathbb{C}^*$ *maps the* Riemannian sphere $\mathbb{S}^2 = \{X \in \mathbb{R}^3 \mid \|X\| = 1\}$ *onto the extended complex plane* $\mathbb{C}^*$, *by assigning every point* $X \in \mathbb{S}^2 \setminus N$, *where* $N := (0, 0, 1)^T$, *to its unique intersection* $P(X)$ *between the straight line through* X *and* N *and the* X_1*-*X_2*-plane. Moreover, we let* $P(N) := \infty$.

The stereographic projection has the following analytical representation.

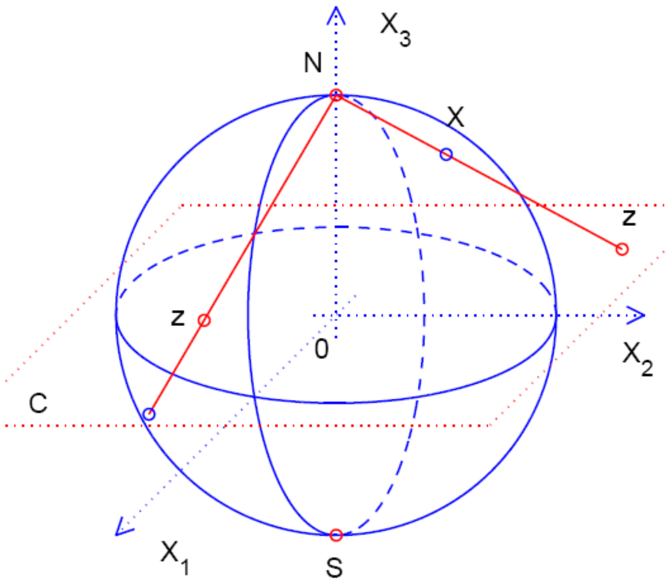
$$z = P(X) = \frac{X_1 + iX_2}{1 - X_3} \in \mathbb{C}^* \qquad \text{for } X = (X_1, X_2, X_3)^T \in \mathbb{S}^2.$$

Remarks:

- The stereographic projection $P : \mathbb{S}^2 \longrightarrow \mathbb{C}^*$ is bijective.
- The inversion P^{-1} of P is determined by

$$X = P^{-1}(z) = \left(\frac{z + \bar{z}}{1 + z\bar{z}}, \frac{z - \bar{z}}{i(1 + z\bar{z})}, \frac{z\bar{z} - 1}{1 + z\bar{z}} \right)^T \in \mathbb{S}^2 \qquad \text{for } z \in \mathbb{C}^*.$$

The stereographic projection $P : \mathbb{S}^2 \longrightarrow \mathbb{C}^*$.



- The projection P maps the upper half sphere of $\mathbb{S}^2$ onto $\{z \in \mathbb{C} \,|\, |z| > 1\}$, whereas the lower half sphere of $\mathbb{S}^2$ is mapped onto $\{z \in \mathbb{C} \,|\, |z| < 1\}$.
  The equator

$$A = \left\{X \in \mathbb{S}^2 \,|\, X = (X_1, X_2, 0)^T\right\}$$

is invariant under P, i.e., every point $a \in A$ is a fixpoint of P, i.e., $P(a) = a$. □

The geometry of the stereographic projection.

A spherical image U of a set $B \subset \mathbb{C}^*$ is the domain $U \subset \mathbb{S}^2$ (i.e., the set of arguments) of the stereographic projection P satisfying $P(U) = B$.

Satz: *The stereographic projection has the following properties.*

- *The spherical image of a straight line in $\mathbb{C}^*$ is a circle on $\mathbb{S}^2$, passing through N.*
- *A circle on $\mathbb{S}^2$, which passes through N, is being mapped by the stereographic projection onto a straight line in $\mathbb{C}^*$.*
- *The spherical image of a circle in $\mathbb{C}$ is a circle on $\mathbb{S}^2$, which does not pass through N.*
- *The stereographic projection maps a circle on $\mathbb{S}^2$, which does not pass through N, onto a circle in $\mathbb{C}$.*
- *The stereographic projection* preserves circles*.*

$\square$

3.2 Möbius transformations

Definition: *A rational function of the form*

$$w = T(z) = \frac{az+b}{cz+d} \qquad \textit{where } ad \neq bc$$

is called Möbius transformation. $\square$

Remark: A Möbius transformation $T : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ has the following properties:

- Nominator and denominator have different zeros (if any).
- $T(-d/c) = \infty$ and $T(\infty) = a/c$.
- $T(z)$ is bijective with inverse $T^{-1} : \mathbb{C}^* \longrightarrow \mathbb{C}^*$,

$$T^{-1}(w) = \frac{dw-b}{-cw+a}.$$

Note that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Composition of Möbius transformations.

Theorem: *The composition between two Möbius transformations is a Möbius transformation. More precisely, we have*

$$\begin{aligned} w = T_1(z) &= \frac{az+b}{cz+d} && \text{for } ad \neq bc \\ u = (T_2 \circ T_1)(z) = T_2(w) &= \frac{\alpha w+\beta}{\gamma w+\delta} && \text{für } \alpha\delta \neq \beta\gamma \\ &= \frac{Az+B}{Cz+D} \end{aligned}$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$\square$

Möbius transformations preserve circles.

Theorem: *Möbius transformations* preserve circles, *i.e., Möbius transformations map (generalized) circles in $\mathbb{C}^*$ onto (generalized) circles in $\mathbb{C}^*$.*

Proof: Let $T(z) = \frac{az+b}{cz+d}$, for $ad \neq bc$, be a Möbius transformation.

Case (a): For $c = 0$, T is linear and so T preserves circles.

Case (b): For $c \neq 0$, we can rewrite T as

$$T(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c} \frac{1}{cz+d}.$$

Next we show that $f(z) = 1/z$ preserves circles.

If so, then $T(z)$ (as a composition between mappings that preserve circles) preserves circles.

Recall that the stereographic projection preserves circles. To show that f preserves circles, we apply the stereographic projection to $w = 1/z$.

We have

$$X = P^{-1}(z) = \left(\frac{z+\overline{z}}{1+z\overline{z}}, \frac{z-\overline{z}}{\mathrm{i}(1+z\overline{z})}, \frac{z\overline{z}-1}{1+z\overline{z}} \right)^T \in \mathbb{S}^2$$

Therefore, for the image of $1/z$ under P^{-1} we obtain

$$\begin{aligned} X' &= F(X) = P^{-1}(1/z) \\ &= \left(\frac{1/z+1/\overline{z}}{1+(1/z)(1/\overline{z})}, \frac{1/z-1/\overline{z}}{\mathrm{i}(1+(1/z)(1/\overline{z}))}, \frac{(1/z)(1/\overline{z})-1}{1+(1/z)(1/\overline{z})} \right)^T \\ &= \left(\frac{z+\overline{z}}{1+z\overline{z}}, -\frac{z-\overline{z}}{\mathrm{i}(1+z\overline{z})}, -\frac{z\overline{z}-1}{1+z\overline{z}} \right)^T \\ &= (X_1, -X_2, -X_3)^T \end{aligned}$$

Observation: $F(X)$ describes a rotation about the X_1 axis with angle π. Obviously, the mapping $F(X)$ preserves circles. Therefore, the composition

$$f(z) = P \circ F \circ P^{-1}$$

preserves circles. ■

Remarks on Möbius tranformations.

Remark: For a Möbius transformation

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{where } ad \neq bc$$

the following properties hold.

- (Generalized) circles passing through the point $-d/c$ are being mapped by T onto straight lines in the w-plane.
- All straight lines in the z-plane are being mapped by T onto (generalized) circles in the w-plane passing through the point a/c.
- Circles that are *not* passing through the point $-d/c$ are being mapped by T onto circles that are *not* passing through the point a/c.

$\square$

Cross-ratios and Möbius tranformations.

Theorem: *Let* $z_1, z_2, z_3 \in \mathbb{C}^*$ *and* $w_1, w_2, w_3 \in \mathbb{C}^*$ *be pairwise distinct, respectively. Then, there is one unique Möbius transformation* $w = T(z)$ *satisfying the interpolation conditions*

$$w_j = T(z_j) \qquad \textit{for} \;\; j = 1, 2, 3.$$

The interpolating Möbius transformation $T(z)$ *is determined by the* three-point-formula

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

□

Definition: *The expression*

$$D(z_0, z_1, z_2, z_3) = \frac{z_0 - z_1}{z_0 - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

is called the cross-ratio *(or:* double ratio*) of the points* z_0, z_1, z_2, z_3. □

Example.

To be determined:

A Möbius transformation $T(z)$ satisfying $T(1) = i$, $T(i) = -i$ and $T(0) = 0$.

According to the three-point-formula, we obtain

$$\frac{w-i}{w+i} : \frac{0-i}{0+i} = \frac{z-1}{z-i} : \frac{0-1}{0-i}$$

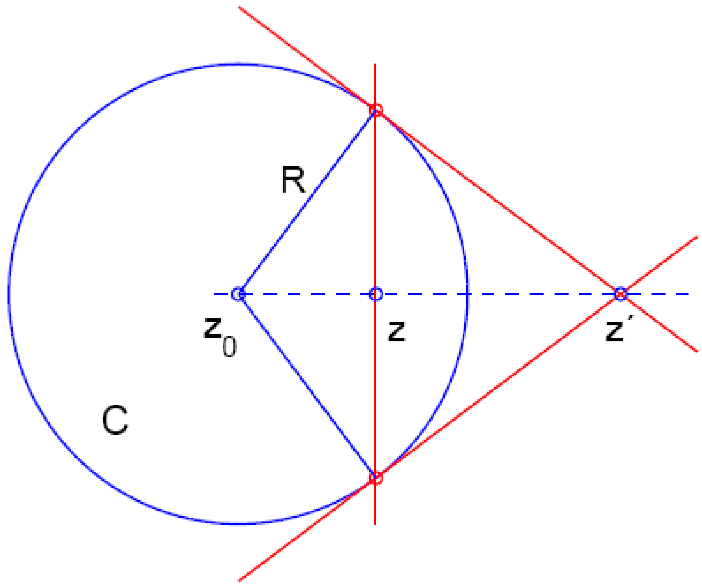
and so (by solving for w):

$$w = T(z) = \frac{(1+i)z}{(1+i)z-2i}.$$

Exercise: Verify the above mentioned interpolation conditions for $T(z)$. $\square$

Symmetry with respect to circles.

If the points z and z' are located as shown in the following figure, then we say that the points z and z' are lying `symmetric with respect to the circle` $C = \{z \in \mathbb{C} \mid |z - z_0| = R\}$.



The points z and z' lie symmetric w.r.t. the circle C.

□

Remarks on symmetries w.r.t. circles.

- The map $z \longmapsto z'$ is called inversion at the circle or reflection at the circle.
- A point z with $|z - z_0| \leq R$ is always symmetric to one (unique) point z' with $|z' - z_0| \geq R$.
- If $|z - z_0| = R$, then z is self-symmetric, i.e., $z' = z$.
- The point $z = z_0$ is symmetric to $z' = \infty$.
- We have $(z - z_0)\overline{(z' - z_0)} = R^2$.

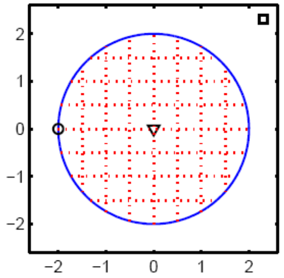
$\square$

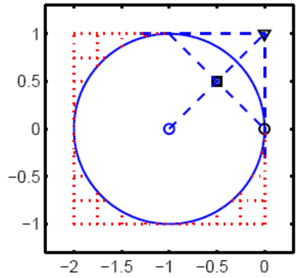
Möbius transformations and circular symmetries.

Theorem:
Möbius transformations preserve symmetries w.r.t. (generalized) circles.

More precisely: *Let* $\mathcal{C}$ *be a (generalized) circle in* $\mathbb{C}^*$ *and* z, z' *be symmetric with respect to* $\mathcal{C}$. *Then, the images of* z, z' *under a Möbius transformation* T *are symmetric with respect to that (generalized) circle in* $\mathbb{C}^*$, *which is the image of* $\mathcal{C}$, *i.e.,* $\mathbb{C}^* = \mathsf{T}(\mathcal{C})$. $\square$

Example. Find a Möbius transformation $w = T(z)$, which maps the circle $|z| = 2$ onto the circle $|w + 1| = 1$ with satisfying $T(-2) = 0$ and $T(0) = i$.





Solution: $z_2 = 0$ and $z_3 = \infty$ lie symmetric w.r.t. $|z| = 2$. Therefore, the images $w_2 = i$ and $w_3 = T(\infty)$ must lie symmetric w.r.t. the circle $|w + 1| = 1$. But in this case we have $(w_2 + 1)\overline{(w_3 + 1)} = 1$ and so $w_3 = 0.5(-1 + i)$.

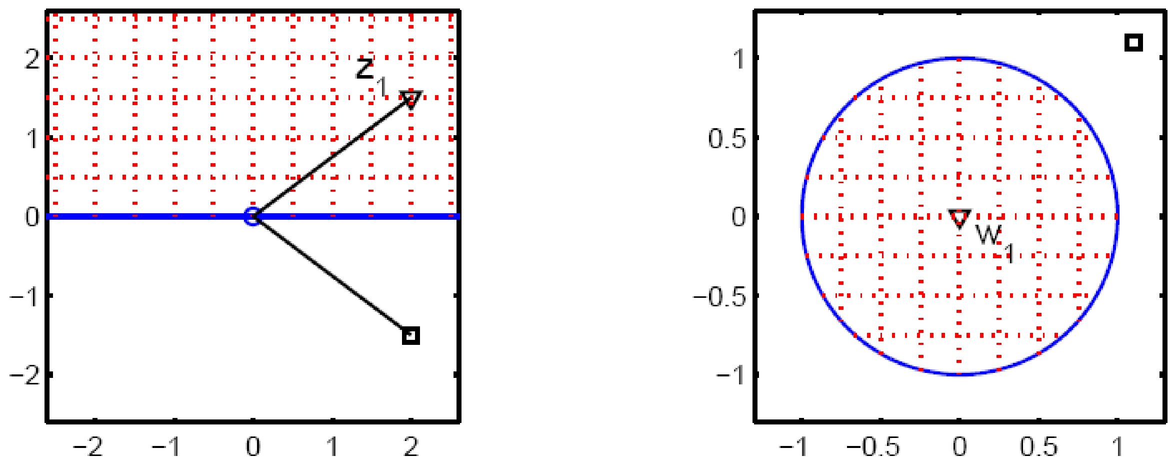
From the three-point-rule we get

$$\frac{w - 0}{w - i} : \frac{w_3 - 0}{w_3 - i} = \frac{z + 2}{z - 0} : \frac{z_3 + 2}{z_3 - 0}\bigg|_{z_3 \to \infty},$$

whereby

$$w = T(z) = -\frac{z + 2}{(1 + i)z + 2i}. \qquad \square$$

Example. Find a Möbius transformation $w = T(z)$, which maps the upper half plane $\mathrm{Im}(z) > 0$ onto the disk $|w| \leq 1$, and, moreover, maps a given point z_1, with $\mathrm{Im}(z_1) > 0$, on $w_1 = 0$.



Solution: For symmetry, the point $z_2 = \overline{z_1}$ must be mapped on $w_2 = \infty$, which implies

$$w = c\,\frac{z - z_1}{z - \overline{z_1}} \qquad \text{with } |c| = 1.$$

$\square$

Example.

For $b > a > 0$ we regard the Möbius transformation

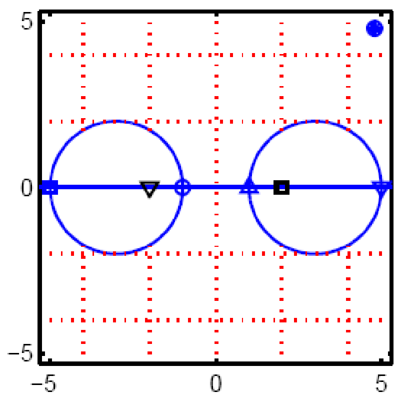
$$w = T(z) = \frac{z+p}{-z+p} \qquad \text{where } p = \sqrt{ab} \in (a,b)$$

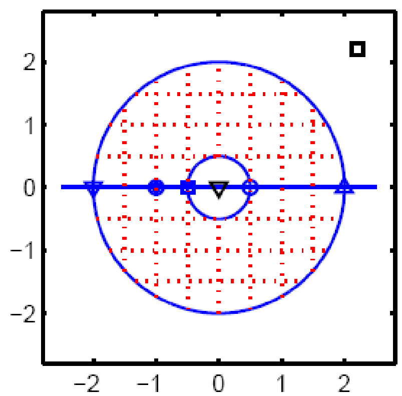
For different evaluations of T we get

$$\begin{aligned}
z_{1,2} &= \pm p & \longrightarrow \quad & w_{1,2} = \infty, 0 \\
z_{3,4} &= a, b & \longrightarrow \quad & w_{3,4} = \pm\frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}-\sqrt{a}} = \pm\rho \text{ with } \rho > 1 \\
z_{5,6} &= -a, -b & \longrightarrow \quad & w_{5,6} = \pm\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \pm\frac{1}{\rho} \\
z_{7,8} &= 0, \infty & \longrightarrow \quad & z_{7,8} = 1, -1.
\end{aligned}$$

Example (continued).

- The Möbius transformation T maps the x-axis onto the u-axis.
- Point pairs that are symmetric w.r.t. the x-axis are being mapped by T onto point pairs that are symmetric w.r.t. the u-axis
- Circles that are symmetric w.r.t. the x-axis are being mapped by T onto circles that are symmetric w.r.t. the u-axis





□

4 Analytic Functions

4.1 Complex Differentiation

Questions:

- How should we *differentiate* complex functions?
- How do define complex *limits*?
- How should we describe *continuity* for complex functions?

Starting point: Let $f(z) : D \longrightarrow \mathbb{C}$ be a complex function of the form

$$f(z) = u(z) + iv(z)$$

where $u, v : D \longrightarrow \mathbb{R}$ are real-valued. Further let $z = x + iy$, so that

$$f(z) \equiv f(x, y) \qquad u(z) \equiv u(x, y) \qquad v(z) \equiv v(x, y).$$

Complex differentials.

Assumptions:

- Let $z_0 = x_0 + iy_0$ be a (fixed) point in Definitionsbereich $D(f)$ von f.
- There be an (open) neighbourhood around z_0, on which the *real* functions $u \equiv u(x,y)$, $v \equiv v(x,y)$ have continuous partial derivatives w.r.t. x, y, respectively,, i.e., the partial derivatives u_x, u_y, v_x and v_y are continuous around (x_0, y_0).

Then:

- The (total) differentials du and dv exist in (x_0, y_0).
- For $dx = x - x_0$ and $dy = y - y_0$ (from real analysis) we have

$$\begin{aligned} du &= u_x(x_0,y_0)dx + u_y(x_0,y_0)dy \\ dv &= v_x(x_0,y_0)dx + v_y(x_0,y_0)dy. \end{aligned}$$

Definition: *A* differential *of a complex function* $f = u + iv$ *at a point* $z_0 = x_0 + iy_0$ *is the linear function (in* dx *and* dy*)* $df = du + idv$.

Differentials and partial derivatives.

For $df = du + idv$ the differential of f at z_0 has the form

$$df = [u_x(x_0, y_0) + iv_x(x_0, y_0)]\, dx + [u_y(x_0, y_0) + iv_y(x_0, y_0)]\, dy.$$

Now we represent the coefficients of df (i.e., dx and dy) by corresponding partial derivatives f_x, f_y of f. In particular, we have

$$\begin{aligned} f_x(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \quad h \to 0 \\ &= \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0) + i\,[v(x_0 + h, y_0) - v(x_0, y_0)]}{h} \\ &= \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \lim_{h \to 0} \frac{[v(x_0 + h, y_0) - v(x_0, y_0)]}{h} \end{aligned}$$

and so

$$f_x(x_0, y_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

On the form of the differential (continued).

Likewise, we have

$$f_y(x_0, y_0) = u_y(x_0, y_0) + iv_y(x_0, y_0).$$

Altogether, we obtain

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Now: Represent df in terms of dz (rather than in terms of dx and dy).
To this end, we write

$$dz = z - z_0 = (x + iy) - (x_0 + iy_0) = dx + idy.$$

Observe: We have

$$\overline{dz} = \overline{z - z_0} = dx - idy$$

and so

$$dx = \frac{1}{2}\left(dz + \overline{dz}\right) \qquad \text{and} \qquad dy = \frac{1}{2i}\left(dz - \overline{dz}\right).$$

Complex differentiation.

This further leads us to the representation

$$df = A dz + B \overline{dz},$$

where

$$A = \frac{1}{2}(f_x(z_0) - i f_y(z_0)) \qquad \text{and} \qquad B = \frac{1}{2}(f_x(z_0) + i f_y(z_0))$$

and we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0) - df}{dz} = 0.$$

Definition: *A function* f *is said to be* `complex differentiable` *at* z_0*, if*

$$df = \frac{1}{2}(f_x(z_0) - i f_y(z_0)) dz$$

i.e., if $B = 0$. □

Cauchy-Riemann differential equations.

If f is complex differentiable at z_0, then we have (by $B = 0$)

$$f_x(z_0) + i f_y(z_0) = 0$$

whereby

$$u_x(z_0) + i v_x(z_0) + i[u_y(z_0) + i v_y(z_0)] = 0$$

or,

$$u_x(z_0) - v_y(z_0) + i[u_y(z_0) + v_x(z_0)] = 0.$$

If we separate by real and imaginary part, we obtain the `Cauchy-Riemann differential equations`

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x.$$

Conclusion: The function $f = u + iv$ is complex differentiable at z_0, if and only if u and v satisfy the Cauchy-Riemann differential equations at z_0. □

Pointwise differentiation.

Observation: If f is complex differentiable at z_0, then we have

$$df = A dz \qquad \text{with } A = (f_x(z_0) - i f_y(z_0))/2.$$

Therefore, the *complex growth* $dz = \ell$ gives

$$f(z_0 + \ell) - f(z_0) = A\ell + \Phi(\ell)$$

where

$$\lim_{\ell \to 0} \frac{\Phi(\ell)}{\ell} = 0 \qquad \text{or} \qquad \lim_{\ell \to 0} \frac{f(z_0 + \ell) - f(z_0)}{\ell} = A.$$

Definition: *The limit*

$$\lim_{\ell \to 0} \frac{f(z_0 + \ell) - f(z_0)}{\ell}$$

is called the derivative *of f at z_0, in short:*

$$f'(z_0), \quad \frac{df}{dz}(z_0), \quad Df(z_0)$$

□

Complex differentiation and derivatives.

Remarks:

- We characterize derivatives as for real functions, i.e., by *difference quotients*.
- For real functions, the geometric interpretation of the derivative is by the tangent's slope. But how is that for complex functions? (details later)
- The complex differentiability implies the existence of the derivative.
- Vice versa: The existence of the derivative implies the complex differentiability.

In fact: From the existence of the derivative at $z_0 = x_0 + iy_0$ we get

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = f_x(z_0) \\ f'(z_0) &= \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{ih} = \frac{1}{i} f_y(z_0) \end{aligned}$$

and so (with $B = 0$)

$$f_x(z_0) = -i f_y(z_0)$$

Summary of the discussion.

Theorem: *Let $f = u + iv$ be a complex function with domain $D(f)$. Moreover, let $z_0 \in D(f)$, such that u, v have continuous partial derivatives w.r.t. x and y in a neighbourhood of z_0. Then, the following statements are equivalent.*

(a) f *is complex differentiable at* z_0*;*

(b) u *and* v *satisfy the Cauchy-Riemann differential equations;*

(c) *The function* f *has a derivative at* z_0*.*

□

Remark: Further (from our previous discussion) we can conclude the relation

$$df = f'(z_0)dz,$$

provided that f is complex differentiable at z_0. Finally, we have

$$f'(z_0) = u_x(z_0) + iv_x(z_0).$$

□

Example.

For $f(z) = z^2$ we have

$$f(x, y) = f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

and so

$$f_x(x, y) = 2x + 2iy \qquad \text{and} \qquad f_y(x, y) = -2y + 2ix = if_x(x, y)$$

For any $z = z_0$ we have $B = 0$ and $A = 2z_0$, i.e.,

$$df = 2z_0 dz.$$

Therefore, $f(z)$ is complex differentiable at z_0, and we have

$$f'(z_0) = 2z_0 \qquad \text{for } z_0 \in \mathbb{C}.$$

More directly:

$$\frac{f(z_0 + \ell) - f(z_0)}{\ell} = \frac{(z_0 + \ell)^2 - z_0^2}{\ell} = \frac{2z_0\ell + \ell^2}{\ell} = 2z_0 + \ell \longrightarrow 2z_0 \text{ for } \ell \to 0.$$

$\square$

Example.

For $f(z) = \overline{z}$ we have

$$f(x, y) = f(z) = \overline{z} = x - iy$$

and so

$$f_x(x, y) = 1 \qquad \text{and} \qquad f_y(x, y) = -i.$$

For any $z = z_0$ we have

$$A = \frac{1}{2}(f_x(x, y) - if_y(x, y)) = 0 \text{ and } B = \frac{1}{2}(f_x(x, y) + if_y(x, y)) = \frac{1 - i^2}{2} = 1,$$

whereby $A \equiv 0$, $B \neq 0$ and $df = \overline{dz}$.

Conclusion: The complex function $f(z) = \overline{z}$ is in *none* of the points in $\mathbb{C}$ complex differentiable, i.e., the Cauchy-Riemann differential equations are *violated* in all points in $\mathbb{C}$, i.e., there is *no* point $z_0 \in \mathbb{C}$ at which the function $f(z) = \overline{z}$ has a derivative. $\square$

Example.

For $f(z) = |z|^2 = z\overline{z}$ we have

$$f(x,y) = |z|^2 = x^2 + y^2, \quad f_x(x,y) = 2x \quad f_y(x,y) = 2y$$

and so, for any $z_0 \in \mathbb{C}$,

$$A = \frac{1}{2}(f_x(z_0) - i f_y(z_0)) = \overline{z_0}$$

and

$$B = \frac{1}{2}(f_x(z_0) + i f_y(z_0)) = z_0$$

whereby

$$df = \overline{z_0}dz + z_0 d\overline{z}.$$

Conclusion: The complex function $f(z) = |z|^2$ is only at $z_0 = 0$ complex differentiable, i.e., the Cauchy-Riemann differential equations are only satisfied at the origin, i.e., the derivative of f does only exist at the origin, where we have $f'(0) = 0$. $\square$

Example.

For $f(z) = \exp(z)$ we obtain by $f = u + iv$ the decomposition

$$f(x, y) = e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)),$$

whereby

$$u(x, y) = e^x \cos(y) \qquad \text{and} \qquad v(x, y) = e^x \sin(y)$$

and, moreover,

$$\begin{aligned} u_x(x, y) &= e^x \cos(y) = v_y(x, y) \\ u_y(x, y) &= -e^x \sin(y) = -v_x(x, y). \end{aligned}$$

Therefore, the Cauchy-Riemann differential equations are satisfied at all point in the complex plane, i.e., the complex exponential function $f(z) = \exp(z)$ is everywhere complex differentiable. For its derivative we get

$$f'(z) = u_x(z) + iv_x(z) = e^x(\cos(y) + i\sin(y)) = e^z = f(z).$$

$\square$

4.2 Analytic functions

From now:

We restrict ourselves to *connected* and *open* domains in the complex plane.

Examples: The following point sets are open and connected.

- the complex plane $\mathbb{C}$;
- the cut complex plane $\mathbb{C}^-$;
- the complex plane without the points $z_1 = 0$, $z_2 = 1$, $z_3 = \mathrm{i}$;
- the open unit disk $\{z \in \mathbb{C} \,|\, |z| < 1\}$;
- the *annulus* (i.e., area between two concentric circles), without boundary, e.g. $\{z \in \mathbb{C} \,|\, 3 < |z| < 7\}$.

But:

The closed disk $\{z \in \mathbb{C} \,|\, |z| \leq 1\}$ is not admissible, since it is not open. $\square$

Analytic (holomorphic) functions.

Definition: *A complex function* $\mathrm{f} : \mathrm{D}(\mathrm{f}) \longrightarrow \mathbb{C}$ *is called* `analytic` *(or:* `holomorphic`*), if the following two conditions are satisfied.*

- $\mathrm{D}(\mathrm{f})$ *is open and connected in the complex plane* $\mathbb{C}$*;*
- f *is complex differentiable at every point* $z \in \mathrm{D}(\mathrm{f})$*.*

$\square$

Remark: Recall that the second of the above two conditions is equivalent to:

- real and imaginary part of f satisfy the Cauchy-Riemann differential equations at every point $z \in \mathrm{D}(\mathrm{f})$
- the complex function f has a derivative at every point $z \in \mathrm{D}(\mathrm{f})$.

Remark: An analytic function f is continuous at all points of its domain $\mathrm{D}(\mathrm{f})$. $\square$

Differentiation rules for analytic functions.

Theorem: *Let the functions* f *and* g *be analytic on a (connected open) domain* G. *Then, the functions* $f+g$ *and* fg *are also analytic on* G. *If* $g(z) \neq 0$ *holds for all* $z \in G$, *then the function* f/g *is also analytic on* G. *The following differentiation rules hold:*

$$
\begin{aligned}
(f+g)' &= f' + g' \\
(fg)' &= f'g + fg' \\
\left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2}
\end{aligned}
$$

□

Entire functions.

Definition: *A complex function, which is analytic on the entire complex plane, is called* entire function □

Remark: Every complex polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \qquad a_0, \ldots, a_n \in \mathbb{C}$$

is an entire function.

In fact: Constants $f_c(z) \equiv c \in \mathbb{C}$ are entire with $f_c'(z) \equiv 0$. Moreover, the identity $g(z) = z$ is entire with $g'(z) = 1$. Now any polynomial $p : \mathbb{C} \longrightarrow \mathbb{C}$ can be written is a composition of the functions f_c and g, and so p is entire, where

$$p'(z) = a_1 + 2a_2 z + \ldots + n a_n z^{n-1}.$$

□

Remark: The complex exponential function $f(z) = \exp(z)$ is entire. □

Composition of analytic functions.

Regard the analytic functions

$$g : D(g) \longrightarrow W(g) \qquad \text{and} \qquad f : D(f) \longrightarrow W(f)$$

where $W(g) \subset D(f)$.

Theorem: *The composition* $f \circ g$ *of two analytic functions* $f : D(f) \longrightarrow W(f)$ *and* $g : D(g) \longrightarrow W(g)$ *satisfying* $W(g) \subset D(f)$ *is analytic, where, moreover, the* `chain rule`

$$(f \circ g)' = (f' \circ g)g'$$

holds, i.e.,

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0) \qquad \text{for all } z_0 \in D(f \circ g) = D(g).$$

$\square$

The inversion of analytic functions.

Regard the bijective analytic function

$$f : D(f) \longrightarrow W(f)$$

with its inverse function

$$f^{-1} : W(f) \longrightarrow D(f).$$

Theorem: *The inverse function* f^{-1} *of a bijective analytic function* f *is also analytic, where we have*

$$\left(f^{-1}\right)' = \frac{1}{f' \circ f^{-1}}$$

i.e.,

$$\left(f^{-1}\right)'(w_0) = \frac{1}{f'(f^{-1})(w_0)} \qquad \textit{for all } w_0 \in D(f^{-1}) = W(f).$$

□

Examples.

Example 1: Regard $f(z) = z^2$ on the right half plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}$, on which f is injective with image $\mathbb{C}^-$. The inverse function $f^{-1}(z) = \sqrt{z}$ is the principle value of the square root function, and we have

$$\left(\sqrt{z}\right)' = \frac{1}{2\sqrt{z}} \qquad \text{for all } z \in \mathbb{C}^-.$$

Example 2: Regard $f(z) = \exp(z)$ on the strip $S = \{z \in \mathbb{C} \,|\, -\pi < \mathrm{Im}(z) < \pi\}$, on which f is injective with image $\mathbb{C}^-$. The inverse function $f^{-1}(z) = \mathrm{Log}(z)$ is the principle value of the logarithm, and we have

$$(\mathrm{Log} z)' = \frac{1}{e^{\mathrm{Log}(z)}} = \frac{1}{z} \qquad \text{für alle } z \in \mathbb{C}^-.$$

Example 3: For $f(z) = z^a$, the principle value of $\{z^a\}$, $z \in \mathbb{C}^-$ and for fixed $a \in \mathbb{C}$ we have

$$(z^a)' = a z^{a-1} \qquad \text{for all } z \in \mathbb{C}^-.$$

□

4.3 On the geometry of complex differentiation

Let $f : D(f) \longrightarrow W(f)$ be an analytic function and $z_0 \in D(f)$. Moreover, let

$$\Gamma = \{z(t) = x(t) + iy(t) \,|\, t \in [\alpha, \beta]\} \subset D(f)$$

be a curve containing z_0, i.e, $z_0 = \Gamma(t_0)$ for some $t_0 \in [\alpha, \beta]$.

Finally, let $x(t)$ und $y(t)$ be differentiable at t_0. Then, the function $z(t)$ is differentiable at t_0 with derivative

$$z'(t_0) = x'(t_0) + iy'(t_0).$$

From now, we assume $z'(t_0) \neq 0$.

Question: How can we describe the image of Γ under the mapping f?

To this end, regard the image

$$\Gamma^* = \{w(t) = f(z(t)) \,|\, t \in [\alpha, \beta]\}$$

with $w(t_0) = f(z(t_0))$, in short: $w_0 = f(z_0)$.

Geometric interpretations.

Observe: The tangent vector $w'(t_0)$ of Γ^* at w_0 can be computed, according to the chain rule, as

$$w'(t_0) = f'(z_0)z'(t_0).$$

Then, for $f'(z_0) \neq 0$ we have

$$\arg(w'(t_0)) = \arg(f'(z_0)) + \arg(z'(t_0)).$$

or

$$\alpha^* = \alpha + \omega$$

for $\alpha^* = \arg(w'(t_0))$, $\alpha = \arg(z'(t_0))$ and $\omega = \arg(f'(z_0))$.

Geometric interpretations:

- We obtain the tangent vector of Γ^* by rotation of Γ about angle ω;
- The rotation angle ω depends on f and z_0, but not on Γ;
- The tangent vector of *any* curve containing z_0 is being rotated by the mapping f about the angle $\omega = \arg(f'(z_0))$. $\square$

Conformal mappings.

Definition: *A map* $f : D(f) \longrightarrow W(f)$, *which preserves all angles (including their orientation) is called* `conformal`. $\square$

Theorem: *An analytic function* $f : D(f) \longrightarrow W(f)$ *is conformal at any point* $z_0 \in D(f)$ *with* $f'(z_0) \neq 0$. $\square$

Theorem: *Let* $f : D(f) \longrightarrow W(f)$ *be conformal at* $z_0 \in D(f)$. *Moreover, let the real and imaginary parts* $u(z)$ *and* $v(z)$ *of* $f = u + iv$ *be continuously differentiable in a neighbourhood of* z_0. *Then,* f *is complex differentiable with* $f'(z_0) \neq 0$. $\square$