Complex functions for engineering study programs

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Content of the lecture on complex functions.

1. Complex functions of a single variable.
2. Möbius–transformation.
3. Complex differentiation.
4. Conformal mappings.
5. Complex integration.
6. Cauchy's intergal formula and applicatons.
7. Taylor– and Laurent–series.
8. Isolated singularities and residue.
9. Residue.
10. Fourier–transform and partial differential equations.

Chapter 1. Complex numbers

Starting point: consider the cubic equation

$$x^3 = 3px + 2q$$

and the solution formula (by Gerolamo Cardano, 16th century)

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

Rafael Bombelli (also 16th century) considers the equation

$$x^3 = 15x + 4$$

and obtains the solution formula

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Bombelli defines the imaginary unit i via $i^2 = -1$, the complex numbers and their summation and multiplication.

First ideas to introduce the complex numbers.

Starting point: Use the symbolic solution i for the equation $x^2 + 1 = 0$, such that

$$i^2 = -1$$

The "number" i is called imaginary unit.
Next step: With the imaginary unit we build the set of numbers

$$\mathbb{C} = \{a + ib \,|\, a, b \in \mathbb{R}\}$$

Then we introduce the following rules on $\mathbb{C}$:

- **Addition**

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

- **Multiplication**

$$(a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

With this $\mathbb{C}$ obtaines an algebraic structure.

Fundamental question about the complex numbers.

- What exactely is i?
- With the above rules can we "calculate" without contradictions?
- Are the above rules consistent with the related rules in $\mathbb{R}$?
- Can we order the complex numbers?
- Is there alternative representations of the complex numbers?
- Is there a geometric interpretation of the operations in $\mathbb{C}$?
- ...
- Why do we introduce the complex numbers?
- ... and later complex functions?
- Is there interesting applications of the complex numbers in eingineering?

On the contruction of the complex numbers.

Starting point: consider the set $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}$ with **addition**

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

and **multiplication**

$$(a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

Observation: The multiplication is associative and commutative; in addition we have

$$(a, b) \cdot (1, 0) = (a, b) \qquad \text{for } (a, b) \in \mathbb{R}^2,$$

i.e. $(1, 0) \in \mathbb{C}$ is neutral element of the multiplication. The equation

$$(a, b) \cdot (x, y) = (1, 0) \qquad \text{for } (a, b) \neq (0, 0)$$

has the unique solution, the multiplicative inverse to (a, b),

$$(x, y) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$$

On the structure of the complex numbers.

Remark: The set $\mathbb{R}^2$ forms together with the addition and the multiplication a field, the field of complex numbers which we denote by $\mathbb{C}$.

Observation: the map $\varphi : \mathbb{R} \to \mathbb{C}$, defined by $\varphi(a) = (a, 0)$ is injectiv. For all $a_1, a_2 \in \mathbb{R}$ we have

$$\begin{aligned} \varphi(a_1 + a_2) &= (a_1 + a_2, 0) = (a_1, 0) + (a_2, 0) = \varphi(a_1) + \varphi(a_2) \\ \varphi(a_1 a_2) &= (a_1 a_2, 0) = (a_1, 0) \cdot (a_2, 0) = \varphi(a_1) \cdot \varphi(a_2) \end{aligned}$$

Conclusion:

- We can identify the real numbers as complex numbers of the form $(a, 0)$;
- The real numbers form a subfield of $\mathbb{C}$;
- The rules for calculation in $\mathbb{C}$ are consistent with the rules in $\mathbb{R}$.

The field of real numbers is ordered.

Remark: The real numbers form a ordered filed; the following order axioms hold.

- For every $x \in \mathbb{R}$ it is $x > 0$ or $x = 0$ or $x < 0$;
- For $x > 0$ and $y > 0$ it is $x + y > 0$;
- For $x > 0$ and $y > 0$ it is $xy > 0$.

Question: Is the field of complex numbers $\mathbb{C}$ ordered?

Answer: NO!

In an ordered field nonzero square numbers are positiv. If $\mathbb{C}$ would be ordered then

$$0 < 1^2 = 1 \qquad \text{and} \qquad 0 < i^2 = -1$$

the contradiction $0 < 1 + (-1) = 0$.

A simpler notation for the complex numbers.

Simplification of the notation:

- For $a \in \mathbb{R}$ we write a instead of $(a, 0)$;
- We denote the complex unit $(0, 1)$ by i;
- With this every complex number (a, b) can be written

$$(a, b) = (a, 0) + (0, b) \cdot (0, 1) = a + b \cdot i = a + ib$$

and is is

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0) = -1.$$

Conclusion: We have constructed a field $\mathbb{C}$ which includes $\mathbb{R}$. The equation

$$x^2 + 1 = 0$$

is solvable in $\mathbb{C}$. The only two solutions are $\pm i$.

Real and imaginary part.

From now on we denote complex numbers by z or w. For

$$z = x + iy \in \mathbb{C} \qquad \text{for } x, y \in \mathbb{R}$$

x is called the real part and y is called the imaginary part of z, shortly

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z)$$

We have the following rules

$$\begin{aligned}
\mathrm{Re}(z+w) &= \mathrm{Re}(z) + \mathrm{Re}(w) && \text{for } z, w \in \mathbb{C} \\
\mathrm{Im}(z+w) &= \mathrm{Im}(z) + \mathrm{Im}(w) && \text{for } z, w \in \mathbb{C} \\
\mathrm{Re}\,(az) &= a\,\mathrm{Re}(z) && \text{for } z \in \mathbb{C}, a \in \mathbb{R} \\
\mathrm{Im}\,(az) &= a\,\mathrm{Im}(z) && \text{for } z \in \mathbb{C}, a \in \mathbb{R}
\end{aligned}$$

and

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \qquad \text{for } z \neq 0.$$

The complex plane.

Geometric rappresentation:

We identify $z = (x, y) \in \mathbb{C}$ as point in the

complex plane (Gauß plane)

given by the cartesian coordinate system of the $\mathbb{R}^2$, with a real axis, $\mathbb{R}$, and an imaginary axis, $i \cdot \mathbb{R}$.

Geometric rappresentation of the addition:

The usual addition of vectors according to the parallelogram rule.

Rappresentation of the addition of two complex numbers on **slide**.

Conjugation of complex numbers.

We obtain for every complex number $z = x + iy$ by mirrowing along the real axis a complex number

$$\bar{z} = x - iy \in \mathbb{C}$$

the conjugate complex number.

We have the following rules

$$\begin{aligned}
\overline{z+w} &= \bar{z} + \bar{w} && \text{for } z, w \in \mathbb{C} \\
\overline{zw} &= \bar{z} \cdot \bar{w} && \text{for } z, w \in \mathbb{C} \\
\overline{(\bar{z})} &= z && \text{for } z \in \mathbb{C} \\
z\bar{z} &= x^2 + y^2 && \text{for } z = x + iy \in \mathbb{C} \\
\mathrm{Re}(z) &= (z + \bar{z})/2 && \text{for } z \in \mathbb{C} \\
\mathrm{Im}(z) &= (z - \bar{z})/2i && \text{for } z \in \mathbb{C}
\end{aligned}$$

In particular it holds $z = \bar{z}$ if an only if $z \in \mathbb{R}$.

The absolute value.

We set

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} \qquad \text{for } z = x + iy \in \mathbb{C}$$

for the absolute value of z and $|z - w|$ for the distance of two numbers $z, w \in \mathbb{C}$ in the complex plane.

- Then $|z| = |z - 0|$ represents the Eucledian distance of z to the origin.
- For $z \in \mathbb{R}$ the absolute value $|z|$ coincides with the usual absolute value for real numbers.
- We have the following estimates.

$$-|z| \leq \mathrm{Re}(z) \leq |z| \qquad \text{and} \qquad -|z| \leq \mathrm{Im}(z) \leq |z| \qquad \text{for } z \in \mathbb{C}$$

Theorem: The absolute value defines a norm on $\mathbb{C}$, since we have the relations

1. $|z| \geq 0$ for all $z \in \mathbb{C}$ and $|z| = 0$ if and only if $z = 0$;
2. $|z + w| \leq |z| + |w|$ for all $z, w \in \mathbb{C}$ (triangle inequality);
3. $|zw| = |z| \cdot |w|$ for all $z, w \in \mathbb{C}$.

The Euler's formula.

In the complex plane we have for $z = x + iy$ using polar coordinates

$$(x, y) = |z|(\cos(\varphi), \sin(\varphi))$$

the Euler's formula

$$z = |z| \exp(i\varphi) = |z|(\cos(\varphi) + i \sin(\varphi))$$

where $\varphi \in [0, 2\pi)$ for $z \neq 0$ represents the (unique) angle between the positive real axis and the ray from 0 through $z = (x, y)$.

The angle $\varphi \in [0, 2\pi)$ is called polar angle (azimuth, argument) of $z \neq 0$, shortly

$$\varphi = \arg(z) \in [0, 2\pi)$$

Example: $i = (0, 1) = \exp(i\pi/2)$, $-1 = i^2 = \exp(i\pi)$, thus $e^{i\pi} + 1 = 0$.

The geometry of multiplikation and division.

Using polar coordinates the multiplication of two complex numbers $z, w \in \mathbb{C}$ can be interpreted as rotational dilation in the complex plane, since for

$$z = |z|\,(\cos(\varphi), \sin(\varphi)) \qquad \text{and} \qquad w = |w|\,(\cos(\psi), \sin(\psi))$$

we have

$$\begin{aligned} z \cdot w &= |z| \cdot |w|\,(\cos(\varphi) + i\sin(\varphi)(\cos(\psi) + i\sin(\psi)) \\ &= |z| \cdot |w|\,(\cos(\varphi + \psi) + i\sin(\varphi + \psi)) = |z| \cdot |w|\, \exp(i(\varphi + \psi)) \end{aligned}$$

and with the Euler's formula

$$z \cdot w = |z| \cdot |w|\, \exp(i\varphi)\, \exp(i\psi) = |z| \cdot |w|\, \exp(i(\varphi + \psi))$$

For the division of two complex numbers $z, w \in \mathbb{C}$ with $z \neq 0$ we have in analogy

$$\frac{z}{w} = \frac{|z|}{|w|} \exp(i(\varphi - \psi)) = \frac{|z|}{|w|} (\cos(\varphi - \psi) + i\sin(\varphi - \psi))$$

Powers and roots of unity.

For the n–th power z^n of $z \in \mathbb{C}$ we have

$$z^n = (|z|\, \exp(i\varphi))^n = |z|^n\, \exp(in\varphi) = |z|^n\, (\cos(n\varphi) + i\sin(n\varphi))$$

The equation

$$z^n = 1$$

has n pairwise different solutions

$$z_k = \exp\left(i\,\frac{2\pi k}{n}\right) \qquad \text{for } k = 0, \ldots, n-1.$$

These solutions are called n–th roots of unity.

A complex function $w = f(z)$ is a map $f : D \to \mathbb{C}$ with $D \subset \mathbb{C}$, i.e. for every $z \in D$ there is a unique $w = f(z) \in \mathbb{C}$.

The set D is the domain (of defintion) of f. The set

$$W = f(D) = \{f(z) \,|\, z \in D\}$$

is called the codomain.

Notation:

$$\begin{aligned} z &= x + iy \\ w &= u + iv \\ u &= u(x, y) = \mathrm{Re}(w) \\ v &= v(x, y) = \mathrm{Im}(w) \end{aligned}$$

For a geometric representation of complex functions often images of coordinate nets are used.

Chapter 2. Complex valued functions of a single variable

2.1 Linear functions

Definition: A complex function f is called linear, if f for fixed complex constants $a, b \in \mathbb{C}, a \neq 0$, has a representation of the following form

$$f(z) = az + b \qquad \text{for } z \in \mathbb{C}.$$

Question: Can we interpet linear functions geometrically?

Special case 1: The choice $a = 1$ leads to a translation of b,

$$f(z) = z + b \qquad \text{for } z \in \mathbb{C}$$

Special case 2: The choice $a \in (0, \infty)$ and $b = 0$ leads to a dilation or contraction,

$$f(z) = az \qquad \text{for } z \in \mathbb{C},$$

i.e. the absolute value of z is dilated ($a > 1$) or contracted ($0 < a < 1$). In general we talk about a scaling with scaling factor $a > 0$.

Other special cases of linear functions.

Special case 3: The choice $a \in \mathbb{C}$ with $|a| = 1$ and $b = 0$ leads to a rotation,

$$f(z) = az \qquad \text{for } z \in \mathbb{C},$$

More precisely: a rotation with angle $\alpha \in [0, 2\pi)$, where $\alpha = \arg(a)$ and $a = \exp(i\alpha)$.

Special case 4: The choice $a \in \mathbb{C}$, $a \neq 0$ and $b = 0$ leads to a rotational dilation

$$f(z) = az \qquad \text{for } z \in \mathbb{C},$$

which we understand as a combination of a rotation and a scaling.

More precisely: For

$$a = |a| \exp(i\alpha) \qquad \text{with } \alpha = \arg(a)$$

we have a rotation with angle $\alpha \in [0, 2\pi)$ and a scaling with factor $|a|$.

The general case of linear functions.

For $a, b \in \mathbb{C}$, $a \neq 0$, every linear function

$$f(z) = az + b = |a| \exp(i\alpha)z + b$$

can be written as composition

$$f = f_3 \circ f_2 \circ f_1$$

of three maps,

1. $f_1(z) = \exp(i\alpha)z$ a rotation with angle $\alpha = [0, 2\pi)$;
2. $f_2(z) = |a|z$ a dilation with scaling factor $|a| > 0$;
3. $f_3(z) = z + b$ a shift with a vector b.

Remark: rotation f_1 and dilation f_2 commute, i.e. can be exchanged since

$$f_2 \circ f_1 = f_1 \circ f_2$$

and thus

$$f = f_3 \circ f_2 \circ f_1 = f_3 \circ f_1 \circ f_2$$

2.2 Quadratic functions

Definition: A complex function f is called quadratic, if f for fixed constants $a, b, c \in \mathbb{C}$, $a \neq 0$, has the following form.

$$f(z) = az^2 + bz + c \qquad \text{for} \quad z \in \mathbb{C}$$

First we consider the geometric behaviour of the function

$$f(z) = z^2 \qquad \text{for} \quad z \in \mathbb{C}$$

To do so we consider the image under f of straight lines parallel to the coordinate axes.

Set $w = z^2$. Then with $z = x + iy$ and $w = u + iv$ we obtain the representation

$$w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

and thus

$$u = x^2 - y^2 \qquad \text{and} \qquad v = 2xy.$$

Images of straight lines parallel to the axes under $z \mapsto z^2$.

For the image of a straight line $y = y_0$ parallel to the x-axis we obtain

$$u = x^2 - y_0^2 \qquad \text{and} \qquad v = 2xy_0$$

For $y_0 = 0$ (the x-axis) we obtain $u = x^2$ and $v = 0$.

For $y_0 \neq 0$ we can eliminate x with $x = v/(2y_0)$ and obtain

$$u = \frac{v^2}{4y_0^2} - y_0^2,$$

a parabola open to the right, symmetric with respect to the u-axes with focus in zero, intersecting the u-axis in $u = -y_0^2$ and the v-axis in $v = \pm 2y_0^2$.

Conclusion: The family of straight lines parallel to the x-axis by the quadratic function $f(z) = z^2$ is mapped on a family of confocal (i.e. same symmetry axis, same focus) parabolas, open to the right.

The lines $y = y_0$ and $y = -y_0$ are mapped onto the same parabola.

Images of straight lines parallel to the axes under $z \mapsto z^2$.

For the image of a straight line $x = x_0$ parallel to the y-axis we obtain

$$u = x_0^2 - y^2 \qquad \text{und} \qquad v = 2x_0 y$$

For $x_0 = 0$ (the y-axis) we obtain $u = -y^2$ and $v = 0$.

For $x_0 \neq 0$ we can eliminate y with $y = v/(2x_0)$ and obtain

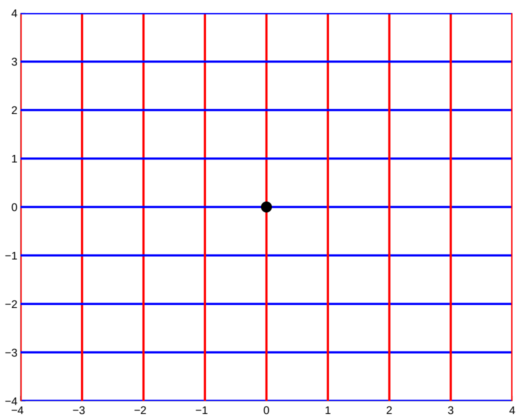
$$u = x_0^2 - \frac{v^2}{4x_0^2}$$

a parabola open to the left, symmetric to the u-axis with focus zero, intersecting the u-axis in $u = x_0^2$ and the v-axis in $v = \pm 2x_0^2$.

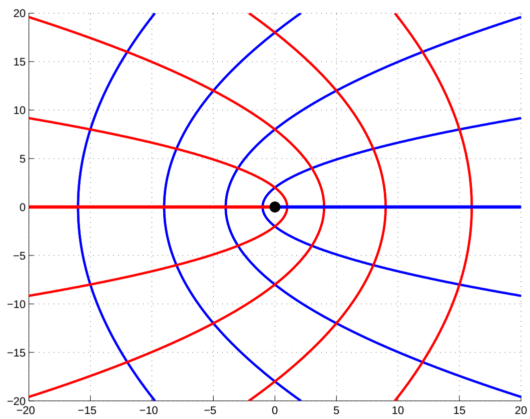
Conclusion: The family of straight lines parallel to the y-axis by the quadratic function $f(z) = z^2$ is mapped on a family of confocal parabolas, open to the left.

The lines $x = x_0$ and $x = -x_0$ are mapped onto the same parabola.

Images of straight lines parallel to the axes under $z \mapsto z^2$.



Domain.



Codomain of $f(z) = z^2$.

General quadratic functions.

For $a, b, c \in \mathbb{C}$, $a, b \neq 0$, and the representation

$$f(z) = az^2 + bz + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

every quadratic function can be written as a composition of 4 maps

$$f = f_4 \circ f_3 \circ f_2 \circ f_1$$

consisting in:

1. a shift $f_1(z) = z + \frac{b}{2a}$;
2. a quadratic function $f_2(z) = z^2$;
3. a rotational dilation $f_3(z) = az$;
4. a shift $f_4(z) = z - \frac{b^2}{4a} + c$.

Chapter 2. Complex valued functions of a single variable

2.3 The exponential function

Definition: The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is defined as

$$\exp(z) = e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)) \qquad \text{for } z = x + iy.$$

We observe: The rule for the addition holds

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \qquad \text{for } z_1, z_2 \in \mathbb{C}.$$

Question: How does the complex exponential function $z \to \exp(z)$ look like?

For $w = \exp(z)$, $z = x + iy$ and $w = u + iv$ we obtain

$$w = u + iv = e^z = e^x(\cos(y) + i\sin(y))$$

and thus

$$u = e^x\cos(y) \qquad \text{and} \qquad v = e^x\sin(y)$$

Images of straight lines parallel to the axes under $z \mapsto \exp(z)$.

For the image of a straight line $y = y_0$ parallel to the x-axis we obtain

$$u = e^x \cos(y_0) \qquad \text{and} \qquad v = e^x \sin(y_0)$$

- For fixed y_0 this gives a ray starting from the origin with angle y_0 with respect to the the x-axis.
- For angles y_0 and y_1, which differ by a multiple of 2π, i.e.

$$y_1 = y_0 + 2\pi k \qquad \text{for a } k \in \mathbb{Z},$$

  we obtain the same ray.
- **More precisely:** Due to the periodicity of $\exp(z)$ we have

$$e^{z+2\pi ik} = e^z e^{2\pi ik} = e^z(\cos(2\pi k) + i\sin(2\pi k)) = e^z \cdot 1 = e^z.$$

  i.e. two points with identical real part, which imaginary parts only differ by a multiple of 2π, are mapped onto the same point.

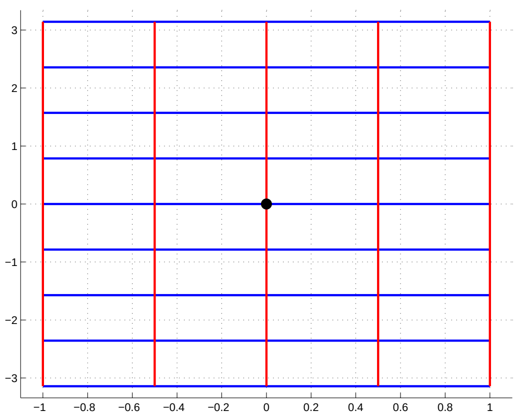
Images of straight lines parallel to the axes under $z \mapsto \exp(z)$.

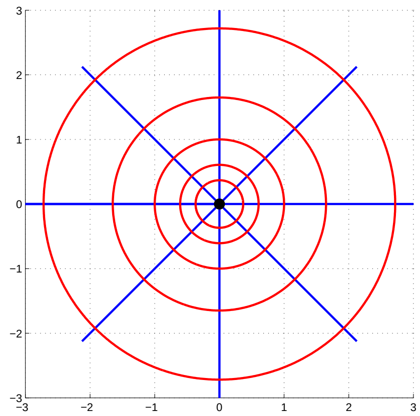
For the image of a straight line $x = x_0$ parallel to the y-axis we obtain

$$u = e^{x_0} \cos(y) \qquad \text{und} \qquad v = e^{x_0} \sin(y)$$

- For fixed x_0 this gives a circle around the origin with radius e^{x_0}.
- **Observe:** The origin does not lie in the codomain of the exponential function, i.e. there is no $z \in \mathbb{C}$ with $\exp(z) = 0$. Therefore $e^z \neq 0$ for all $z \in \mathbb{C}$.
- **Observation:** The exponential function maps rectangular lattices in the cartesian coordinate system onto lattices of curves which intersect orthogonally.
- **More precisely:** Curves which intersect orthogonally in the cartesian coordinate system, are mapped by the exponential function exp onto curves, which intersect orthogonally (in the images of the interesction point)
- **Even more general:** The exponential function is isogonal or conformal in $\mathbb{C} \setminus \{0\}$. More details later.

Images of straight lines parallel to the axes under $z \mapsto \exp(z)$.





Domain. **Codomain of $f(z) = \exp(z)$.**

Chapter 2. Complex valued functions of a single variable

2.4 The inverse function

Definition: A complex function $f = f(z)$ is called injective, if for every point $w \in \mathbb{C}$ in the domain there is exactely one point $z \in \mathbb{C}$ in the codomain with $f(z) = w$.

Remark: A non-injective function might become injective if the domain is appropriately restricted.

Examples.

1. the linear function $f(z) = az + b$, $a \neq 0$ is injective.
2. the quadratic function $f(z) = z^2$ is not injective, since we have $f(z) = f(-z)$ for all $z \in \mathbb{C}$.
3. the complex exponential function $\exp(z)$ is not injective, since we have $\exp(z) = \exp(z + 2\pi ik)$ for all $k \in \mathbb{Z}$ and all $z \in \mathbb{C}$.

Restriction of the domain.

Remark: A non-injective function might become injective if the domain is appropriately restricted.

Example: Consider the quadratic function

$$f(z) = z^2 \qquad \text{for } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0$$

on the right halfplane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}$. There f is injective.

In this case the codomain is given by the "partly cutted" complex plane

$$\begin{aligned} \mathbb{C}^- &= \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \neq 0 \text{ or } \mathrm{Re}(z) > 0\} \\ &= \mathbb{C} \setminus \{z \in \mathbb{R} \,|\, z \leq 0\} \end{aligned}$$

Graphical representation of the domain and codomain on a slight.

The inverse function.

Definition: Let f be an injective function with domain $D(f)$ and codomain $W(f)$. Then the inverse function $f^{-1} : W(f) \to D(f)$ to f is the function, which maps every point $w \in W(f)$ onto the (unique) point $z \in D(f)$ with $f(z) = w$, i.e. it is $f^{-1}(w) = z$ and

$$\begin{aligned} (f^{-1} \circ f)(z) &= z \qquad \text{for all } z \in D(f) \\ (f \circ f^{-1})(w) &= w \qquad \text{for all } w \in W(f) \end{aligned}$$

Example: For the domain

$$D(f) = \{z = re^{i\varphi} \in \mathbb{C} \,|\, r > 0 \text{ and } -\pi/2 < \varphi < \pi/2\}$$

the exists an inverse function f^{-1} of $f(z) = z^2$ with codomain $W(f) = \mathbb{C}^-$.

For the main value uf the root $f^{-1} : W(f) \to D(f)$ it is

$$w = f^{-1}(z) = \sqrt{r} e^{i\varphi/2} \qquad \text{for } z = re^{i\varphi} \text{ with } \varphi = \arg(z) \in (-\pi, \pi).$$

2.5 The complex logarithm

Aim: To inverte the complex exponential function

$$f(z) = \exp(z).$$

Observe: The exponential function $\exp(z)$ is defined for all $z \in \mathbb{C}$ and we have

$$D(\exp) = \mathbb{C} \qquad \text{and} \qquad W(f) = \mathbb{C} \setminus \{0\}$$

for the domain and the codomain.

But: The exponential function is not injective on $\mathbb{C}$.

Also: For the construction of the inverse function $\exp^{-1}$ of exp we need to restrict the domain of exp appropriately.

Question: Let $z = x + iy \in W(\exp)$. Which values $w = u + iv$ are possible such that

$$e^w = z?$$

Construction of the complex logarithm.

Starting point: For $z = x + iy \in W(\exp)$ it should be

$$e^w = z \qquad \text{for a } w = u + iv \in \mathbb{C}.$$

Then

$$|e^w| = |e^u| = |z|$$

and thus $u = \ln(|z|)$, where $\ln : (0, \infty) \to \mathbb{R}$ denotes the real logarithm.
In addition we have

$$\arg(e^w) = \arg(e^{u+iv}) = \arg(e^u e^{iv}) = v$$

and thus $v = \arg(z) + 2\pi k$ for a $k \in \mathbb{Z}$.
Therefore the set of solutions of $e^w = z$ consists of complex numbers

$$w = \ln(|z|) + i(\arg(z) + 2\pi k) \qquad \text{with a } k \in \mathbb{Z}.$$

The set of solutions of $e^w = z$ is called complex logarithm of z.

Examples.

The function $\mathrm{Log}(z)$ denotes the complex logarithm of z.

Example 1: How does the set $\mathrm{Log}(-1)$ look like? We have $\ln(|-1|) = \ln(1) = 0$ and the argument of -1 is $\arg(-1) = \pi$. Thus

$$\mathrm{Log}(-1) = \{i(2k+1)\pi \mid k \in \mathbb{Z}\}$$

for the values of the logarithm of -1.

Example 2: How does the set $\mathrm{Log}(-1+i)$ look like? We have $|-1+i| = \sqrt{2}$ and it is $\arg(-1+i) = \frac{3\pi}{4}$ the argument of $-1+i$. Thus

$$\mathrm{Log}(-1+i) = \left\{ \ln(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right) \,\middle|\, k \in \mathbb{Z} \right\}$$

for the values of the logarithm of $-1+i$.

Example 3: For $x > 0$ it is $\mathrm{Log}(x) = \{\ln(x) + 2\pi i k \mid k \in \mathbb{Z}\}$.

The principal value of the logarithm.

The previous consoderations for the equation

$$z = e^w$$

show that the exponantial function is injective on the strip

$$S = \{w \in \mathbb{C} \mid -\pi < \mathrm{Im}(w) < \pi\}.$$

The related codomain is $\mathbb{C}^-$.
The unique value of $\mathrm{Log}(z)$ being element in the strip S is

$$w = \log(|z|) + i \arg(z) \qquad \text{with } -\pi < \arg(z) < \pi.$$

This value is called principal value of the logarithm of z, shortly $\ln(z)$.

Remark: The principal value is only defined in the "opened" complex plane $\mathbb{C}^-$. On the negative real axis and at $z = 0$ the $\ln(z)$ is not defined. On the positive real axis $\ln(z)$ coincides with the real logarithm $\ln(x)$.

2.6 The Joukowski–function

The Joukowski–function is defined as

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{for } z \neq 0,$$

and has an interesting connection to fluid mechanics.

Observation: We have the symmetry

$$f(z) = f(1/z) \qquad \text{for } z \neq 0.$$

Aim: Analyse the geometric behaviour of the Joukowski–function.

To do so determine for

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

the images of the circles $|z| = \mathrm{const.}$ and the rays $\arg(z) = \mathrm{const.}$.

Geometric behaviour of the Joukowski–function.

For $z = re^{i\varphi}$ and $w = u + iv$ we obtain

$$u + iv = \frac{1}{2}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)$$

and thus

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\varphi) \qquad \text{and} \qquad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\varphi).$$

For the images of the circles $r \equiv r_0 > 0$ we obtain the parameterized form

$$\left.\begin{array}{rcl} u & = & \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right)\cos(\varphi) \\ v & = & \frac{1}{2}\left(r_0 - \frac{1}{r_0}\right)\sin(\varphi) \end{array}\right\} \qquad 0 \leq \varphi < 2\pi.$$

For the unit circle $r_0 \equiv 1$ we have $u = \cos(\varphi)$, for $0 \leq \varphi < 2\pi$, and $v \equiv 0$, i.e. the line between -1 and 1, which is reached twice.

For $r_0 \neq 1$ we can eliminate φ and we obtain the ellipse

$$\frac{u^2}{\frac{1}{4}\left(r_0 + \frac{1}{r_0}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r_0 - \frac{1}{r_0}\right)^2} = 1$$

with the semi axes

$$a = \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right) \qquad \text{and} \qquad b = \frac{1}{2}\left|r_0 - \frac{1}{r_0}\right|$$

and the foci ± 1.

Conclusion: The Joukowski–function maps a collection of circles $r \equiv \mathrm{const.}$ onto a collection of kofocal ellipses. The two circles $r \equiv r_0$ and $r \equiv 1/r_0$ are mapped onto the same ellipse.

Geometric behaviour of the Joukowski–function.

For the image of the ray $\varphi \equiv \varphi_0$ we obtain

$$\left.\begin{array}{rcl} u & = & \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\varphi_0) \\ v & = & \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\varphi_0) \end{array}\right\} \qquad 0 < r < \infty,$$

and therefore for the positive x-axis $\varphi_0 = 0$

$$\left.\begin{array}{rcl} u & = & \frac{1}{2}\left(r + \frac{1}{r}\right) \\ v & = & 0 \end{array}\right\} \qquad 0 < r < \infty,$$

the subset $\{(u, 0) \,|\, 1 \leq u < \infty\}$ of the u-axes.

In analogy we obtain for the negative x-axis $\varphi_0 = \pi$ the piece $-\infty < u < -1$.

The rays $\varphi_0 = \pi/2$ (positive y-axis) and $\varphi_0 = 3\pi/2$ (negative y-axis) are mapped onto the (complete) v-axis.

Geometric behaviour of the Joukowski–function.

If $\varphi_0 \notin \{0, \pi/2, \pi, 3\pi/2\}$ we can eliminate r. Thus we obtain the hyperbola

$$\frac{u^2}{\cos^2(\varphi_0)} - \frac{v^2}{\sin^2(\varphi_0)} = 1$$

with the semiaxes

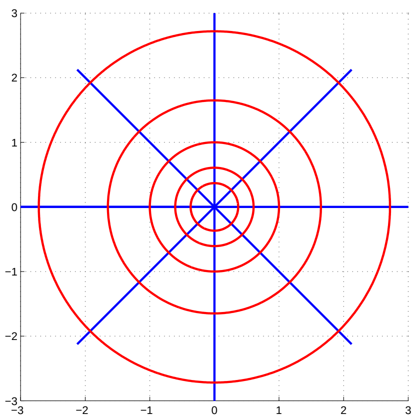
$$a = |\cos(\varphi_0)| \qquad \text{and} \qquad b = |\sin(\varphi_0)|.$$

The distance of the foci from the origin is

$$\sqrt{a^2 + b^2} = \sqrt{\cos^2(\varphi_0) + \sin^2(\varphi_0)} = 1.$$

Therefore the two foci are in ± 1.

Images of the Joukowski–function.



Domain.

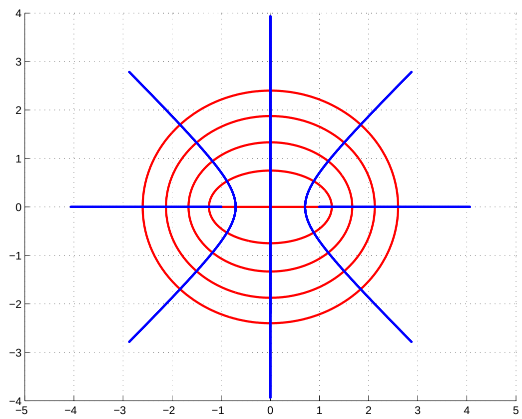


Image under the Joukowski–function.

Additional remarks to the Joukowski–function.

1. The Joukowski–function maps the net of polar coordinates onto a net of ellipses and hyperbolas which intersect orthogonally. Thus the Joukowski–function is isogonal.
2. The Joukowski–function is not injective on its domain $\mathbb{C} \setminus \{0\}$ since for every $z \in \mathbb{C} \setminus \{\pm 1, 0\}$ it is $z \neq 1/z$, but $f(z) = f(1/z)$.
3. On the following two restrictions of the domain the Joukowski–function becomes injectiv.
   - On the complement of the unit circle $D(f) = \{z \in \mathbb{C} \,|\, |z| > 1\}$.
   - On the upper half plane $D(f) = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$.
4. The inverse function $w = f^{-1}(z)$ of the Joukowski–function $f(w)$ is obtained by solving the related quadratic equation

$$w^2 - 2zw + 1 = 0$$

w.r.t. w in the related domain $D(f)$, thus $w = z + \sqrt{z^2 - 1}$.

Chapter 3. The Möbius–transform

3.1 The stereographic projection

Preliminaries: In analysing rational functions

$$R(z) = \frac{p(z)}{q(z)} \qquad \text{with polynomials } p, q : \mathbb{C} \to \mathbb{C}$$

it is reasonable to close the gaps in the domain (i.e. the zero's of $q(z)$) by attributing to $R(z)$ in theese points the "value" ∞ if at such point not at the same time the nominator $p(z)$ vanishes.

Notation: If $z^* \in \mathbb{C}$ is a zero of q, i.e. $q(z^*) = 0$, and $p(z^*) \neq 0$, then $R(z^*) = \infty$, i.e. the codomain of R is enlarged by adding the "number" ∞.

Definition: In the extension $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ of the complex plane ∞ is denoted as infinitely far point.

Extension of the rules of calculus for $\mathbb{C}^*$.

In the extended complex plane $\mathbb{C}^*$ in addition to the usual rules in $\mathbb{C}$ we define the following rules.

$$\begin{aligned} a + \infty &:= \infty \qquad \text{for } a \in \mathbb{C} \\ a \cdot \infty &:= \infty \qquad \text{for } a \in \mathbb{C} \setminus \{0\} \\ a/\infty &:= 0 \qquad \text{for } a \in \mathbb{C} \end{aligned}$$

Warning: The combintions $0 \cdot \infty$ and $\infty \pm \infty$ cannot be defined reasonably (i.e. without contradictions).

Topological meaning: The extended complex plane $\mathbb{C}^*$ is a topological space. For a complex sequence $\{z_n\}_n$, $z_n \neq 0$, we have

$$z_n \to \infty \quad \text{for } n \to \infty \qquad \Longleftrightarrow \qquad 1/z_n \to 0 \quad \text{for } n \to \infty$$

The space $\mathbb{C}^*$ is sequentially compact, i.e. every sequence in $\mathbb{C}^*$ as (at least) one limit point. Thus $\mathbb{C}^*$ is denoted as compactification of $\mathbb{C}$.

The stereographic projection.

Definition: The stereographic projection is the map $P : \mathbb{S}^2 \to \mathbb{C}^*$ which maps the Riemann sphere

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

on the extended complex plan $\mathbb{C}^*$, in particular it maps a point $x \in \mathbb{S}^2$, $x \neq N = (0,0,1)^T$, onto the point in the x_1–x_2–plane (considered to lie below the sphere) which lies on a straight line from the north pole N of the sphere through the point x on the sphere. And N is mapped to $P(N) := \infty$.

The stereographic projection has the following analytical representation

$$z = P(x) = \frac{x_1 + ix_2}{1 - x_3} \in \mathbb{C}^* \qquad \text{for } = (x_1, x_2, x_3)^T \in \mathbb{S}^2.$$

Remark:

1. The stereographic projection $P : \mathbb{S}^2 \to \mathbb{C}^*$ is bijective.
2. The inverse map P^{-1} of P is given by

$$x = P^{-1}(z) = \left(\frac{z + \overline{z}}{1 + z\overline{z}}, \frac{z - \overline{z}}{i(1 + z\overline{z})}, \frac{z\overline{z} - 1}{1 + z\overline{z}} \right)^T \in \mathbb{S}^2 \qquad \text{for } z \in \mathbb{C}^*.$$

The geometry of the stereographic projection.

By a sperical image U of a set $B \subset \mathbb{C}^*$ in the following we unterstand the (original) domain which under the stereographic projection is mapped on B, i.e. $P(U) = B$.

Theorem: The stereographic projection has the following properties.

a) The spherical image of a straight line in $\mathbb{C}^*$ is a circle on $\mathbb{S}^2$ containing N.

b) A circle on $\mathbb{S}^2$, passing through N, is mapped under the stereographic projection on a straight line in $\mathbb{C}^*$.

c) The spherical image of a circle in $\mathbb{C}$ is a circle in $\mathbb{S}^2$, NOT passing through N.

d) A circle on $\mathbb{S}^2$, NOT passing through N, is mapped under the stereographic projection on a circle in $\mathbb{C}$.

e) The stereographic projection is conformal.

Chapter 3. The Möbius–transform

3.2 Möbius–transforms

Definition: A rational map of the form

$$w = T(z) = \frac{az + b}{cz + d} \qquad \text{with} \quad ad \neq bc$$

is called Möbius–transform.

Remark: For the Möbius–transform $T : \mathbb{C}^* \to \mathbb{C}^*$ it holds:

1. Nominator and denominator have no common zero.
2. It is $T(-d/c) = \infty$ and $T(\infty) = a/c$.
3. The map $T(z)$ is bijective with inverse map $T^{-1} : \mathbb{C}^* \to \mathbb{C}^*$

$$T^{-1}(w) = \frac{dw - b}{-cw + a}.$$

4. Analogy to the inverse of a (2×2)–matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Composition of Möbius-transforms.

Theorem: The composition of two Möbius-transforms is again a Möbius-transform. More precisely

$$\begin{aligned} w = T_1(z) &= \frac{az+b}{cz+d} \qquad \text{for } ad \neq bc \\ u = (T_2 \circ T_1)(z) = T_2(w) &= \frac{\alpha w+\beta}{\gamma w+\delta} \qquad \text{for } \alpha\delta \neq \beta\gamma \\ &= \frac{Az+B}{Cz+D} \end{aligned}$$

The coefficients A, B, C and D can be obtained from the matrix product

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Due to $\det(AB) = \det A \cdot \det B$ we have

$$AD - BC = (ad - bc) \cdot (\alpha\delta - \beta\gamma) \neq 0$$

Conformality of Möbius-transforms.

Theorem: Möbius-transforms are conformal, i.e. (generalized) circles in $\mathbb{C}^*$ are mapped by Möbius-transforms in (generalized) circles.

Proof: Use an appropriate decomposition for $c \neq 0$

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c} \cdot \frac{1}{cz+d}$$

Now we set

$$\begin{aligned} w_1 &= cz+d \\ w_2 &= \frac{1}{w_1} \\ w_3 &= \frac{a}{c} - \frac{ad-bc}{c} \cdot w_2 \end{aligned}$$

The maps w_1 and w_3 are linear and thus conformal.

Continuation of the proof.

It remains to show:

The inversion $w = f(z) = 1/z$ is a conformal map.

We use the detour via the stereographic projection, i.e. instead of $z \to 1/z$ we consider the three of maps

$$z \to \mathrm{x} := P^{-1}(z) \to \tilde{\mathrm{x}} \to P(\tilde{\mathrm{x}}) = \frac{1}{z}$$

Then we have

$$\mathrm{x} = P^{-1}(z) = \left(\frac{z+\bar{z}}{z\bar{z}+1}, \frac{z-\bar{z}}{i(z\bar{z}+1)}, \frac{z\bar{z}-1}{z\bar{z}+1} \right)^T$$

and

$$\begin{aligned} \tilde{\mathrm{x}} &:= P^{-1}\left(\frac{1}{z}\right) \\ &= \left(\frac{\frac{1}{z}+\frac{1}{\bar{z}}}{\frac{1}{z}\frac{1}{\bar{z}}+1}, \frac{\frac{1}{z}-\frac{1}{\bar{z}}}{i(\frac{1}{z}\frac{1}{\bar{z}}+1)}, \frac{\frac{1}{z}\frac{1}{\bar{z}}-1}{\frac{1}{z}\frac{1}{\bar{z}}+1} \right)^T \end{aligned}$$

Completion of the proof.

A simplification gives

$$\begin{aligned} \tilde{\mathrm{x}} &= \left(\frac{z+\bar{z}}{z\bar{z}+1}, -\frac{z-\bar{z}}{i(z\bar{z}+1)}, -\frac{z\bar{z}-1}{z\bar{z}+1} \right) \\ &= (x_1, -x_2, -x_3)^T \end{aligned}$$

Thus we obtain a map $F : S^2 \to S^2$ with

$$F(\mathrm{x}) = (x_1, -x_2, -x_3)^T$$

This map is a rotation of the sphere arount the x_1–axis by 180° and apparentely confromal.

Therefore we have proofed that the three maps

$$z \to \mathrm{x} := P^{-1}(z) \to \tilde{\mathrm{x}} \to P(\tilde{\mathrm{x}}) = \frac{1}{z}$$

are conformal. With this the inversion $z \to 1/z$ is conformal.

Remarks on the Möbius-transform.

Remark: The Möbius-transform

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{with } ad \neq bc$$

has the follwoing properties.

- (Generalized) circles through the point $-d/c$ are mapped by T on straight lines in the w-plane.
- All straight lines in the z-plane are mapped by T on (generalized) circles in the w-plane containing the point a/c.
- Circles not containing the point $-d/c$ are mapped by T on circles not containing the point a/c.

Cross-ratio's and Möbius-tranforms.

Theorem: Let $z_1, z_2, z_3 \in \mathbb{C}^*$ and $w_1, w_2, w_3 \in \mathbb{C}^*$ be pairwise different. Then there exists exactely one Möbius-transform $w = T(z)$ satisfying the interpolations

$$w_j = T(z_j) \qquad \text{für } j = 1, 2, 3.$$

The interpolating Möbius-transform $T(z)$ is given by the three-point-formula

$$\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2}.$$

Definition: The expression

$$D(z_0, z_1, z_2, z_3) = \frac{z_0-z_1}{z_0-z_2} : \frac{z_3-z_1}{z_3-z_2}.$$

is called cross-ratio of the points z_0, z_1, z_2, z_3.

Example.

We are looking for the Möbius–transform with interpolation properties

z_i	1	i	0
w_i	i	$-i$	0

We obtain a unique Möbius–transform using the Ansatz

$$\frac{w-i}{w+i} : \frac{0-i}{0+i} = \frac{z-1}{z-i} : \frac{0-1}{0-i}$$

A simplification gives

$$-\frac{w-i}{w+i} = i\,\frac{z-1}{z-i}$$

or

$$(z-i)(w-i) = -i\,(z-1)(w+i)$$

This finally leads to gives

$$w = \frac{(1+i)z}{(1+i)z-2i}$$

Symmetry w.r.t. the circle.

Definition:
Let C in $\mathbb{C}$ be circle with center $z_0 \in \mathbb{C}$ and radius R. Two points $z, z' \in \mathbb{C}$ are called symmetric w.r.t. the circle C, if

$$(z-z_0)\overline{(z'-z_0)} = R^2$$

The map $z \to z'$ is called circle inversion on C or plane inversion on C.

Graphical representation of the plane inversion in the slide!

Remarks:

- A point z with $|z-z_0| \leq R$ is symmetrric w.r.t. a point z' with $|z'-z_0| \geq R$.
- If $|z-z_0| = R$, then z is symmetric to itself, i.e. $z' = z$.
- The point $z = z_0$ is symmetric to $z' = \infty$.

Möbius-transforms a circle symmetries.

Definition: Two points z, z' are called symmetric with respect to a straight line in $\mathbb{C}$, if z' is obtained from z by reflection across a line.

Theorem:
Möbius-transforms conserve symmetries w.r.t. (generalized) circles.

More precisely:
If C is a (generalized) circle in $\mathbb{C}^*$ and if z and z' are symmetric w.r.t. C, then the images z, z' of a Möbius-transform are symmetric w.r.t the to the (generalized) circle in $\mathbb{C}^*$, which is the image of C.

Example: We look for a Möbius-transform $w = T(z)$, such that the circle $|z| = 2$ is mapped on the circle $|w+1| = 1$ with $T(-2) = 0$ and $T(0) = i$.
A Möbius–transform is uniquely determined if the transformation is given for three points. But we only have

$$z_1 = -2, \; z_2 = 0 \quad \text{and} \quad w_1 = 0, \; w_2 = i$$

Therefore one point is missing!

Continuation of the example.

According to the last theorem Möbius–transforms conserve symmetries w.r.t. generalized circles.

$$z_2 = 0 \quad \Rightarrow \quad z_3 = \infty \quad \text{is symmetric to } z_2 \text{ w.r.t. the circle } |z| = 2$$

Thus w_3 is the point symmetric to $w_2 = i$ w.r.t the circle $|w+1| = 1$ and therefore given by the condition $(w_2+1)\overline{(w_3+1)} = 1$, i.e.

$$w_3 = \frac{1}{2}(-1+i)$$

Application of the three point formula gives

$$\frac{w-0}{w-i} : \frac{w_3-0}{w_3-i} = \frac{z+2}{z-0} : \frac{z_3+2}{z_3-0}$$

What happens to

$$\frac{z_3+2}{z_3-0}$$

as $z_3 \to \infty$?

Completion of the example.

What happens to

$$\frac{z_3 + 2}{z_3 - 0}$$

as $z_3 \to \infty$?

It is

$$\frac{z_3 + 2}{z_3 - 0} = \frac{1 + \frac{2}{z_3}}{1 + \frac{0}{z_3}} \to 1 \quad \text{for } z_3 \to \infty$$

We obtain

$$\left(\frac{w}{w - i}\right) : \left(\frac{\frac{1}{2}(-1 + i)}{\frac{1}{2}(-1 + i) - i}\right) = \left(\frac{z + 2}{z}\right)$$

and solving w.r.t w gives

$$w = T(z) = -\frac{z + 2}{(1 + i)z + 2i}$$

Example.

For $b > a > 0$ we consider the Möbius-transform

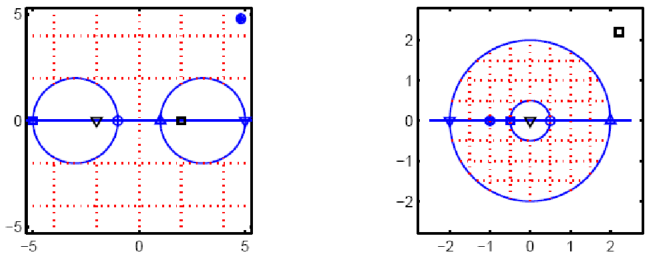
$$w = T(z) = \frac{z + p}{-z + p} \qquad \text{where } p = \sqrt{ab} \in (a, b)$$

Using T we obtain

$$\begin{array}{rcl}
z_{1,2} = \pm p & \to & w_{1,2} = \infty, 0 \\
z_{3,4} = a, b & \to & w_{3,4} = \pm \dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{b} - \sqrt{a}} = \pm \varrho \quad \text{with } |\varrho| > 1 \\
z_{5,6} = -a, -b & \to & w_{5,6} = \pm \dfrac{\sqrt{b} - \sqrt{a}}{\sqrt{a} + \sqrt{b}} = \pm \dfrac{1}{\varrho} \\
z_{7,8} = 0, \infty & \to & z_{7,8} = 1, -1.
\end{array}$$

Continuation of the example.

- The x-axis is mapped by T onto the u-axis.
- Points which are symmetric with respect to the x-axis are mapped onto points which are symmetric w.r.t. the u-axis.
- Circles being symmetric w.r.t the x-axis are mapped onto circles beiing symmetric wi



Important applications: The electrostatic field in the exterior of two parallel conducting lines is mapped on the field of a cylindrical condensator.

Chapter 4. Differential calculus in the complex numbers

4.1 Complex differentiation

Definition: Let $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ be a complex function. $f(z)$ is called complex differentiable in the point $z_0 \in D^0$ with derivative $f'(z_0)$, if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If $f(z)$ is complex differentiable in every point in the domain D, we call $f(z)$ holomorphic or analytic on D.

Remark:

1. The limit process $z \to z_0$ is intended in the complex plane, i.e. the approach $z \to z_0$ is arbitrary.
2. The division in the limit is a division in complex numbers.

4.1 Complex differentiation

Lemma: If $f(z)$ is real valued, i.e. $f : D \to \mathbb{R}$, $D \subset \mathbb{C}$ a domain, and if $f(z)$ is holomorphic on D, then $f(z)$ is a constant function.

Proof: We first consider the sequence $z_n \to z_0$ given by

$$z_n = z_0 + \frac{1}{n}$$

The the differential quotient is real for all $n \in \mathbb{N}$ since

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = n(f(z_n) - f(z_0)) \in \mathbb{R}$$

On the other hand the sequence $z_n \to z_0$ with $z_n = z_0 + i/n$ gives a purely imanginary differential quotient

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = \frac{n}{i}(f(z_n) - f(z_0)) \in \mathbb{C}$$

Since the function is holomorphic on D it follows

$$f'(z_0) = 0 \qquad \text{for all } z_0 \in D.$$

The Cauchy–Riemannschen equations.

Remark: If the function $f(z)$ is complex differentiable in z_0, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0} = 0$$

or equivalentely

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|)$$

Let $f(z)$ be complex differentiable in z_0. We set

$$\gamma := f'(z_0),$$

then we obtain the equivalent formulation

$$f(z) = f(z_0) + \gamma(z - z_0) + \varepsilon(z)|z - z_0|$$

with $\varepsilon(z) \to 0$ as $z \to z_0$.

The Cauchy–Riemannschen equations.

We now use with $z = x + iy$ the formulation

$$f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$$

and

$$\gamma = \alpha + i\beta$$

Thus we obtain

$$\begin{aligned} u(z) &= u(z_0) + \alpha(x - x_0) - \beta(y - y_0) + \mathrm{Re}(\varepsilon(z)) \cdot |z - z_0| \\ v(z) &= v(z_0) + \beta(x - x_0) + \alpha(y - y_0) + \mathrm{Im}(\varepsilon(z)) \cdot |z - z_0| \end{aligned}$$

In matrix formulation this reads as

$$\begin{pmatrix} u(z) \\ v(z) \end{pmatrix} = \begin{pmatrix} u(z_0) \\ v(z_0) \end{pmatrix} + \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \varepsilon(z) \cdot |z - z_0|$$

The Cauchy–Riemannschen equations.

We interpret $f(z)$ as vector valued, **totally differentiable** function of two variables, i.e.

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

with the Jacobian–matrix

$$Jf(x_0, y_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\Bigg|_{(x_0, y_0)} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

Theorem: The function $f(z)$ is complex differentiable in $z_0 \in D$ if and only if $f(z)$ as function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is totally differentiable and if the Cauchy–Riemannschen equations hold

$$\begin{aligned} u_x(z_0) &= v_y(z_0) \\ u_y(z_0) &= -v_x(z_0) \end{aligned}$$

Representation of the complex differentiation.

Corollary: If $f(z)$ is complex differentiable in $z_0 \in D$, then

$$f'(z_0) = u_x(z_0) + iv_x(z_0)$$

Proof: Since $f'(z_0) \in \mathbb{C}$ we can write

$$f'(z_0) = \tilde{u}(z_0) + i\tilde{v}(z_0)$$

From this we obtain

$$\begin{aligned} f'(z_0) \cdot (z - z_0) &= (\tilde{u}(z_0) + i\tilde{v}(z_0)) \cdot \Big[(x - x_0) + i(y - y_0)\Big] \\ &= \tilde{u} \cdot (x - x_0) - \tilde{v} \cdot (y - y_0) + i\left(\tilde{v} \cdot (x - x_0) + \tilde{u} \cdot (y - y_0)\right) \end{aligned}$$

Since f is totally differentiable in z_0 and since the Cauchy–Riemannschen equations are satisfied we have on the other side

$$\begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} u_x \cdot (x - x_0) - v_x(y - y_0) \\ v_x \cdot (x - x_0) + u_x(y - y_0) \end{pmatrix}$$

Holomorphic functions and the Laplace's equation.

Theorem: It $f \in \mathcal{C}^2$ is holomorphic on D, then

$$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0,$$

i.e. both real and imaginary part of f satisfy the Laplace's equation.

Proof: If $f(z)$ is holomorphic, then

$$\begin{aligned} \Delta u &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \stackrel{C.R.}{=} \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0 \\ \Delta v &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \stackrel{C.R.}{=} -\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = 0 \end{aligned}$$

Also, the following inversion holds true: If $u = u(x, y)$ satisfies the Laplace's equation $\Delta u = 0$ on a connected domain, then there exists a differentiable function $v = v(x, y)$ such that $f(z) = u(z) + iv(z)$ on D is holomorphic.

Proof of the inversion.

Let $u = u(x, y)$ be given with $\Delta u = 0$. We are looking for a function $v = v(x, y)$, such that the Cauchy–Riemannschen equations are satisfied. Thus

$$v_x = -u_y \qquad v_y = u_x$$

From the C.R. equations it follows

$$\operatorname{grad} v = (v_x, v_y) = (-u_y, u_x) =: V = (V_1, V_2)$$

Therefore we are looking for a potential v with $\operatorname{grad} v = V$. If the integrability conditions

$$\frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial x} = 0$$

are satisfied, the existence of such a potential if guaranteed.

This is true since

$$\frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial x} = -u_{yy} - u_{xx} = -\Delta u = 0$$

Rules for the differentiation.

- The following rules hold:

$$\begin{aligned}
(f \pm g)'(z_0) &= f'(z_0) \pm g'(z_0) \\
(f \cdot g)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0) \\
\left(\frac{f}{g}\right)'(z_0) &= \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}
\end{aligned}$$

- Chain rule: If $f(z)$ is differentiable in z_0 and if $g(w)$ is differentiable in $w_0 = f(z_0)$, then

$$(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0)$$

- Derivation of the inverse function: If $f(z)$ is holomorphic and if $f'(z_0) \neq 0$, then $f(z_0)$ is locally bijective around z_0 and we have

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}, \quad w_0 = f(z_0)$$

The modified chain rule.

Lemma: If $f(z)$ is holomorphic on D and if $c : [a, b] \to D$ is a $\mathcal{C}^1$–curve in D, then

$$\frac{d}{dt} f(c(t)) = f'(c(t)) \cdot \dot{c}(t)$$

Proof: We have

$$\begin{aligned} \frac{d}{dt} f(c(t)) &= \frac{d}{dt} u(c(t)) + i\, \frac{d}{dt} v(c(t)) \\ &= (u_x \dot{c}_1 + u_y \dot{c}_2) + i\,(v_x \dot{c}_1 + v_y \dot{c}_2) \end{aligned}$$

In addition we have

$$\begin{aligned} f'(c(t)) \cdot \dot{c}(t) &= (u_x + i\, v_x) \cdot (\dot{c}_1 + i\, \dot{c}_2) \\ &= (u_x \dot{c}_1 - v_x \dot{c}_2) + i\,(v_x \dot{c}_1 + u_x \dot{c}_2) \end{aligned}$$

Both expressions are identical due to the C.R. equations.

Examples.

Example 1:

For $f(z) = z$ we obtain due to $u(x, y) = x$ and $v(x, y) = y$

$$f'(z) = u_x(z) + i\, v_x(z) = 1$$

Thus complex polynomials on $\mathbb{C}$ are holomorphic with

$$\frac{d}{dz}\left(\sum_{k=0}^{n} a_k z^k\right) = \sum_{k=1}^{n} a_k k z^{k-1}$$

Explicit calculation for $f(z) = z^2$: with

$$f(z) = z^2 = (x^2 - y^2) + i\, 2xy$$

we calculate

$$f'(z) = u_x(z) + i\, v_x(z) = 2x + i\, 2y = 2z$$

Examples.

Example 2: Rational functions, i.e. functions of the form

$$f(z) = \frac{p(z)}{q(z)}, \qquad p, q \text{ complex polynomials}$$

are complex differentiable at every point with $q(z) \neq 0$.

Example 3: The exponential function $f(z) = e^z = e^x(\cos y + i \sin y)$ is complex differentiable with $f'(z) = e^z$, since with

$$u(x, y) = e^x \cos y, \qquad v(x, y) = e^x \sin y$$

the C.R. equations are satisfied

$$u_x = v_y = e^x \cos y, \qquad u_y = -v_x = -e^x \sin y$$

and we have

$$f'(z) = u_x + i\, v_x = e^x \cos y + i\, e^x \sin y = e^z$$

More examples.

Example 4: The trigonometrc functions

$$\sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \qquad \cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

are according to example 3 holomorphic on $\mathbb{C}$ and we have the formulas for the derivatives in analogy to the real valued functions.

Example 5: Functions defined as complex power series,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

are holomorphic on the domain of convergence $K_r(z_0)$ with

$$f'(z) = \sum_{k=1}^{\infty} a_k k (z - z_0)^{k-1}$$

and thus on $K_r(z_0)$ at the same time arbitrary many times complex differentiable.

4.2 Conformal mappings

Theorem: Let $f : D \to \mathbb{C}$ be a holomorphic function on the domain $D \subset \mathbb{C}$ with $f'(z) \neq 0$ for all $z \in D$. Then locally in a point $z_0 \in D$ we have:

a) Angles between curves which intersect in z_0 are conserved under the transformation $w = f(z)$, including the rotational direction,

b) the expression $|f'(z_0)|$ is for all directions "leaving" z_0 the common scaling. In particular relations of lenghtes are conserved.

Mappings with these properties are called conformal mappings.

For conformal mappings we have the following inversion of the theorem.

Theorem: If $w = f(z)$ is a conformal mapping and if the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable, then $f(z)$ is complex differentiable and we have $f'(z) \neq 0$.

Proof of the first theorem.

Let c and d be two curves which at $t = 0$ go through z_0. The two tangential vectors in this point are $\dot{c}(0)$ and $\dot{d}(0)$ and for the angle γ between the tangential vectors we have

$$\gamma = \measuredangle\left(\dot{c}(0), \dot{d}(0)\right) = \arg\left(\dot{d}(0)\right) - \arg\left(\dot{c}(0)\right)$$

With f we obtain the two curves $f \circ c$ and $f \circ d$ in the codomain. Th angle $\tilde{\gamma}$ between these two curves in $f(z_0)$ in the codomain is

$$\begin{aligned}
\tilde{\gamma} &= \measuredangle\left(f'(z_0)\dot{c}(0), f'(z_0)\dot{d}(0)\right) \\
&= \arg\left(f'(z_0)\dot{d}(0)\right) - \arg\left(f'(z_0)\dot{c}(0)\right) \\
&= \arg\left(f'(z_0)\right) + \arg\left(\dot{d}(0)\right) - \arg\left(f'(z_0)\right) - \arg\left(\dot{c}(0)\right) = \gamma
\end{aligned}$$

and w.r.t the scaling of lenghtes we calculate

$$\|\frac{d}{dt}\Big(f \circ c\Big)\| = |f'(z_0)\dot{c}(0)| = |f'(z_0)| \cdot |\dot{c}(0)|$$

Conformal transformations.

Definition: Let $f : D \to D'$ be a bijective and conformal mapping bewteen the domains $D \subset \mathbb{C}$ and $D' \subset \mathbb{C}$. Let $\Phi : D \to \mathbb{R}$ be a real valued twice continuously differentiable function on D. Then we call the function $\Psi : D' \to \mathbb{R}$ defined by

$$\Psi = \Phi \circ f^{-1}$$

the conformal transformation of Φ with mapping f.

Physical Applications: If $\Phi(z)$ is an unknown potential defined in the in the physical plane D, then Ψ is the related function in the modell plane D'.
In the following Φ and Ψ are potentials, i.e.

- electrostatic potentials;
- fluid dynamic potentials;
- temperature fields etc.

The vectors (Φ_x, Φ_y) and (Ψ_u, Ψ_v) are of particular interest.

The complex gradient.

Definition: For a real valued function $\Phi : D \to \mathbb{R}$ on a domain $D \subset \mathbb{C}$ we call with $z = x + iy$ the expression

$$\operatorname{grad} \Phi(z) = \frac{\partial \Phi}{\partial x} + i \, \frac{\partial \Phi}{\partial y}$$

the complex gradient of $\Phi(z)$.

Theorem: Let Ψ be the conformal transformation of Φ with mapping f. Then the two relations

$$\begin{aligned} \operatorname{grad}_z \Phi(z) &= \operatorname{grad}_w \Psi(f(z)) \cdot \overline{f'(z)} \\ \Delta_z \Phi(z) &= \Delta_w \Psi(f(z)) \cdot |f'(z)|^2 \end{aligned}$$

hold. **Proof:** By definition the conformal transformation of Φ with mapping f is given by

$$\Psi = \Phi \circ f^{-1}$$

Continuation of the proof.

We conclude $\Phi = \Psi \circ f$ and with $f(x,y) = u(x,y) + i\,v(x,y)$

$$\Phi(x,y) = \Psi(u(x,y), v(x,y))$$

We calculate

$$\begin{aligned} \Phi_x &= \Psi_u u_x + \Psi_v v_x \\ \Phi_y &= \Psi_u u_y + \Psi_v v_y \end{aligned}$$

For the complex gradient gwe have with $f'(z) = u_x + iv_x$

$$\begin{aligned} \operatorname{grad}\Phi(z) &= (\Psi_u u_x + \Psi_v v_x) + i\,(\Psi_u u_y + \Psi_v v_y) \\ &= \Psi_u(u_x + i\,u_y) + \Psi_v(v_x + i\,v_y) \\ &\overset{C.R.}{=} \Psi_u(u_x - i\,v_x) + i\,\Psi_v(u_x - i\,v_x) \\ &= \operatorname{grad}\Psi(f(z)) \cdot \overline{f'(z)} \end{aligned}$$

Completion of the proof.

Calculating the second derivative gives

$$\begin{aligned} \Phi_{xx} &= \Psi_{uu}u_x^2 + 2\Psi_{uv}u_x v_x + \Psi_{vv}v_x^2 + \Psi_u u_{xx} + \Psi_v v_{xx} \\ \Phi_{yy} &= \Psi_{uu}u_y^2 + 2\Psi_{uv}u_y v_y + \Psi_{vv}v_y^2 + \Psi_u u_{yy} + \Psi_v v_{yy} \end{aligned}$$

Thus

$$\begin{aligned} \Delta\Phi &= \Psi_{uu}(u_x^2 + u_y^2) + 2\Psi_{uv}(u_x v_x + u_y v_y) \\ &\quad + \Psi_{vv}(v_x^2 + v_y^2) + \Psi_u \Delta u + \Psi_v \Delta v \end{aligned}$$

We use again the C.R. equations and obtain

$$\begin{aligned} u_x^2 + u_y^2 &= v_x^2 + v_y^2 = u_x^2 + v_x^2 = |f'(z)|^2 \\ u_x v_x + u_y v_y &= 0 \\ \Delta u &= \Delta v = 0 \end{aligned}$$

and therefore the desired result

$$\Delta\Phi = \Delta\Psi \cdot |f'(z)|^2$$

Practical applications of conformal transformations.

Corollary: Conformal transformations transform harmonic functions into harmonic functions.

Applications of conformal transformations: Lets consider the Dirichlet–problem for the Laplace equation, i.e. the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$

where $D \subset \mathbb{R}^2$ is a "complicated" two-dimensional domain.

With an appropriate conformal transformation we can solve the problem explicitely.

1. identify a conformal transformation which maps the physical domain D on a "simple" model domain D';
2. transform the boundary conditions on ∂D to boundary conditions on $\partial D'$ and solve the Dirichlet–problem on D';
3. Transform the solution back on the physical domain D.

An application: plain potential flow.

We would like to determine the velocity field of a stationary, curl– and source–free flow around a cylinder. Let $\mathrm{w} : \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity field to be determined.

Then we have the equations

$$\begin{aligned} \operatorname{rot} \mathrm{w} &= \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} = 0 \\ \operatorname{div} \mathrm{w} &= \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0 \end{aligned}$$

If $D \subset \mathbb{R}^2$ is simply connected we obtain from the first condition

$$\text{there exists a function } u : D \to \mathbb{R} \text{ with } \nabla u = -\mathrm{w}$$

and from the second condition

$$\text{there exists a function } v : D \to \mathbb{R} \text{ with } \nabla v = (w_2, -w_1)^T$$

The complex flow potential.

We call

- the function u the velocity potential;
- the function v the stream function.

Related to the stream function we have stream lines which are solutions of the ordinary differential equations $y'(x) = w_2/w_1$ and given by

$$v(x,y) = \text{const.}$$

Definition: The complex funktion $\Phi = \Phi(x,y)$ defined by

$$\Phi(x,y) = u(x,y) + i\,v(x,y)$$

is called complex flow potential.

The complex flow potential $\Phi(z)$ is a holomorphic function, since we have the Cauchy–Riemann equations

$$\begin{aligned} u_x - v_y &= -w_1 - (-w_1) = 0 \\ u_y + v_x &= -w_2 + w_2 = 0 \end{aligned}$$

Continuation: plain potential flow.

The velocity field w can be calculated directly: due to

$$\Phi'(z) = u_x + i\,v_x = -w_1 + i\,w_2$$

it follows

$$\mathrm{w} = w_1 + i\,w_2 = -\overline{\Phi'(z)}$$

Our physical domain is diven by $D = \{z \in \mathbb{C} : |z| > R\}$ and the related model domain is

$$D' = \{z \in \mathbb{C} \mid \operatorname{Im} z \neq 0 \,\text{und}\, |\operatorname{Re} z| > 1\}$$

The Joukowski–function $f(z)$ given by

$$f(z) = \frac{1}{2}\left(\frac{z}{R} + \frac{R}{z}\right)$$

is a conformal transformation from D on D'.

Continuation: plain potential flow.

In the model plane we can assume a homogeneous velocity field, i.e. in D' we have

$$\mathsf{W} = \text{const.} = (V_\infty, 0)^T$$

since a infinitely flat plate is not interacting with a given homogeneous flow in the direction of the real axis with velocity V_∞.

For the velocity potential $U(W)$ we have the equation

$$\text{grad}\, U(W) = -(V_\infty, 0)^T$$

and from this follows

$$U(w) = -V_\infty W_1$$

Also there is a stream function $V(W)$

$$\text{grad}\, V(W) = (0, -V_\infty)^T \qquad \Rightarrow \qquad V(w) = -V_\infty W_2$$

Continuation: plain potential flow.

In the physical plane we can assume that

$$\lim_{z\to\infty} \text{grad}\, \Phi(z) = -v_\infty$$

i.e. at infinity the undisturbed flow does not "feel" any obstacle.

Because of the relation

$$\text{grad}\, \Phi(z) = \text{grad}\, \Psi(f(z)) \cdot \overline{f'(z)}$$

it follows with

$$f'(z) = \frac{1}{2}\left(\frac{1}{R} - \frac{R}{z^2}\right)$$

the relation $V_\infty = 2Rv_\infty$.

For the complex flow potential we have

$$\Psi(W) = -2Rv_\infty(\text{Re}\, W + i\, \text{Im}\, W)$$

Continuation: plain potential flow.

Now we consider the back–transformation in the physical plane, i.e.

$$\Phi(z) = (\Psi \circ f)(z) = -2Rv_\infty(\operatorname{Re} f(z) + i \operatorname{Im} f(z))$$

For the Joukowski–function

$$f(z) = \frac{1}{2}\left(\frac{z}{R} + \frac{R}{z}\right)$$

it is

$$\operatorname{Re} f(z) = \frac{1}{2}\left(\frac{x}{R} + \frac{Rx}{x^2+y^2}\right) \qquad \operatorname{Im} f(z) = \frac{1}{2}\left(\frac{y}{R} - \frac{Ry}{x^2+y^2}\right)$$

With this in the physical plane we obtain the velocity potential $u(z)$

$$u(z) = u(x,y) = -v_\infty\left(x + \frac{R^2x}{x^2+y^2}\right)$$

Continuation: plain potential flow.

We obtain for the stream function

$$v(z) = v(x,y) = -v_\infty\left(y - \frac{R^2y}{x^2+y^2}\right)$$

The velocity field w around the cylinder is given by

$$\mathrm{w} = -\nabla u = -v_\infty\left(\frac{(x^2+y^2)^2 - R^2(x^2-y^2)}{(x^2+y^2)^2}, -\frac{2R^2xy}{(x^2+y^2)^2}\right)$$

In particular we have:

- In the two points $(-R,0)$ and $(R,0)$ the velocity is zero,

$$\mathrm{w}(-R,0) = \mathrm{w}(R,0) = (0,0)^T$$

- The velocity is maximal in the two points $(0,-R)$ and $(0,R)$ with

$$w_{\max} = 2v_\infty$$

Chapter 5. Complex integration

5.1 Examples for complex integration

Definition: A complex valued function $f : [a, b] \to \mathbb{C}$ of a real variable is integrable, if real– and imaginary part of f are integrable, and we have:

$$\int_a^b f(t)\,dt := \int_a^b \mathrm{Re}(f(t))\,dt + i\int_a^b \mathrm{Im}(f(t))\,dt = Re^{i\varphi}$$

The following properties in analogy to the intergration in the real numbers are valid Linearity. In addition we have

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$$

Proof: We calculate

$$\begin{aligned}\left|\int_a^b f(t)\,dt\right| &= R = e^{-i\varphi}\int_a^b f(t)\,dt = \int_a^b e^{-i\varphi} f(t)\,dt = \int_a^b \mathrm{Re}(e^{-i\varphi} f(t))\,dt \\ &\leq \int_a^b |e^{-i\varphi} f(t)|\,dt = \int_a^b |f(t)|\,dt\end{aligned}$$

Complex integration in analogy to curve integrals.

Real analysis: Let $c : [a, b] \to D \subset \mathbb{R}^n$ a piecewise $\mathcal{C}^1$–curve, $f : D \to \mathbb{R}$ and $\mathrm{F} : D \to \mathbb{R}^n$ are given. Then we have defined in Analysis II and III the line integrals of scalar and vector fileds

$$\int_c f(x)ds := \int_a^b f(c(t))\|\dot{c}\|\,dt$$

or

$$\int_c \mathrm{F}(x)\,dx := \int_a^b \langle \mathrm{F}(c(t)), \dot{c}(t)\rangle\,dt$$

Definition: Let $D \subset \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ continuous and $c : [a, b] \to D$ a piecewise $\mathcal{C}^1$–curve. Then

$$\int_c f(z)\,dz := \int_a^b f(c(t))\dot{c}(t)\,dt$$

is the complex integral of $f(z)$ along the curve c.

Properties of the complex integral.

- The value of the compelx integral is independent of the parameterisation of the curve.
- Changing the orientation we have

$$\int_{-c} f(z)\, dz = -\int_c f(z)\, dz$$

We denote $(-c)(t) := c(b + t(a-b))$, $0 \le t \le 1$.

- Linearity

$$\int_c (\alpha f(z) + \beta g(z))\, dz = \alpha \int_c f(z)\, dz + \beta \int_c g(z)\, dz \qquad \text{für } \alpha, \beta \in \mathbb{C}$$

- Additivity with respect to the path of integration:

$$\int_{c_1+c_2} f(z)\, dz = \int_{c_1} f(z)\, dz + \int_{c_2} f(z)\, dz$$

Additional properties of the complex integral

We have the estimate

$$\left|\int_c f(z)\, dz\right| \le \sup_{z \in image(c)} |f(z)| \cdot \underbrace{\int_a^b |\dot{c}(t)|\, dt}_{lenghtofthepath \;\; L(c)}$$

Proof We calculate directly

$$\begin{aligned}\left|\int_c f(z)\, dz\right| &= \left|\int_a^b f(c(t))\dot{c}(t)\, dt\right| \\ &\le \int_a^b |f(c(t))|\, |\dot{c}(t)|\, dt \\ &\le \sup_{a \le t \le b} |f(c(t))| \cdot \int_a^b |\dot{c}(t)|\, dt\end{aligned}$$

An example of complex integration.

Example 1:

Let $f(z) = z$ and $c(t) = re^{it}$ with $0 \leq t \leq 2\pi$. Then we have

$$\begin{aligned}
\oint_c z\, dz &= \int_0^{2\pi} re^{it} \cdot \left(rie^{it}\right) dt \\
&= ir^2 \int_0^{2\pi} e^{2it}\, dt \\
&= ir^2 \int_0^{2\pi} (\cos(2t) + i \sin(2t))\, dt \\
&= -r^2 \int_0^{2\pi} \sin(2t))\, dt + i\, r^2 \int_0^{2\pi} \cos(2t))\, dt \\
&= 0
\end{aligned}$$

Additional examples of complex integration.

Example 2:

Let $f(z) = \bar{z}$ and $c(t) = re^{it}$ with $0 \leq t \leq 2\pi$. then it is

$$\oint_c \bar{z}\, dz = \int_0^{2\pi} re^{-it} \cdot \left(rie^{it}\right) dt = ir^2 \int_0^{2\pi} dt = r^2 \cdot 2\pi i$$

Example 3:

Let $f(z) = 1/z$ and $c(t) = re^{it}$ with $0 \leq t \leq 2\pi$. Then it is

$$\oint_c \frac{1}{z}\, dz = \oint_c \frac{\bar{z}}{|z|^2}\, dz = \frac{1}{r^2} \oint_c \bar{z}\, dz = 2\pi i$$

Example 4: With $c(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ we have the relation

$$\oint_c (z - z_0)^n\, dz = \begin{cases} 2\pi i & : \quad \text{for } n = -1 \\ 0 & : \quad \text{for } n \in \mathbb{Z} \setminus \{-1\} \end{cases}$$

Continuation of the last example.

Example 4:

$$\begin{aligned}
\oint_c (z-z_0)^n\,dz &= \int_0^{2\pi} \left(re^{it}\right)^n \cdot \left(rie^{it}\right) dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\,dt \\
&= r^{n+1}\left(-\int_0^{2\pi} \sin((n+1)t))\,dt + i\int_0^{2\pi} \cos((n+1)t))\,dt\right) \\
&= \begin{cases} 2\pi i & : \text{ für } n=-1 \\ 0 & : \text{ for } n \in \mathbb{Z}\setminus\{-1\} \end{cases}
\end{aligned}$$

Only for $n=-1$ the integral is not vanishing and we have

$$\oint_c \frac{1}{z-z_0}\,dz = 2\pi i$$

Question: Why this?

Uniform convergence and complex integration.

Theorem: Let $f(z) = \sum_{k=0}^{\infty} f_k(z)$ be a series of continuous functions, which on a domain $D \subset \mathbb{C}$ converges uniformly. Let $c : [a,b] \to D$ be a piecewise $\mathcal{C}^1$–curve, then

$$\int_c f(z)\,dz = \sum_{k=0}^{\infty} \int_c f_k(z)\,dz$$

Proof: SInce the series of continuous functions converges uniformly also the limit function $f(z)$ is continuous and thus integrable

$$\int_c f(z)\,dz - \sum_{k=0}^{n} \int_c f_k(z)\,dz = \int_c R_n(z)\,dz$$

Uniform convergence means

$$\forall\,\varepsilon > 0\ :\ \exists\,N(\varepsilon)\ :\ \forall\,n \geq N,\ z \in D\ :\ |R_n(z)| < \varepsilon$$

Continuation of the proof.

From the uniform convergence we conclude

$$\left| \int_c R_n(z)\,dz \right| \le \varepsilon \cdot L(c)$$

and thus

$$\lim_{n\to\infty} \int_c R_n(z)\,dz = 0$$

Example: Let

$$c(t) = re^{it} \qquad \text{with } 0 \le t \le 2\pi$$

and $|z_0| > r$. Then:

$$\oint_{|z|=r} \frac{dz}{z - z_0}\,dz = 0$$

Note: The point z_0 lies outside the circle $c(t)$.

Continuation of the example.

We calculate directly using the geometric series

$$\oint_{|z|=r} \frac{dz}{z - z_0} = -\frac{1}{z_0} \oint_{|z|=r} \frac{dz}{1 - \frac{z}{z_0}} = -\frac{1}{z_0} \oint_{|z|=r} \sum_{k=0}^{\infty} \frac{1}{z_0^k}\, z^k\, dz$$

since it is

$$\left| \frac{z}{z_0} \right| < 1$$

Due to the uniform convergence it is

$$\frac{1}{z_0} \oint_{|z|=r} \sum_{k=0}^{\infty} \frac{1}{z_0^k}\, z^k\, dz = \sum_{k=0}^{\infty} \frac{1}{z_0^{k+1}} \oint_{|z|=r} z^k\, dz = 0$$

since we can exchange integration and summation.

Anticipation of the Laurent–series.

Example: A series of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \underbrace{\sum_{k=0}^{\infty} a_k (z-z_0)^k}_{\text{in analogy to the Taylor–series}} + \underbrace{\sum_{k=-\infty}^{-1} a_k (z-z_0)^k}_{\text{negativ powers}}$$

is called a Laurent–serie.

It is comverging locally uniformly and absolutely in the ring

$$0 \le R_1 < |z - z_0| < R_2$$

For $R_1 < r < R_2$ and $c(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$ we have

$$\oint_{|z-z_0|=r} f(z)\,dz = \sum_{k=-\infty}^{\infty} a_k \oint_{|z-z_0|=r} (z-z_0)^k\,dz = 2\pi\,i\,a_{-1}$$

Kapitel 5. Komplexe Integration

5.2 Der Cauchysche Hauptsatz

Wir hatten im Abschnitt 5.1 mit der Kurve $c(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$ die Aussage

$$\oint_c (z-z_0)^n\,dz = \begin{cases} 2\pi i & : \text{ für } n = -1 \\ 0 & : \text{ für } n \in \mathbb{Z} \setminus \{-1\} \end{cases}$$

Frage: Wann verschwindet das Integral über geschlossene Kurven?

Satz: (Cauchyscher Integralsatz, Hauptsatz der Funktionentheorie)

Ist $G \subset \mathbb{C}$ ein einfach zusammenhängendes Gebiet, $f : G \to \mathbb{C}$ eine holomorphe Funktion und $c : [a, b] \to G$ eine geschlossene stückweise $\mathcal{C}^1$–Kurve, so gilt stets

$$\oint_c f(z)\,dz = 0$$

Bemerkung zum Cauchyschen Integralsatz.

Alle drei Voraussetzungen sind wichtig und zusammen hinreichend

1. Die Funktion $f(z) = \bar{z}$ ist nicht holomorph und es gilt

$$\oint_{|z|=1} \bar{z}\, dz \neq 0$$

2. Das Gebiet $G = \{z \in \mathbb{C} \,:\, z \neq 0\}$ ist nicht einfach zusammenhängend und es gilt

$$\oint_{|z|=1} \frac{1}{z}\, dz \neq 0$$

3. Die Kurve $c(t) = e^{(1+i)t}$ mit $0 \leq t \leq 2\pi$ ist nicht geschlossen und es gilt

$$\int_c z\, dz \neq 0$$

Beweis des Cauchyschen Integralsatzes.

Beweis: Wir setzen $c(t) = (x(t), y(t))^T$ und $f(x, y) = u(x, y) + i\, v(x, y)$:

$$\begin{aligned} \oint_c f(z)\, dz &= \int_a^b (u\dot{x} - v\dot{y})\, dt + i \int_a^b (u\dot{y} + v\dot{x})\, dt \\ &= \oint_c \begin{pmatrix} u \\ -v \end{pmatrix} dx + i \oint_c \begin{pmatrix} v \\ u \end{pmatrix} dx \end{aligned}$$

Bei beiden Vektorfelder $(u, -v)^T$ und $(v, u)^T$ ist wegen der C.R. DGL's die Integrabilitätsbedingung erfüllt:

$$\mathrm{rot} \begin{pmatrix} u \\ -v \end{pmatrix} = u_y + v_x = 0, \quad \mathrm{rot} \begin{pmatrix} v \\ u \end{pmatrix} = v_y - u_x = 0$$

Daher existiert ein Potential und beide Integrale sind wegen der geschlossenen Kurve c identisch gleich Null.

Die Stammfunktion einer holomorphen Funktion.

Korollar: Ist $G \subset \mathbb{C}$ einfach zusammenhängend, $f(z)$ holomorph auf G und $c_1, c_2 : [a, b] \to G$, so folgt aus $c_1(a) = c_2(a)$ und $c_1(b) = c_2(b)$ direkt

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)\, dz$$

d.h. das Integral $\int_c f(z)\, dz$ ist wegunabhängig.

Satz: (Existenz einer Stammfunktion)
Sei $G \subset \mathbb{C}$, $f(z)$ holomorph auf G, $z_0 \in G$ ein fester Punkt und setze für $z \in G$

$$F(z) := \int_{c_z} f(\xi)\, d\xi$$

mit einer beliebigen stückweisen $\mathcal{C}^1$–Kurve, die z_0 und z verbindet.
Dann ist $F(z)$ eine Stammfunktion von $f(z)$, d.h. es gilt

$$F'(z) = f(z)$$

Beweis des letzten Satzes.

Beweis: Es gilt

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\int_z^{z+h} f(\xi)\, d\xi = \frac{1}{h}\int_0^1 f(z+th)h\, dt \\ &= \int_0^1 f(z+th)\, dt\end{aligned}$$

Daraus folgt

$$\begin{aligned}\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\int_0^1 (f(z+th) - f(z))\, dt\right| \\ &\leq \sup_{t\in[0,1]} |f(z+th) - f(z)| \to 0\end{aligned}$$

für $h \to 0$.

Berechnung komplexer Integrale mittels Stammfunktion.

Korollar: Ist $f(z)$ auf einem einfach zusammenhängenden Gebiet G holomorph und $F(z)$ eine Stammfunktion von $f(z)$, so gilt für alle stückweisen $\mathcal{C}^1$–Kurven $c : [a, b] \to G$

$$\int_c f(z)\, dz = F(c(b)) - F(c(a))$$

Beispiel: Wir betrachten mit $a, b \in \mathbb{R}$, $a, b > 0$ das Integral

$$\int_{a-i\,b}^{a+i\,b} \frac{dz}{z^2}$$

Die Funktion $f(z) = 1/z^2$ ist holomorph auf dem einfach zusammenhängenden Gebiet

$$G = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\},$$

also die komplexe Ebene ohne die negative reelle Achse.

Insbesondere ist das obenstehende Integral wegunabhängig.

Forsetzung des Beispiels.

Direkte Integration: Wir setzen den Integrationsweg

$$c(t) = a + i\,t, \qquad \text{für } -b \leq t \leq b$$

und erhalten

$$\begin{aligned} \int_c \frac{dz}{z^2} &= \int_{-b}^{b} \frac{i}{(a+i\,t)^2}\, dt = -\left.\frac{1}{a+i\,t}\right|_{-b}^{b} \\ &= \frac{1}{a-i\,b} - \frac{1}{a+i\,b} = \frac{2ib}{a^2+b^2} \end{aligned}$$

Auf der anderen Seite berechnet man mit Hilfe der Stammfunktion

$$\int_{a-i\,b}^{a+i\,b} \frac{dz}{z^2} = \left(-\frac{1}{z}\right)\Bigg|_{a-i\,b}^{a+i\,b} = \frac{2ib}{a^2+b^2}$$

Homotope Kurven.

Definition: Sei $G \subset \mathbb{C}$ ein Gebiet, $c, \tilde{c} : [a, b] \to G$ zwei geschlossene Kurven in G. Man nennt c und $\tilde{c}$ homotop, falls eine stetige Abbildung $\Phi : [a, b] \times [0, 1] \to G$ existiert mit

$$\begin{aligned} \Phi(t, 0) &= c(t) && \text{für alle } t \in [a, b] \\ \Phi(t, 1) &= \tilde{c}(t) && \text{für alle } t \in [a, b] \\ \Phi(a, s) &= \Phi(b, s) && \text{für alle } s \in [0, 1] \end{aligned}$$

Eine geschlossene Kurve c heißt nullhomotop, falls sie sich in G stetig auf einen Punkt zusammenziehen läßt.

Damit folgt aus dem Cauchyschen Integralsatz:

Sei $f(z)$ holomorph auf einem Gebiet G. Dann gilt für zwei geschlossene Wege c und $\tilde{c}$:

$$c, \tilde{c} \text{ homotop} \quad \Rightarrow \quad \oint_c f(z)\,dz = \oint_{\tilde{c}} f(z)\,dz$$

Beispiel und Definition der Umlaufzahl.

Beispiel: Für jede einfach geschlossene Kurve c, die den Punkt $z_0 \in \mathbb{C}$ (einmal) im positiven Sinn umläuft, gilt

$$\oint_c \frac{dz}{z - z_0} = 2\pi i$$

Denn $c(t)$ ist homotop zu $\tilde{c}(t) = z_0 + e^{it}$, $0 \leq t \leq 2\pi$.

Definition: Für eine geschlossene, stückweise $\mathcal{C}^1$–Kurve $c : [a, b] \to \mathbb{C} \setminus \{z_0\}$ heißt

$$\text{Uml}\,(c, z_0) := \frac{1}{2\pi i} \oint_c \frac{dz}{z - z_0}$$

die Umlaufzahl von c bezüglich des Punktes z_0.

Die Umlaufzahl ist stets eine ganze Zahl und gibt an, wie oft der Weg c den Punkt z_0 in mathematisch positivem Sinne umläuft.

5.3 Die Cauchysche Integralformel, Taylor–Entwicklung

Satz: (Cauchysche Integralformel)

Sei $f(z)$ holomorph auf einem Gebiet G, $z_0 \in G$ und $c : [a, b] \to G \setminus \{z_0\}$ ein zum Punkt z_0 homotoper Weg, der z_0 im positiven Sinn einmal umläuft. Dann gilt

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0}\, dz$$

Beweis:

Der Weg c läßt sich innerhalb von $G \setminus \{z_0\}$ auf einen Kreis $K_r(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$ zusammenziehen. Daher gilt

$$\oint_c \frac{f(z)}{z - z_0}\, dz = \oint_{K_r} \frac{f(z)}{z - z_0}\, dz = \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\, dt$$

Daher gilt

$$\begin{aligned}\oint_c \frac{f(z)}{z - z_0}\, dz &= \oint_{K_r} \frac{f(z)}{z - z_0}\, dz = \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\, dt \\ &= i \int_0^{2\pi} f(z_0 + re^{it})\, dt\end{aligned}$$

Im Grenzfall $r \to 0$ erhalten wir offensichtlich die Beziehung

$$i \int_0^{2\pi} f(z_0 + re^{it})\, dt \longrightarrow 2\pi i f(z_0)$$

Da das Integral $\oint_c \frac{f(z)}{z - z_0}\, dz$ aber unabhängig von r ist, folgt

$$\oint_c \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$

Bemerkungen zur Cauchyschen Integralformel.

- Für einen beliebigen z_0–homotopen Weg in $G \setminus \{z_0\}$, der den Punkt z_0 nicht notwendigerweise genau einmal durchläuft, gilt entsprechend

$$\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i \cdot \mathrm{Uml}\,(c, z_0) \cdot f(z_0)$$

- Nützlich ist folgende heuristische Herleitung: aus der Taylor–Formel

$$f(z) = f(z_0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

folgt

$$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1}$$

Formal erhalten wir damit

$$\oint_c \frac{f(z)}{z-z_0}\,dz = \oint_c \frac{f(z_0)}{z-z_0}\,dz + \sum_{k=1}^{\infty} \underbrace{\oint_c \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1}\,dz}_{=0} = 2\pi i \cdot f(z_0)$$

Ein Beispiel zur Cauchyschen Integralformel.

Zu berechnen sei das Integral

$$\oint_c \frac{1}{1+z^2}\,dz,$$

wobei c die Achterkurve bezeichnet, die den Punkt $z_1 = i$ einmal im positiven Sinn, den Punkt $z_2 = -i$ einmal im negativen Sinn umläuft.

1. Berechnung mittels Partialbruchzerlegung

$$\begin{aligned}\oint_c \frac{1}{1+z^2}\,dz &= \oint_c \frac{1}{(z+i)(z-i)}\,dz = \frac{i}{2}\oint_c \left(\frac{1}{z+i} - \frac{1}{z-i}\right)dz \\ &= \frac{i}{2}\oint_c \frac{dz}{z+i} - \frac{i}{2}\oint_c \frac{dz}{z-i} = \frac{i}{2}(-2\pi i) - \frac{i}{2}(2\pi i) = 2\pi\end{aligned}$$

2. Berechnung mittels Cauchyscher Integralformel

$$\oint_c \frac{1}{1+z^2}\,dz = \oint_{c_1} \frac{\left(\frac{1}{z+i}\right)}{z-i}\,dz + \oint_{c_2} \frac{\left(\frac{1}{z-i}\right)}{z+i}\,dz = 2\pi$$

Folgerungen aus der Cauchyschen Integralformel.

Korollar 1: (Mittelwerteigenschaft)

Ist $f(z)$ holomorph auf dem Gebiet G, so gilt für $z_0 \in G$, $\overline{K_r(z_0)} \subset G$ die Mittelwertformel

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

Korollar 2: (Maximumprinzip)

1) Ist $f(z)$ holomorph auf G und besitzt $|f(z)|$ sein Maximum in $z_0 \in G$, dann ist $f(z)$ eine konstante Funktion.

2) Ist $f(z)$ stetig auf $\overline{G}$ und holomorph auf G, so nimmt $|f(z)|$ sein Maximum stets auf dem Rand ∂G an.

Korollar 3: (Fundamentalsatz der Algebra)

Ist $p(z) = \sum\limits_{k=0}^{n} a_k z^k$ ein Polynom vom Grad $n \geq 1$ und $a_n \neq 0$, so besitzt $p(z)$ wenigstens eine Nullstelle in $\mathbb{C}$.

Beweis des Fundamentalsatzes der Algebra.

Wir nehmen an, dass das Polynom keine Nullstelle besitzt. Dann ist die Funktion $f(z) := 1/p(z)$ holomorph auf ganz $\mathbb{C}$. Nun gilt

$$\begin{aligned} |f(z)| &= \left| \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0} \right| \\ &= \frac{1}{|z|^n} \cdot \left| \frac{1}{a_n + a_{n-1}\frac{1}{z} + \cdots + a_0 \frac{1}{z^n}} \right| \end{aligned}$$

Im Grenzfall $z \to \infty$ erhalten wir also $|f(z)| \to 0$.

Daher muss $|f(z)|$ in einem Punkt $z_0 \in \mathbb{C}$ das Maximum annehmen und nach dem Maximumprinzip folgt $f(z) = \mathrm{const.}$

Demnach ist auch $p(z) = \mathrm{const.} =: \alpha$, aber dann gilt

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = \alpha$$

und wir erhalten durch Koeffizientenvergleich $a_n = 0$, also einen Widerspruch.

Taylor–Entwicklung komplexer Funktionen.

Satz: Ist $f(z)$ auf einem Gebiet $G \subset \mathbb{C}$ holomorph und $z_0 \in G$, so ist $f(z)$ in jedem Kreis $\overline{K_r(z_0)} \subset G$ in eine Potenzreihe entwickelbar,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{mit } |z - z_0| < r.$$

Den Punkt z_0 nennt man den Entwicklungspunkt.

Insbesondere ist $f(z)$ auf G beliebig oft differenzierbar mit

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$$

Die Koeffizienten a_k der Potenzreihe sind gegeben durch

$$a_k = \frac{1}{k!} f^{(k)}(z_0)$$

Für den Konvergenzradius R der Taylor–Reihe gilt

$$R \geq \sup\{r > 0 \, : \, K_r(z_0) \subset G\}$$

Verallgemeinerte Cauchysche Integralformel.

Satz: Analog zur Cauchyschen Integralformel

$$f(z_0) = \frac{1}{2\pi i} \oint\limits_{|z-z_0|=r} \frac{f(z)}{z - z_0} \, dz$$

gilt für die Ableitungen von $f(z)$ die Formel

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint\limits_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

Beweis: Nach der Cauchyschen Integralformel gilt

$$f(z_0) = \frac{1}{2\pi i} \oint\limits_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta$$

wobei der Kreis $|\zeta - z_0| = r$ einmal im positiven Sinn durchlaufen wird.

Fortsetzung des Beweises.

Liegt nun z innerhalb dieses Kreises, d.h. $|z - z_0| < r$, so folgt ebenfalls

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Wir schreiben nun

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \cdot \frac{\zeta - z_0}{(\zeta - z_0) - (z - z_0)} \\ &= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} \end{aligned}$$

Der letzte Ausdruck auf der rechten Seite ist der Grenzwert der geometrischen Reihe

$$\frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} = \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k$$

Fortsetzung des Beweises.

Wir erhalten damit

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k$$

Für den Integranden in obenstehendem Integral folgt

$$\frac{f(\zeta)}{\zeta - z} = \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \cdot (z - z_0)^k$$

und damit kann die Integralformel in der Form

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \cdot (z - z_0)^k \, d\zeta$$

geschrieben werden.
Da die Potenzreihe gleichmäßig konvergiert, kann man Summation und Integration vertauschen.

Komplettierung des Beweises.

Wir erhalten damit

$$f(z) = \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \right] (z - z_0)^k$$

und ein Koeffizientenvergleich in der Potenzreihe ergibt

$$\frac{1}{k!} f^{(k)}(z_0) = a_k = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta$$

und damit die verallgemeinerte Cauchysche Integralformel.

Die Cauchysche Ungleichung.

Satz: Sei $f(z)$ holomorph auf G, $z_0 \in G$, $\overline{K_r(z_0)} \subset G$. Für die Koeffizienten der Taylor–Entwicklung von $f(z)$ um z_0 gilt dann die Abschätzung

$$\left| \frac{1}{n!} f^{(n)}(z_0) \right| \leq \frac{M(r)}{r^n}$$

mit

$$M(r) = \max_{|z - z_0| = r} |f(z)|$$

Beweis: Aus der verallgemeinerten Cauchyschen Integralformel

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{|z - z_0| = r} \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

folgt direkt

$$\begin{aligned} \left| \frac{1}{n!} f^{(n)}(z_0) \right| &\leq \frac{1}{2\pi} \cdot \max_{|z - z_0| = r} \left(\frac{|f(z)|}{|z - z_0|^{n+1}} \right) \cdot 2\pi r \\ &= \frac{1}{r^n} \cdot M(r) \end{aligned}$$

Der Satz von Liouville.

Satz: Ist $f(z)$ holomorph und beschränkt auf $\mathbb{C}$, so ist $f(z)$ konstant.

Beweis: Aus der Cauchyschen Ungleichung folgt im Grenzwert $r \to \infty$

$$f^{(n)}(z) = 0 \qquad \text{für alle } z \in \mathbb{C},\ n \geq 1$$

Damit ist auch $f'(z) = 0$ für alle $z \in \mathbb{C}$ und $f(z) = \mathrm{const.}$

Kapitel 5. Komplexe Integration

5.4 Singularitäten und Residuen

Satz: (Laurent–Entwicklung)
Sei $f(z)$ auf einem Gebiet $G \subset \mathbb{C}$ holomorph, $z_0 \in \mathbb{C}$ und $0 \leq R_1 < R_2$ mit

$$K_{R_1,R_2}(z_0) := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\} \subset G$$

Dann ist $f(z)$ auf $K_{R_1,R_2}(z_0)$ in eine Laurent–Reihe mit Entwicklungspunkt z_0 entwickelbar,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad \text{für } R_1 < |z - z_0| < R_2.$$

Für die Koeffizienten gilt mit $R_1 < \rho < R_2$

$$a_k = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta \qquad \text{für } k \in \mathbb{Z}.$$

Die Laurent–Entwicklung.

Weiterhin gilt: Die Reihe konvergiert innerhalb des größten Kreisringes $K_{r,R}(z_0)$, der noch innerhalb von G liegt, in jedem kleineren kompakten Kreisring $\overline{K_{\rho_1,\rho_2}(z_0)}$ ist die Konvergenz absolut und gleichmäßig.

Beweis des Satzes: Gegeben sei ein Kreisring $K_{r,R}(z_0) \subset G$ mit $R_1 < r < R < R_2$ und den beiden Rändern

$$
\begin{aligned}
c_r(t) &:= z_0 + re^{it}, \quad 0 \le t \le 2\pi \\
c_R(t) &:= z_0 + Re^{it}, \quad 0 \le t \le 2\pi
\end{aligned}
$$

Behauptung: Nach dem Cauchyschen Integralsatzes gilt für einen Punkt $z \in K_{r,R}(z_0)$ die Beziehung

$$
f(z) = \frac{1}{2\pi i} \oint_{c_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \oint_{c_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta
$$

Fortsetzung des Beweises.

Seien dazu die beiden Kurven c_1 und c_2 definiert wie an der Tafel angegeben. Dann gilt

$$
\begin{aligned}
\oint_{c_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \oint_{c_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \oint_{c_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \oint_{c_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\
&= 2\pi i \cdot f(z) + 0
\end{aligned}
$$

Wir versuchen nun, für die beiden Kurvenintegrale entlang c_R und c_r eine Reihendarstellung herzuleiten.

Sei zunächst ζ ein Punkt auf c_R, also $|\zeta - z_0| > |z - z_0|$. Dann gilt

$$
\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \cdot \frac{\zeta - z_0}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} = \frac{1}{\zeta - z_0} \cdot \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k
\end{aligned}
$$

Fortsetzung des Beweises.

Setzt man diese Formel in das Kurvenintegral ein, so folgt direkt

$$\frac{1}{2\pi i} \oint_{c_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{c_R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta \right] (z - z_0)^k$$

Sei nun ζ ein Punkt auf c_r, d.h. $|\zeta - z_0| < |z - z_0|$. Dann gilt

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{z - z_0} \cdot \frac{z - z_0}{(\zeta - z_0) - (z - z_0)} \\
&= -\frac{1}{z - z_0} \cdot \frac{1}{1 - \left(\frac{\zeta - z_0}{z - z_0}\right)} = -\frac{1}{z - z_0} \cdot \sum_{k=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^k \\
&= -\sum_{m=-\infty}^{-1} \frac{(z - z_0)^m}{(\zeta - z_0)^{m+1}} \qquad \text{mit } m = -(k+1)
\end{aligned}$$

Fortsetzung des Beweises.

Einsetzen in das Kurvenintegral über c_r ergibt demnach

$$\frac{1}{2\pi i} \oint_{c_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\sum_{k=-\infty}^{-1} \left[\frac{1}{2\pi i} \oint_{c_r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta \right] (z - z_0)^k$$

Addiert man nun beide Reihendarstellungen, so folgt

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \qquad r \leq |z - z_0| \leq R$$

wobei die Koeffizienten durch

$$a_k = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\oint_{c_R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta & : \quad k = 0, 1, 2, \ldots \\ \dfrac{1}{2\pi i} \displaystyle\oint_{c_r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta & : \quad k = -1, -2, -3, \ldots \end{cases}$$

gegeben sind.

Komplettierung des Beweises.

Nach dem Cauchyschen Integralsatz gilt dann auch

$$a_k = \frac{1}{2\pi i} \oint\limits_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \qquad (k \in \mathbb{Z})$$

für ein beliebiges $\rho \in [r, R]$.

Bemerkung:

- Die Laurent–Entwicklung von $f(z)$ ist bei vorgegebenem Kreisring eindeutig bestimmt.
- Ist $f(z)$ holomorph im gesamten Kreis $\overline{K_{R_2}(z_0)}$, so gilt aufgrund des Cauchyschen Integralsatzes für $k = -1, -2, -3, \ldots$ die Beziehung

$$a_k = 0$$

  und die Laurent–Entwicklung stimmt dann mit der Taylor–Entwicklung überein.

Beispiel.

Wir betrachten die Funktion

$$f(z) = \frac{\sin z}{z^2}$$

mit Entwicklungspunkt $z_0 = 0$ und Kreisring $0 < |z| < \infty$.

Mit der Taylor–Entwicklung

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

erhalten wir die Laurent–Reihe

$$f(z) = \frac{\sin z}{z^2} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-1}}{(2k+1)!} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \ldots$$

Beispiel.

Wir betrachten die Funktion

$$f(z) = \frac{1}{(z+1)(z-2)}$$

mit Entwicklungspunkt $z_0 = 0$. Der Nenner hat zwei Nullstellen in $z = -1$ und $z = 2$. Es existieren daher drei Laurent–Entwicklungen, nämlich in $|z| < 1$, in $1 < |z| < 2$, und in $|z| > 2$.

Für den Kreisring $1 < |z| < 2$ gilt etwa:

$$\begin{aligned} f(z) &= \frac{1}{3} \cdot \frac{1}{z-2} - \frac{1}{3} \cdot \frac{1}{z+1} = -\frac{1}{6} \cdot \frac{1}{1-z/2} - \frac{1}{3z} \cdot \frac{1}{1+1/z} \\ &= \left(-\frac{1}{6}\right) \cdot \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k - \frac{1}{3z} \cdot \sum_{k=0}^{\infty} \left(-\frac{1}{z}\right)^k \\ &= \sum_{k=-\infty}^{-1} \frac{(-1)^k}{3} \cdot z^k + \sum_{k=0}^{\infty} \left(\frac{-1}{3 \cdot 2^{k+1}}\right) \cdot z^k \end{aligned}$$

Isolierte Singularitäten.

Definition: Sei $f(y)$ holomorph auf einem Gebiet G. Ein Punkt $z_0 \in \mathbb{C}$ heißt isolierte Singularität von $f(z)$, falls ein $r > 0$ existiert mit $K_{0,r}(z_0) \subset G$.
Ist $f(z) = \sum a_k(z-z_0)^k$ die Laurent–Entwicklung in $K_{0,r}(z_0)$, so nennt man den Punkt z_0

1) eine hebbare Singularität, falls $a_k = 0$ für alle $k < 0$ gilt,

2) einen Pol der Ordnung $m \in \mathbb{N}$, falls gilt

$$a_{-m} \neq 0 \quad \wedge \quad \forall\, k < -m \; : \; a_k = 0$$

3) eine wesentliche Singularität, falls $a_k \neq 0$ für unendlich viele $k < 0$ gilt.

Einige Beispiele.

1. Der Punkt $z_0 = 0$ ist eine hebbare Singularität der Funktion
$$f(z) = \frac{\sin z}{z},$$
denn die Taylor–Entwicklung lautet
$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$
2. Die Funktion
$$f(z) = \frac{\sin z}{z^2}$$
hat in $z_0 = 0$ einen Pol der Ordnung 1.
3. Die Funktion $f(z) = e^{1/z}$ hat in $z_0 = 0$ eine wesentliche Singularität, denn es gilt
$$e^{1/z} = 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \ldots$$

Ein weiteres Beispiel: Rationale Funktionen.

Rationale Funktionen haben keine wesentlichen Singularitäten: sei
$$f(z) = \frac{p(z)}{q(z)}$$
eine rationale Funktion. Die Singularitäten sind die Nullstellen von $q(x)$. Ist nun z_0 eine m–fache Nullstelle von $q(z)$, so gilt
$$q(z) = (z - z_0)^m r(z),\ r(z_0) \neq 0 \quad \to \quad f(z) = \frac{1}{(z - z_0)^m} \cdot g(z),$$
wobei g holomorph in z_0 ist. Daraus folgt
$$f(z) = \frac{1}{(z - z_0)^m} \sum_{k=0}^{\infty} a_k (z - z_0)^k$$
d.h. z_0 ist ein Pol der Ordnung $\leq m$ oder eine hebbare Singularität, falls $a_0 = a_1 = \ldots a_{m-1} = 0$.

Klassifikation von Singularitäten.

Satz:

a) Ist z_0 eine hebbare Singularität, so existiert der Grenzwert $\lim_{z\to z_0} f(z)$. Die Funktion

$$\tilde{f}(z) = \begin{cases} f(z) & : \quad z \neq z_0 \\ \lim\limits_{z\to z_0} f(z) & : \quad z = z_0 \end{cases}$$

ist eine holomorphe Fortsetzung von $f(z)$.

b) Ist $f(z)$ in einer Umgebung von z_0 beschränkt, so ist z_0 eine hebbare Singularität.

c) Ist z_0 ein Pol von $f(z)$, so gilt

$$\lim_{z\to z_0} f(z) = \infty$$

d) Ist z_0 eine wesentliche Singularität von $f(z)$, so bildet $f(z)$ jeden Kreisring $K_{0,\varepsilon}(z_0)$ auf $\mathbb{C}$ oder $\mathbb{C} \setminus \{a\}$ ab.

Kapitel 5. Komplexe Integration

5.5 Komplexe Partialbruchzerlegung, Residuensatz

Definition: Besitzt die Funktion $f(z)$ bei z_0 die Laurent–Entwicklung

$$f(z) = \sum_{-\infty}^{\infty} a_k(z-z_0)^k \quad \text{mit } 0 < |z-z_0| < r,$$

so nennt man

$$h(z; z_0) := \sum_{-\infty}^{-1} a_k(z-z_0)^k$$

den Hauptteil von $f(z)$ zum Entwicklungspunkt z_0.

Satz: Ist $r(z) = p(z)/q(z)$ eine rationale Funktion, wobei der Grad des Zählers echt kleiner als der Grad des Nenners ist, und sind $z_1, \ldots, z_m$ die (verschiedenen) Nullstellen von $q(z)$, so gilt

$$r(z) = h(z; z_1) + \cdots + h(z; z_m)$$

Beweis des letzten Satzes.

Idee: Wir zeigen, dass die Funktion $g(z)$ definiert durch

$$g(z) := r(z) - \sum_{j=1}^{m} h(z; z_j)$$

beschränkt und auf ganz $\mathbb{C}$ holomorph ist. Nach dem Satz von Liouville folgt dann, dass $g(z)$ konstant ist. Mit $\lim_{z\to\infty} g(z) = 0$, folgt dann die Behauptung.

Offensichtlich ist $g(z)$ holomorph auf dem Definitionsbereich $\mathbb{C} \setminus \{z_1, \ldots, z_m\}$ und die Hauptteile der Laurent–Entwicklungen zu den Entwicklungspunkten $z_1, \ldots, z_m$ verschwinden identisch.

Demnach sind die Punkte $z_1, \ldots, z_m$ hebbare Singularitäten und $g(z)$ ist holomorph auf **ganz** $\mathbb{C}$. Wegen $\operatorname{grad} p < \operatorname{grad} q$ folgt

$$\lim_{z\to\infty} r(z) = 0$$

und damit auch

$$\lim_{z\to\infty} g(z) = 0$$

Fortsetzung des Beweises.

Also ist $g(z)$ beschränkt und holomorph auf ganz $\mathbb{C}$. Nach dem Satz von Liouville folgt

$$g(z) = \text{const}$$

und aufgrund des Grenzverhaltens für $z \to \infty$ folgt

$$g(z) = 0$$

Anwendung des Satzes: Die Partialbruchzerlegung einer komplexen rationalen Funktion kann über die Hauptteile der Laurent–Entwicklungen berechnet werden, wobei die Entwicklungspunkte gerade die Singularitäten der rationalen Funktion sind.

Ein Beispiel zur Partialbruchzerlegung.

Man bestimme die Partialbruchzerlegung der Funktion

$$f(z) = \frac{4}{(z+1)^2(z-1)}$$

Der Nenner besitzt die beiden Nullstellen $z = -1$ und $z = 1$. Wir bestimmen daher die Hauptteile der Laurent–Entwicklungen um gerade diese beiden Punkte.

1) Für $z = -1$ schreibt man

$$f(z) = \frac{1}{(z+1)^2} \cdot \underbrace{\frac{4}{z-1}}_{g(z)}$$

Nun ist $g(z)$ in einer Umgebung des Punktes $z = -1$ holomorph und kann in eine Taylor–Reihe entwickelt werden. Es gilt daher

$$f(z) = \frac{1}{(z+1)^2} \cdot \left(-2 - (z+1) + O((z+1)^2)\right)$$

Fortsetzung des Beispiels.

und wir erhalten damit

$$f(z) = \underbrace{-\frac{2}{(z+1)^2} - \frac{1}{z+1}}_{h(z;-1)} + \dots$$

2) Analog schreiben wir für den Punkt $z = 1$

$$f(z) = \frac{1}{z-1} \cdot \underbrace{\frac{4}{(z+1)^2}}_{g(z)}$$

und erhalten durch Taylor–Entwicklung

$$f(z) = \frac{1}{z-1} \cdot \left(1 - (z-1) + O((z-1)^2)\right) = \underbrace{\frac{1}{z-1}}_{h(z;-1)} - 1 + \dots$$

Demnach ist die komplexe Partialbruchzerlegung von $f(z)$ gegeben durch

$$f(z) = -\frac{2}{(z+1)^2} - \frac{1}{z+1} + \frac{1}{z-1}$$

Der Residuensatz.

Definition: Ist z_0 eine isolierte Singularität von $f(z)$, so besitzt $f(z)$ eine Laurent–Entwicklung zum Punkt z_0, d.h.

$$f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k \quad \text{mit } 0 < |z - z_0| < r$$

Man nennt dann

$$\text{Res}\,(f; z_0) := a_{-1}$$

das Residuum von $f(z)$ in z_0.

Satz: Sei G ein Gebiet, $f : G \setminus \{z_1, \ldots, z_m\} \to \mathbb{C}$ sei holomorph, c eine geschlossene, stückweise $\mathcal{C}^1$–Kurve in $G \setminus \{z_1, \ldots, z_m\}$, die in G nullhomotop ist, d.h. innerhalb von c liegen höchstens die isolierten Singularitäten $z_1, \ldots, z_m$. Dann gilt

$$\oint_c f(z)\, dz = 2\pi i \sum_{k=1}^{m} \text{Uml}\,(c; z_k) \cdot \text{Res}\,(f; z_k)$$

Beweisskizze zum Residuensatz.

1) Zunächst genügt es, nur die Singularitäten zu betrachten, die innerhalb von c liegen, da sonst die Umlaufzahl Null ist.

2) Man zerlegt c in geschlossene Kurven $c_1, \ldots, c_s$, sodass jede dieser Kurven c_j nur Singularitäten mit gleicher Umlaufzahl l_j enthält.

3) Jede Kurve c_j ist innerhalb von $G \setminus \{z_1, \ldots, z_m\}$ homotop zu einer l_j–fach durchlaufenen einfach geschlossenen Kurve $\tilde{c}_j$. Daraus folgt

$$\oint_c f(z)\, dz = \sum_{j=1}^{s} \oint_{c_j} f(z)\, dz = \sum_{j=1}^{s} l_j \cdot \oint_{\tilde{c}_j} f(z)\, dz$$

4) Jeder einfach geschlossene Weg $\tilde{c}_j$ kann in eine Summe von Kreisen um die Singularitäten innerhalb von $\tilde{c}_j$ zerlegt werden. Daraus folgt

$$\oint_c f(z)\, dz = \sum_{k=1}^{m} \text{Uml}\,(c; z_k) \oint_{|z-z_k|=\rho_k} f(z)\, dz$$

Fortsetzung der Beweisskizze zum Residuensatz.

Mit der Laurent–Entwicklung um z_k gilt aber

$$\begin{aligned}\oint_{|z-z_k|=\rho_k} f(z)\,dz &= \oint_{|z-z_k|=\rho_k} \sum_{j=-\infty}^{\infty} a_j(z-z_k)^j\,dz \\ &= \sum_{j=-\infty}^{\infty} a_j \cdot \oint_{|z-z_k|=\rho_k} (z-z_k)^j\,dz \\ &= 2\pi i \cdot a_{-1} = 2\pi i \cdot \text{Res}\,(f; z_k)\end{aligned}$$

Für die Kurve an der Tafel erhält man also

$$\begin{aligned}\oint_c f(z)\,dz &= 2\pi i \cdot [2\text{Res}\,(f; z_1) + \text{Res}\,(f; z_2) \\ &\quad +\text{Res}\,(f; z_3) + 2\text{Res}\,(f; z_4) + 2\text{Res}\,(f; z_5)]\end{aligned}$$

Methoden zur Berechnung von Residuen.

Satz:

a) Ist z_0 ein einfacher Pol von $f(z)$, so besitzt $f(z)$ eine Darstellung der Form

$$f(z) = \frac{g(z)}{z - z_0}$$

mit einer in z_0 holomorphen Funktion $g(z)$. Für das Residuum gilt dann

$$\text{Res}\,(f; z_0) = g(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

b) Ist $f(z) = p(z)/q(z)$ mit auch in z_0 holomorphen Funktionen p und q eine rationale Funktion und z_0 eine einfache Nullstelle von $q(z)$, so gilt

$$\text{Res}\,(f; z_0) = \frac{p(z_0)}{q'(z_0)}$$

c) Gilt $f(z) = g(z)/(z - z_0)^m$, $m \geq 1$ mit einer in einer Umgebung von z_0 holomorphen Funktion $g(z)$, so gilt

$$\text{Res}\,(f; z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

Beweis des letzten Satzes.

Wie man leicht sieht, ist Teil a) ein Spezialfall von Teil c).

Weiterhin kann Teil c) über eine Taylor–Entwicklung bewiesen werden, da $g(z)$ in einer Umgebung von z_0 holomorph ist.

Wir schreiben zunächst

$$f(z) = \frac{g(z)}{(z-z_0)^m} = \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!}(z-z_0)^{k-m}$$

Daraus kann man direkt das Residuum ablesen und es gilt

$$a_{-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!} = \operatorname{Res}(f; z_0)$$

Für Teil b) definieren wir

$$q(z) =: (z-z_0)r(z)$$

Dann ist $r(z)$ im Punkt z_0 holomorph fortsetzbar mit $r(z_0) \neq 0$.

Fortsetzung des Beweises.

Für Teil b) definieren wir

$$q(z) =: (z-z_0)r(z)$$

Dann ist $r(z)$ im Punkt z_0 holomorph fortsetzbar mit $r(z_0) \neq 0$.
Damit ist die Funktion

$$g(z) = \frac{p(z)}{r(z)}$$

bei $z = z_0$ holomorph und wir erhalten für $f(z)$ die Darstellung

$$f(z) = \frac{g(z)}{z-z_0}$$

Nach Teil a) folgt wegen

$$q'(z) = r(z) + (z-z_0)r'(z)$$

gerade

$$\operatorname{Res}(f; z_0) = g(z_0) = \frac{p(z_0)}{r(z_0)} = \frac{p(z_0)}{q'(z_0)}$$

Drei Beispiele zur Berechnung von Residuen.

Beispiel 1: Für die Funktion

$$f(z) = \frac{1}{(z+1)(z-2)}$$

hat man nach Teil a) des letzten Satzes

$$\begin{aligned} \operatorname{Res}(f;-1) &= \left.\frac{1}{z-2}\right|_{z=-1} = -\frac{1}{3} \\ \operatorname{Res}(f;2) &= \left.\frac{1}{z+1}\right|_{z=2} = \frac{1}{3} \end{aligned}$$

Beispiel 2: Für

$$f(z) = \frac{1}{1+z^2}$$

gilt nach b)

$$\operatorname{Res}(f;i) = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i} \qquad \text{und} \qquad \operatorname{Res}(f;-i) = \left.\frac{1}{2z}\right|_{z=-i} = -\frac{1}{2i}$$

Drei Beispiele zur Berechnung von Residuen.

Beispiel 3: Die Funktion

$$f(z) = \frac{e^{iz}}{z(z^2+1)^2}$$

hat bei $z_0 = i$ einen Pol zweiter Ordnung. Nach dem letzten Satz, Teil c), gilt

$$\operatorname{Res}(f;i) = g'(i) = -\frac{3}{4e}$$

wobei die Funktion $g(z)$ aufgrund der Darstellung

$$f(z) = \frac{e^{iz}}{z(z+i)^2(z-i)^2}$$

durch

$$g(z) = \frac{e^{iz}}{z(z+i)^2}$$

gegeben ist.

Kapitel 5. Komplexe Integration

5.6 Berechnung reeller Integrale mittels Residuensatz

Satz: Sei $r(x) = p(x)/q(x)$ eine rationale Funktion, die keine Singularitäten auf $\mathbb{R}$ besitzt, und es gelte

$$\text{grad}\,(q) \geq \text{grad}\,(p) + 2$$

Dann folgt

$$\int_{-\infty}^{\infty} r(x)\,dx = 2\pi i \sum_{\text{Im}\,z>0} \text{Res}\,(r;z)$$

Beweis: Wegen $\text{grad}\,(q) \geq \text{grad}\,(p) + 2$ existiert nach dem Majorantenkriterium das oben stehende uneigentliche Integrale, denn für $x \gg 1$ gilt

$$\left|\frac{p(x)}{q(x)}\right| \leq \frac{c}{x^2}$$

Wir approximieren jetzt das uneigentliche reelle Integral durch ein komplexes Integral entlang einer Kurve (siehe Tafel).

Fortsetzung des Beweises.

Ist r hinreichend groß, so liegen alle Singularitäten z_k von $r(z)$ mit strikt positivem Imaginärteil innerhalb der Kurve $c_1 + c_2$.

Daraus folgt

$$2\pi i \sum_{\text{Im}\,z_k>0} \text{Res}\,(r;z_k) = \oint_{c_1+c_2} r(z)\,dz = \int_{c_1} r(z)\,dz + \int_{c_2} r(z)\,dz$$

Nun gilt

$$\int_{c_1} r(z)\,dz = \int_{-r}^{r} r(z)\,dz \longrightarrow \int_{-\infty}^{\infty} r(z)\,dz \qquad \text{für } r \to \infty$$

Weiter berechnet man

$$\left|\int_{c_2} r(z)\,dz\right| \leq \max_{|z|=r} |r(z)| \cdot \pi r \leq \pi r \frac{c}{r^2} \longrightarrow 0 \qquad \text{für } r \to \infty$$

Daraus folgt direkt die Behauptung.

Beispiel.

Wir untersuchen das uneigentliche Integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$$

Die Funktion $r(z) = 1/(1+z^6)$ besitzt sechs Polstellen, von denen drei in der oberen Halbebene liegen, nämlich $e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{6}}$. Ferner gilt

$$\operatorname{Res}(r; z_k) = \left.\frac{1}{6z^5}\right|_{z=z_k} = -\frac{z_k}{6}$$

Damit folgt

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^6} &= 2\pi i \left(-\frac{1}{6}e^{i\frac{\pi}{6}} - \frac{1}{6}e^{i\frac{\pi}{2}} - \frac{1}{6}e^{i\frac{5\pi}{6}}\right) \\ &= \frac{\pi}{3}\left(\sin\frac{\pi}{6} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{6}\right) = \frac{\pi}{3}\left(2\sin\frac{\pi}{6} + 1\right)\end{aligned}$$

Beispiel.

Wir untersuchen das Integral

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2+a^2}\,dx \qquad \text{mit } a > 0, \omega > 0$$

Der letzte Satz läßt sich nicht direkt anwenden, aber wegen

$$\left|\frac{e^{i\omega z}}{z^2+a^2}\right| = \frac{e^{-\omega y}}{|z^2+a^2|} \leq \frac{1}{|z^2+a^2|} \leq \frac{c}{|z|^2} \qquad \text{für } y \geq 0$$

entlang des Weges c_2, gilt die Aussage analog.
Wir erhalten also

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2+a^2}\,dx &= 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}\left(\frac{e^{i\omega z}}{z^2+a^2}; z_k\right) = 2\pi i \operatorname{Res}\left(\frac{e^{i\omega z}}{z^2+a^2}; ia\right) \\ &= 2\pi i \left.\frac{e^{i\omega z}}{2z}\right|_{z=ia} = \frac{\pi}{a}e^{-\omega a}\end{aligned}$$

Weitere Anwendungen.

Satz: Sei $f(z)$ holomorph auf $\{z : \operatorname{Im} z > -\varepsilon\}$, $\varepsilon > 0$, mit Ausnahme endlich vieler Singularitäten in der oberen Halbebene $\operatorname{Im} z > 0$.
Gilt

$$\lim_{|z|\to\infty, y\geq 0} f(z) = 0,$$

so folgt

$$\text{CHW} \int_{-\infty}^{\infty} f(x)e^{ix}\, dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}\left(f(z)e^{iz}; z_k\right)$$

Beispiel: Es gilt

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

und

$$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\, dx = 0$$

Weitere Anwendungen.

Satz: Sei $r(z)$ eine rationale Funktion ohne Polstellen in $0 \leq x < \infty$ und es gelte $\operatorname{grad} q > \operatorname{grad} p$. Für $0 < \alpha < 1$ gilt dann

$$\int_0^{\infty} \frac{r(x)}{x^\alpha}\, dx = \frac{2\pi i}{1 - e^{-2\pi\alpha i}} \sum_{z_k \in \mathbb{C}\setminus\{0\}} \operatorname{Res}\left(\frac{r(z)}{z^\alpha}; z_k\right)$$

Dabei ist folgender Zweig von z^α zu wählen

$$z = re^{i\phi} \quad \text{mit } 0 < \phi < 2\pi \quad \Rightarrow \quad z^\alpha = r^\alpha e^{i\alpha\phi}$$

Beispiel: Man berechnet

$$\int_0^{\infty} \frac{1}{x^\alpha(1+x)}\, dx = \frac{\pi}{\sin(\pi\alpha)}$$

Kapitel 6. Die Fourier–Transformation

Wiederholung aus Analysis II: Fourier–Reihenentwicklung

Sei $f : \mathbb{R} \to \mathbb{C}$ eine T–periodische, stückweise stetige Funktion.

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos(k\omega t) + b_k \sin(k\omega t) \Big)$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(\tau) \cos(k\omega\tau)\, d\tau$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(\tau) \sin(k\omega\tau)\, d\tau$$

Reelle Darstellung der Fourier–Entwicklung von f mit $\omega = 2\pi/T$.

Konvergenzsatz zur Fourier–Entwicklung.

Satz: Sei $f : \mathbb{R} \to \mathbb{C}$ T–periodisch, stückweise stetig differenzierbar und betrachte die zugehörige Fourier–Reihe

$$F_f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos(k\omega t) + b_k \sin(k\omega t) \Big)$$

Dann gilt:

a) Die Reihe konvergiert punktweise und für alle $t \in \mathbb{R}$ gilt

$$F_f(t) = \frac{1}{2} \left(f(t^+) + f(t^-) \right)$$

b) In allen kompakten Intervallen $[a, b]$, in denen $f(t)$ stetig ist, ist die Konvergenz gleichmäßig.

Komplexe Darstellung

$$f(t) = \sum_{k=-\infty}^{\infty} \gamma_k e^{ik\omega t} \qquad \text{mit} \quad \gamma_k = \frac{1}{T} \int_{-T/2}^{T/2} f(\tau) e^{-ik\omega\tau}\, d\tau$$

Die Fourier–Transformierte und das Fourier–Integral.

Betrachte jetzt den formalen Grenzwert $T \to \infty$, um eine Fourier–Entwicklung für nicht–periodische Funktionen zu erhalten

$$f(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{i\omega_k t} \left(\int_{-T/2}^{T/2} f(\tau) e^{-i\omega_k \tau} \, d\tau \right) \Delta\omega$$

Riemannsche Summe mit $\Delta\omega := \omega_{k+1} - \omega_k = 2\pi/T$.

Definition: Die Funktion $F(\omega)$ gegeben durch

$$F(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} \, d\tau$$

heißt die Fourier–Transformierte oder Spektralfunktion von $f(t)$. Die Darstellung

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

nennt man das Fourier–Integral oder Spektrale Zerlegung von $f(t)$.

Diskretes und kontinuierliches Sepktrum.

Bemerkung:

- Die Fourierkoeffizienten $(\gamma_k)_{k\in\mathbb{Z}}$ einer periodischen Funktion $f(t)$ bilden das diskrete Spektrum von f.

- Die Fourier–Transformation $(F(\omega))_{\omega\in\mathbb{R}}$ einer nicht–periodischen Funktion liefert das kontinuierliche Spektrum.

Andere Schreibweisen:

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} \, d\tau$$

$$\mathcal{F}^{-1}[F](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

Fourier–Transformation und inverse Fourier–Transformation.

Reelle Darstellung der Fourier–Transformation.

Bemerkung: Durch Zerlegung in Real– und Imaginärteil erhält man die reelle Darstellung der Fourier–Transformation

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(\tau)\Big(\cos(\omega\tau) - i\sin(\omega\tau)\Big)\,d\tau \\ &= a(\omega) - ib(\omega) \end{aligned}$$

mit

$$\begin{aligned} a(\omega) &= \int_{-\infty}^{\infty} f(\tau)\cos(\omega\tau)\,d\tau \qquad \text{(gerade Funktion)} \\ b(\omega) &= \int_{-\infty}^{\infty} f(\tau)\sin(\omega\tau)\,d\tau \qquad \text{(ungerade Funktion)} \end{aligned}$$

Entsprechend gilt folgende reelle Darstellung des Fourier–Integrals.

Reelle Darstellung des Fourier–Integrals.

Dann gilt auch die folgende Darstellung des Fourier–Integrals

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (a(\omega) - ib(\omega))\Big(\cos(\omega t) + i\sin(\omega t)\Big)\,d\omega \\ &= \frac{1}{\pi}\int_{0}^{\infty} \Big(a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\Big)\,d\omega \end{aligned}$$

Zusammenfassung: (Sinus–, Cosinus–Spektrum)

$$\begin{aligned} f(t) &= \frac{1}{\pi}\int_{0}^{\infty} \Big(a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\Big)\,d\omega \\ a(\omega) &= \int_{-\infty}^{\infty} f(\tau)\cos(\omega\tau)\,d\tau \\ b(\omega) &= \int_{-\infty}^{\infty} f(\tau)\sin(\omega\tau)\,d\tau \end{aligned}$$

Beispiel: Fourier–Transformierte des Rechteckimpulses.

Wir betrachten den Rechteckimpuls der Form

$$f(t) = \begin{cases} 1 & : \quad -a \leq t \leq a \\ 0 & : \quad |t| > a \end{cases}$$

Dann berechnet man

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = \int_{-a}^{a} e^{-i\omega t}\, dt \\ &= -\frac{1}{i\omega} e^{-i\omega t}\Big|_{-a}^{a} = -\frac{1}{i\omega}\left(e^{-i\omega a} - e^{i\omega a}\right) \\ &= \begin{cases} \dfrac{2}{\omega}\sin(\omega a) & : \quad \omega \neq 0 \\ 2a & : \quad \omega = 0 \end{cases} \end{aligned}$$

Fortsetzung des Beispiels.

Für die Umkehrung erhalten wir

$$\begin{aligned} \mathcal{F}^{-1}[F](t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(\omega a)}{\omega} e^{i\omega t}\, d\omega \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega a)}{\omega} \cos(\omega t)\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega(a+t))}{\omega}\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega(a-t))}{\omega}\, d\omega \end{aligned}$$

Dies sind sogenannte Dirichlet–Integrale der Form

$$\text{CHW} \int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{x}\, dx$$

und man verwendet zur Berechnung den Residuensatz.

Komplettierung des Beispiels.

Es folgt

$$\text{CHW} \int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{x}\, dx = \begin{cases} \pi & : \quad \alpha > 0 \\ 0 & : \quad \alpha = 0 \\ -\pi & : \quad \alpha < 0 \end{cases}$$

Damit ergibt sich die Umkehrung in der Form

$$\mathcal{F}^{-1}[F](t) = \begin{cases} 1 & : \quad |t| < a \\ \frac{1}{2} & : \quad t = a \\ 0 & : \quad |t| > a \end{cases}$$

Bemerkung:
Man beachte insbesondere die Mittelwerteigenschaft

$$F_f(t) = \frac{1}{2}\left(f(t^+) + f(t^-)\right)$$

Ein weiteres Beispiel.

Gegeben sei mit $a > 0$ die Funktion

$$f(t) = \begin{cases} e^{-at} & : \quad t \geq 0 \\ 0 & : \quad t < 0 \end{cases}$$

Dann gilt

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = \int_0^{\infty} e^{-(a+i\omega)t}\, dt \\ &= -\frac{1}{a+i\omega}\, e^{-(a+i\omega)t}\Big|_0^{\infty} = \frac{1}{a+i\omega} \end{aligned}$$

Umkehrung wieder mit Hilfe des Residuensatzes

$$\mathcal{F}^{-1}[F](t) = \begin{cases} e^{-at} & : \quad t > 0 \\ 0 & : \quad t < 0 \end{cases}$$

und $\mathcal{F}^{-1}[F](0) = 1/2$.

Fortsetzung des Beispiels.

Konkrete Berechnung

$$
\begin{aligned}
\mathcal{F}^{-1}[F](t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a+i\omega} e^{i\omega t}\, d\omega \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - ia}\, d\omega \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ix}}{x - iat}\, d\omega \\
&= \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}\left(\frac{e^{iz}}{z-iat}; z_k\right) \\
&= \begin{cases} e^{-at} & : \quad t > 0 \\ 0 & : \quad t < 0 \end{cases}
\end{aligned}
$$

Ein weiteres Beispiel.

Gegeben sei mit $a > 0$ die Funktion

$$f(t) = e^{-a|t|}$$

Dann gilt

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{(a-i\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+i\omega t)}\, dt \\
&= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}
$$

Existenz und Eindeutigkeit der Fourier–Transformierten.

Satz:

a) Ist $f(t)$ stückweise stetig und absolut integrierbar, d.h. $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$, so existiert die Fourier–Transformierte

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

für alle $\omega \in \mathbb{R}$. Das Integral konvergiert gleichmäßig und $F(\omega)$ ist stetig.

b) Ist $f(t)$ eine stückweise $\mathcal{C}^1$–Funktion und absolut integrierbar, so gilt für alle $t \in \mathbb{R}$

$$\frac{1}{2}\Big(f(t^-) + f(t^+)\Big) = \text{CHW} \int_{-\infty}^{\infty} F(\omega) \frac{e^{i\omega t}}{2\pi}\, d\omega$$

c) Sind f_1, f_2 wie in b) mit $F_1(\omega) = F_2(\omega)$ für alle $\omega \in \mathbb{R}$, so folgt $f_1(t) = f_2(t)$ in allen Punkten t, in denen f_1 und f_2 stetig sind.

Rechenregeln der Fourier–Transformation.

Im Folgenden seien $f, g, \cdots : \mathbb{R} \to \mathbb{C}$ stückweise stetig und absolut integrierbar. Mit $F(\omega), G(\omega), \ldots$ bezeichnen wir die entsprechenden Fourier–Transformierten.

- Linearität

$$\begin{aligned} \mathcal{F}[f+g](\omega) &= \mathcal{F}[f](\omega) + \mathcal{F}[g](\omega) \\ \mathcal{F}[\alpha f](\omega) &= \alpha \mathcal{F}[f](\omega) \qquad (\alpha \in \mathbb{C}) \end{aligned}$$

- Konjugation

$$\mathcal{F}[\bar{f}](\omega) = \overline{F(-\omega)}$$

denn

$$\mathcal{F}[\bar{f}](\omega) = \int_{-\infty}^{\infty} \overline{f(t)} e^{-i\omega t}\, dt = \overline{\int_{-\infty}^{\infty} f(t) e^{-i(-\omega)t}\, dt}$$

- Streckung

$$\mathcal{F}[f(ct)](\omega) = \frac{1}{|c|} F\left(\frac{\omega}{c}\right)$$

Rechenregeln der Fourier–Transformation.

- Streckung

$$\mathcal{F}[f(ct)](\omega) = \frac{1}{|c|} F\left(\frac{\omega}{c}\right)$$

denn

$$\begin{aligned}\int_{-\infty}^{\infty} f(ct)e^{-i\omega t}\,dt &= \operatorname{sgn}(c)\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\frac{\tau}{c}}\,\frac{1}{c}\,d\tau \\ &= \frac{1}{|c|}\int_{-\infty}^{\infty} f(\tau)e^{-i\frac{\omega}{c}\tau}\,d\tau\end{aligned}$$

- Verschiebungssätze

$$\mathcal{F}[f(t-a)](\omega) = e^{-i\omega a}F(\omega)$$

$$\mathcal{F}[e^{iat}f(t)](\omega) = F(\omega - a)$$

Folgt durch direktes Einsetzen

Rechenregeln der Fourier–Transformation.

- Faltungssatz
  Man nennt

$$(f*g)(t) := \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau$$

die Faltung der Funktionen f und g. Es gilt

$$\mathcal{F}[f*g](\omega) = F(\omega)\cdot G(\omega) \quad \text{und} \quad \mathcal{F}[f\cdot g](\omega) = \frac{1}{2\pi}\Big(F*G\Big)(\omega)$$

denn

$$\begin{aligned}\mathcal{F}[f*g] &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau\right)e^{-i\omega t}\,dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t-\tau)e^{-i\omega(t-\tau)}\,dt\right)g(\tau)e^{-i\omega\tau}\,d\tau \\ &= F(\omega)\cdot\int_{-\infty}^{\infty} g(\tau)e^{-i\omega\tau}\,d\tau dt = F(\omega)\cdot G(\omega)\end{aligned}$$

Ein Beispiel für den Faltungssatz.

Für die Faltung $g = f * f$ des Rechteck–Impulses

$$f(t) = \begin{cases} 1 & : \quad -1 \leq t \leq 1 \\ 0 & : \quad |t| > 1 \end{cases}$$

gilt

$$g(t) = \int_{-1}^{1} f(t-\tau)\,d\tau = \begin{cases} 2-|t| & : \quad -2 \leq t \leq 2 \\ 0 & : \quad |t| > 2 \end{cases}$$

Nach dem Faltungssatz folgt mit dem vorherigen Beispiel direkt

$$G(\omega) = F(\omega) \cdot F(\omega) = \begin{cases} \frac{4}{\omega^2} \sin^2 \omega & : \quad \omega \neq 0 \\ 4 & : \quad \omega = 0 \end{cases}$$

Rechenregeln der Fourier–Transformation.

- Differentiation

  Ist $f(t)$ eine stückweise $\mathcal{C}^1$–Funktion mit endlich vielen Unstetigkeitsstellen $\tau_1, \ldots, \tau_m$ und sind $f(t)$, $f'(t)$ absolut integrierbar, so gilt

  $$\mathcal{F}[f'](\omega) = i\omega F(\omega) - \sum_{k=1}^{m} \Big(f(\tau_k^+) - f(\tau_k^-) \Big) e^{-i\omega\tau_k}$$

  Beweis mittels partieller Integration.

  Ist $f(t)$ sogar stetig, so folgt

  $$\mathcal{F}[f'](\omega) = i\omega F(\omega)$$

  und unter entsprechenden Voraussetzungen gilt sogar

  $$\mathcal{F}[f^{(r)}](\omega) = (i\omega)^r F(\omega)$$

  Wesentliche Eigenschaft zum Einsatz der Fourier–Transformation bei Differentialgleichungen.

Beispiel.

Wir betrachten die gewöhnliche Differentialgleichung

$$y''(t) + ay'(t) + by(t) = c(t)$$

mit den Grenzbedingungen

$$\lim_{|t|\to\infty} y(t) = 0, \qquad \lim_{|t|\to\infty} y'(t) = 0$$

Man berechnet nun die Fourier–Transformation der Differentialgleichung

$$\begin{aligned} \mathcal{F}[y](\omega) &= Y(\omega) \\ \mathcal{F}[y'](\omega) &= -i\omega Y(\omega) \\ \mathcal{F}[y''](\omega) &= -\omega^2 Y(\omega) \end{aligned}$$

Die Fourier–Transformation der Differentialgleichung lautet damit

$$(-\omega^2 + i\omega a + b)Y(\omega) = C(\omega)$$

Fortsetzung des Beispiels.

Die Fourier–Transformation der Differentialgleichung lautet damit

$$(-\omega^2 + i\omega a + b)Y(\omega) = C(\omega)$$

und es ergibt sich

$$Y(\omega) = \frac{C(\omega)}{-\omega^2 + i\omega a + b}$$

Nach Rücktransformation ergibt sich

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C(\omega)}{-\omega^2 + i\omega a + b} e^{i\omega t}\, d\omega$$

und damit

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{c(\tau)}{-\omega^2 + i\omega a + b} e^{i\omega(t-\tau)}\, d\tau d\omega$$

Anwendung der Fourier–Transformation bei partiellen Differentialgleichungen.

Wir betrachten das Anfangswertproblem für die Wärmeleitungsgleichung

$$\begin{cases} u_t = u_{xx} & \text{für } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & \text{für } x \in \mathbb{R} \end{cases}$$

Eine Fourier–Transformation bezüglich der x–Variablen liefert

$$U(\omega,t) = \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x}\, dx$$

Damit folgt für die Wärmeleitungsgleichung

$$U_t = (i\omega)^2 U, \qquad \text{für } t > 0,\ \omega \in \mathbb{R}$$

Dies ist eine gewöhnliche Differentialgleichung in t mit Parameter ω.

Anwendung der Fourier–Transformation bei partiellen Differentialgleichungen.

Daraus folgt

$$U(\omega,t) = U(\omega,0)e^{-\omega^2 t} \qquad \text{mit} \qquad U(\omega,0) = U_0(\omega)$$

und damit

$$U(\omega,t) = U_0(\omega)e^{-\omega^2 t}$$

Rücktransformation: Mit der gegebenen Anfangsbedingung haben wir die Beziehung

$$\mathcal{F}^{-1}[U_0] = u_0(x)$$

Weiter gilt

$$\begin{aligned} \mathcal{F}^{-1}[e^{-\omega^2 t}] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 t}\, e^{i\omega x}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 t + i\omega x}\, d\omega \\ &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \end{aligned}$$

Anwendung der Fourier–Transformation bei partiellen Differentialgleichungen.

Aus dem Faltungssatz folgt dann

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[U_0 e^{-\omega^2 t}] = u_0 * \left(\frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \right) \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} \, u_0(\xi) \, d\xi \end{aligned}$$

Weiteres Beispiel: Potentialproblem für die Halbebene

$$\left\{ \begin{array}{rcll} u_{xx} + u_{yy} &=& 0 & \text{für } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) &=& u_0(x) & \text{für } x \in \mathbb{R} \end{array} \right.$$

Fourier–Transformation bezüglich x (bei festem y) liefert die Poissonsche Integralformel für die Halbebene

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-\xi)^2} \, u_0(\xi) \, d\xi$$

Ende der Vorlesung.