Differential equations II for engineering study programs

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Content of the course Differential equations II.

1. Examples for partial differential equations.
2. First–order partial differential equations.
3. Scalar conservation laws.
4. Second–order partial differential equations.
5. Normal forms and well–posed problems.
6. The Laplace equation.
7. The heat/diffusion equation.
8. The wave equation.
9. Fourier methods for partial differential equations.
10. Numerical methods for partial differential equations.

Chapter 1. What are partial differential equations?

1.1 General notations

Definition: An equation (or a system of equations) of the form

$$\mathbf{F}\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial \mathbf{u}}{\partial x_n}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}\right) = 0$$

where $\mathbf{u} : D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$ is an unknown function is called (system of) partial differential equations (PDE) for the m functions $u_1(\mathbf{x}), \ldots, u_m(\mathbf{x})$.

Does one of the partial derivative $\dfrac{\partial^p \mathbf{u}}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}}$ of order p occurs explicitely, we call the system a partial differential equation of order p.

In most of the applications we deal with (systems of) partial differential equations of first– and second–order.

1.1 General notations

Definition:

a) A PDE is called linear, if $F(\mathbf{x}, \mathbf{u}, \ldots)$ is an affine linear function in the variables $\mathbf{u}, \dfrac{\partial \mathbf{u}}{\partial x_1}, \ldots, \dfrac{\partial^p \mathbf{u}}{\partial x_n^p}$ ist.

b) A PDE is called semilinear, if $F(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in the variables $\dfrac{\partial^p \mathbf{u}}{\partial x_1^p}, \dfrac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \ldots, \dfrac{\partial^p \mathbf{u}}{\partial x_n^p}$ and the coefficients only depend upon $\mathbf{x} = (x_1, \ldots, x_n)^T$.

c) A PDE is called quasi–linear, if $F(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in the variables $\dfrac{\partial^p \mathbf{u}}{\partial x_1^p}, \dfrac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \ldots, \dfrac{\partial^p \mathbf{u}}{\partial x_n^p}$. The coefficients may depend upon $\left(\mathbf{x}, \mathbf{u}, \dfrac{\partial \mathbf{u}}{\partial x_1}, \ldots, \dfrac{\partial^{p-1} \mathbf{u}}{\partial x_n^{p-1}}\right)$.

d) The PDE is called nonlinear if it depends nonlinearly upon the highest order derivatives.

Examples.

- A scalar linear first order PDE in two independent variables is given by

$$a_1(x, y)u_x + a_2(x, y)u_y + b(x, y)u = c(x, y)$$

- A scalar quasi–linear first order PDE in two independent variables is given by

$$a_1(x, y, u)u_x + a_2(x, y, u)u_y = g(x, y, u)$$

- A semilinear system of second–order PDEs in n variables is

$$\sum_{i,j=1}^{n} a_{ij}(x_1, \ldots, x_n)\mathbf{u}_{x_i x_j} = b(x_1, \ldots, x_n, \mathbf{u}, \mathbf{u}_{x_1}, \ldots, \mathbf{u}_{x_n})$$

- A nonlinear scalar first order PDE in two independent variables is given by

$$(u_x)^2 + (u_y)^2 = f(x, y, u, u_x \cdot u_y)$$

A remark on the general notation for PDEs.

In applications we typically have space variables $\mathbf{x} = (x_1, \ldots, x_n)^T$ (often $n = 3$) as well as a time variable $t \in \mathbb{R}$.

In this case we consider a general PDE given by

$$\mathbf{F}\left(\mathbf{x}, t, \mathbf{u}(\mathbf{x}, t), \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial \mathbf{u}}{\partial t}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1}\partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial t^p}\right) = 0$$

using $(n+1)$ variables. Differential operators like

$$\nabla, \quad \text{div}, \quad \text{rot} \quad \text{oder} \quad \Delta$$

always refer to n space variables, e.g.,

$$\text{div}\, u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}$$

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

1.2 Motivation: Why partielle differential equations?

The Reynolds transport theorem:

Let us assume that at time $t = 0$ a physical quantity (like charge, mass etc.) occupies a bounded and open set $D_0 \subset \mathbb{R}^n$.

Moreover the function $\Phi(\mathbf{y}, t)$ should describe the change of a point $\mathbf{y} \in D_0$ in time,

$$\Phi : D_0 \times [0, T] \to D_t \subset \mathbb{R}^n,$$

such that

$$D_t := \{\Phi(\mathbf{y}, t) \; : \; \mathbf{y} \in D_0\}$$

The trajectory of $\mathbf{y} \in D_0$ is the mapping $t \to \Phi(\mathbf{y}, t) \in D_T$ and let

$$\frac{\partial}{\partial t}\Phi(\mathbf{y}, t) =: \mathbf{v}(\Phi(\mathbf{y}, t), t)$$

denote the velocity field $\mathbf{v}$ of the given physical quantity.

The Reynolds transport theorem.

Satz: For any differentiable scalar function $f : D_t \times [0, T] \to \mathbb{R}$ we have

$$\frac{d}{dt}\int_{D_t} f(\mathbf{x}, t)\, d\mathbf{x} = \int_{D_t} \left\{\frac{\partial}{\partial t} f + \operatorname{div}(f\mathbf{v})\right\}(\mathbf{x}, t)\, d\mathbf{x}$$

Proof idea:
Let $J(\mathbf{y}, t) = \det(D_{\mathbf{y}}\Phi(\mathbf{y}, t))$ be the Jacobian matrix of $\Phi(\mathbf{y}, t)$ wrt. $\mathbf{y}$.
Using this matrix we transform D_t to D_0:

$$\int_{D_t} f(\mathbf{x}, t)\, d\mathbf{x} = \int_{D_0} f(\Phi(\mathbf{y}, t), t) J(\mathbf{y}, t)\, d\mathbf{y}$$

Now compute the time derivative of the rhs

$$\frac{d}{dt}\int_{D_0} f(\Phi(\mathbf{y}, t), t) J(\mathbf{y}, t)\, d\mathbf{y}$$

and transform back to the time–dependent domain D_t.

The continuity equation.

Let $u(\mathbf{x}, t)$ be the mass density of a physical quantity and assume that it applies a Erhaltungsprinzip in the form

$$\frac{d}{dt} \int_{D_t} u(\mathbf{x}, t)\, d\mathbf{x} = 0$$

Then by Reynolds transport theorem we get

$$\int_{D_t} \left\{ \frac{\partial}{\partial t} u + \operatorname{div}(u\mathbf{v}) \right\} (\mathbf{x}, t)\, d\mathbf{x} = 0$$

Because D_t can be any subset of $\mathbb{R}^n$, we obtain the differential equation

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) + \operatorname{div}(u\mathbf{v})(\mathbf{x}, t) = 0$$

This equation is called continuity equation.

Continuity equation and corresponding flux function.

If we rewrite the continuity equation using the flux function $q(\mathbf{x}, t)$

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) + \operatorname{div}(\mathbf{q}(\mathbf{x}, t)) = 0,$$

we have one equation for two unknown functions $u(\mathbf{x}, t)$ und $\mathbf{q}(\mathbf{x}, t)$.

Mathematical modelling:

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}(u(\mathbf{x}, t), \nabla u(\mathbf{x}, t), \dots)$$

Simplest modelling Ansatz: The flux $\mathbf{q}$ is proportional to the density u

$$\mathbf{q}(x, t) = \mathbf{a} \cdot u(x, t) \quad \text{for some } \mathbf{a} \in \mathbb{R}^n$$

It follows the so–called linear transport equation or even linear advection equation

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) + \mathbf{a} \cdot \nabla u(x, t) = 0$$

Example: The heat/diffusion equation.

The density $u(x, t)$ describes

1. the concentration of a chemical substance,
2. the temperature of a solid body or
3. a electrostatic potential.

Physical modelling: the flux $\mathbf{q}$ is proportional to the gradient of the density u, but pointing in the opposite direction,

$$\mathbf{q}(x, t) := -a\nabla u(x, t) \quad \text{für ein } a > 0$$

Hence it follows

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) + \operatorname{div}\left(-a\nabla u(x, t)\right) = 0$$

and we get the PDE

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) = a\Delta u(x, t)$$

Continuation of the example.

If we use $a = 1$, we get the classical heat equation or even the linear diffusion equation

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) = \Delta u(x, t)$$

The closure relation

$$\mathbf{q}(x, t) = -a\nabla u(x, t) \quad \text{mit einem } a > 0$$

is noted either as

1. Fick's law of diffusion,
2. Fourier's law of heat conduction or
3. Ohm's law of electric charge.

Note that we have three different physical problems that are descibed using the same partial differential equation.

Example: The Laplace and Poisson's equation.

If the solution of the heat equation does not depend on the time variable t, i.e.

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) = 0$$

we obtain the Laplace equation

$$\Delta u(x) = 0$$

Solutions of the Laplace equation are called harmonic functions.

The equation

$$-\Delta u(x) = f$$

with given function f is called Poisson's equation.

Here the inhomogeneous part describes, e.g., the spatial charge distribution f and the solution u is the thereby generated potential.

Chapter 2: First–order partial differential equations

2.1 The method of characteristics

We first consider a scalar quasi–linear first–order PDE given by

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u) \qquad \text{mit } \mathbf{x} \in \mathbb{R}^n.$$

A solution can be computed using the method of characteristic, which we demonstrate first for the homogeneous and linear case.

Definition: The autonomous system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t))$$

is called characteristic system of differential equations for a homogeneous linear PDE

$$\sum_{i=1}^{n} a_i(\mathbf{x}) u_{x_i} = 0 \qquad \text{mit } \mathbf{x} \in \mathbb{R}^n.$$

2.1 The method of characteristics

We now compute

$$\frac{d}{dt}u(\mathbf{x}(t)) = \sum_{i=1}^{n} a_i(\mathbf{x}(t))u_{x_i}(x(t)) = 0$$

Conclusion:
The function $u(\mathbf{x})$ is a solution of a homogeneous linear PDE iff u is constant along any solution $\mathbf{x}(t)$ of the characteristic system of differential equations,

$$u(\mathbf{x}(t)) = \text{const.}$$

Definition: For the situation above we call the solution $u(\mathbf{x})$ a first integral of the characteristic system of differential equations.

The method of characteristics is therefore nothing else than to reduce a given PDE to a system of ordinary differential equations.

Example.

We consider the following PDE depending upon three independent variables

$$xu_x + yu_y + (x^2 + y^2)u_z = 0$$

The characteristic system of differential equations is given by

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \\ \dot{z} &= x^2 + y^2 \end{aligned}$$

and has the general solution

$$\begin{aligned} x(t) &= c_1 e^t \\ y(t) &= c_2 e^t \\ z(t) &= \frac{1}{2}\left(c_1^2 + c_2^2\right)e^{2t} + c_3 \end{aligned}$$

We even call these solutions the characteristic curves.

Continuation of the example.

For the solution or the inital equation we therefore have

$$u(x(t), y(t), z(t)) = u\left(c_1 e^t, c_2 e^t, \frac{1}{2}\left(c_1^2 + c_2^2\right) e^{2t} + c_3\right) = \text{const.}$$

But the characteristic curves fulfill the relations

$$e^t = x(t)/c_1 = y(t)/c_2 \quad \Rightarrow \quad y(t)/x(t) = c_2/c_1 = c \in \mathbb{R}$$

and

$$z(t) = \frac{1}{2}(x^2 + y^2) + c_3 \quad \Rightarrow \quad z(t) - \frac{1}{2}(x(t)^2 + y(t)^2) = d \in \mathbb{R}$$

i.e. both constants c and d alone define the value of u along the characteristic curves. Hence we have the solution representation

$$u(x, y, z) = \Phi\left(\frac{y}{x}, z - \frac{1}{2}(x^2 + y^2)\right)$$

with an arbitrary $\mathcal{C}^1$–function $\Phi : \mathbb{R}^2 \to \mathbb{R}$.

Quasi-linear inhomogeneous equations.

The method of characteristics can be extended to equations of the form

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u), \qquad \mathbf{x} \in \mathbb{R}^n$$

One considers the extended problem

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) U_{x_i} + b(\mathbf{x}, u) U_u = 0, \qquad \mathbf{x} \in \mathbb{R}^n$$

with the unkown function $U = U(\mathbf{x}, u)$ depending upon the $(n+1)$ independent variables $\mathbf{x}$ and u.

One has: if $U(\mathbf{x}, u)$ is a solution with $U_u \neq 0$, then $U(\mathbf{x}, u) = 0$ implicitely defines a solution $u = u(\mathbf{x})$ of the initial problem.

Proof of the last statement.

If $U_u \neq 0$, we can use the implicit function theorem to get a locally defined function $u(\mathbf{x})$ and differentiating $U(\mathbf{x}, u(\mathbf{x})) = 0$ wrt. x_i we obtain

$$U_{x_i} + U_u u_{x_i} = 0$$

Moreover

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) U_{x_i} + b(\mathbf{x}, u) U_u = 0$$

and therefore

$$-\left(\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} \right) U_u + b(\mathbf{x}, u) U_u = 0$$

Hence with $U_u \neq 0$ we obtain the differential equation

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u)$$

Example.

We are looking for the general solution of the quasi–linear equation

$$(1 + x)u_x - (1 + y)u_y = y - x$$

The extended problem reads

$$(1 + x)U_x - (1 + y)U_y + (y - x)U_u = 0$$

and the characteristic system of differential equations is

$$\begin{aligned} \dot{x} &= 1 + x \\ \dot{y} &= -(1 + y) \\ \dot{u} &= y - x \end{aligned}$$

with general solution

$$\begin{aligned} x(t) &= c_1 e^t - 1 \\ y(t) &= c_2 e^{-t} - 1 \\ u(t) &= c_3 - c_2 e^{-t} - c_1 e^t \end{aligned}$$

Continuation of the example.

We proceed like in the last example and solve the characteristic system:

$$e^t = \frac{x+1}{c_1} = \frac{c_2}{y+1} \quad \Rightarrow \quad (x+1)(y+1) = c_1 \cdot c_2 = c \in \mathbb{R}$$

and

$$u = c_3 - (x+1) - (y+1) \quad \Rightarrow \quad u + x + y = d \in \mathbb{R}$$

Both constants c and d again determine the solution behaviour.

Hence we get the (however) implicit solution representation

$$\Phi\Big((x+1)(y+1), u + x + y\Big) = 0$$

with an arbitrary $\mathcal{C}^1$–function $\Phi : \mathbb{R}^2 \to \mathbb{R}$.

Note that in contrast to the linear case for quasi–linear equations we do not get an explicit solution representation and the solution may exists only locally.

Chapter 2: First–order partial differential equations

2.2 Initial value problems for first–order equations

We now consider the case of one time variable t and n space variables $\mathbf{x} \in \mathbb{R}^n$.

Definition: The following initial value problem defined on the whole $\mathbb{R}^n$

$$\begin{cases} u_t + \sum_{i=1}^{n} a_i(\mathbf{x}, t, u) u_{x_i} = b(\mathbf{x}, t, u) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

is called Cauchy–Problem.

At time $t = 0$ the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x})$$

is given explicitly.

Concrete solutions again can be derived using the method of charateristics.

Example: The transport equation.

A typical example is the transport equation from Chapter 1

$$\begin{cases} u_t + \mathbf{a} \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $\mathbf{a} \in \mathbb{R}^n$ is a given constant vector. Using the method of characteristics, we first get the $(n+1)$ differential equations

$$\frac{dt}{d\tau} = 1, \qquad \frac{d\mathbf{x}}{d\tau} = \mathbf{a}$$

and without restrivtion we may assume $t = \tau$.
The solution of the second equation reads

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{a} \cdot t,$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

Hence the characteristic curves are straight line, which run at time $t = 0$ through the point $\mathbf{x}_0$ in the direction $\mathbf{a}$.

Continuation of the example.

If we want to know the solution at an arbitrary point $(\mathbf{x}, t)$, we first look for the characteristic running through this point and determine the corresponding value $\mathbf{x}_0$ at time $t = 0$:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{a}t \quad \Rightarrow \quad \mathbf{x}_0 = \mathbf{x} - \mathbf{a}t$$

Because the solution remains constant along the characteristics, we directly get the solution representation

$$u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t)$$

Interpretation of the solution:

The given initial profile $u_0(\mathbf{x})$ is transported with constant velocity $\mathbf{a} \in \mathbb{R}^n$ without changing its shape.

Check: It holds

$$u_t(\mathbf{x}, t) = -\mathbf{a}\nabla u_0, \;\; \nabla u(\mathbf{x}, t) = \nabla u_0 \quad \Rightarrow \quad u_t + \mathbf{a} \cdot \nabla u = 0$$

Example.

We consider the initial value problem

$$\begin{cases} u_t + txu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = \sin x & \text{auf } \mathbb{R} \times \{t = 0\} \end{cases}$$

The characteristic equation reads

$$\dot{x} = tx, \qquad x(0) = x_0$$

with solution

$$x(t) = x_0 \exp\left(\frac{t^2}{2}\right)$$

and the solution of the inital value problems is given by

$$u(x,t) = \sin\left[x \exp\left(-\frac{t^2}{2}\right)\right]$$

Problem: Solutions may exist only local in time.

We return the Cauchy–Problem defined above,

$$\begin{cases} u_t + \sum_{i=1}^{n} a_i(\mathbf{x}, t, u) u_{x_i} = b(\mathbf{x}, t, u) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

The characteristic system reads

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{a}(\mathbf{x}, t, u) \\ \dot{u} &= b(\mathbf{x}, t, u) \end{aligned}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $u(0) = u_0(\mathbf{x}_0)$.

This is a nonlinear system of differential equations, which may have only local solutions in time.

In general we will have for quasi–linear 1st order PDEs only local solutions in time.

Nonlinear scalar conservation laws.

An important class of first order partial differential equations are nonlinear scalar conservation laws in one space dimension.

The corresponding Cauchy–Problem reads

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R} \times \{t = 0\} \end{cases}$$

and the given function $f = f(u)$ is called flux function.

Such equations are quasi–linear, because (assuming f is differentiable) they may be written as

$$u_t + a(u)u_x = 0$$

with $a(u) = f'(u)$.

In analogy to the transport equation we call the function $a(u)$ even local speed of propagation.

The Burgers equation.

The Burgers equation (Johannes Martinus Burgers, 1895–1981, Dutch physicist) is a conservation law with flux function $f(u) = u^2/2$ and the associated Cauchy problem is given by

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R} \times \{t = 0\} \end{cases}$$

We choose the initial condition

$$u_0(x) = \begin{cases} 1 & : \quad x \leq 0 \\ 1 - x & : \quad 0 < x < 1 \\ 0 & : \quad x \geq 1 \end{cases}$$

and use the method of characteristics to compute the solution.

The characteristic equation reads

$$\dot{x} = u, \quad x(0) = x_0$$

The Burgers equation: Characteristic curves.

Because the solution of Burgers equations remains constant along the curve $x(t)$, we have

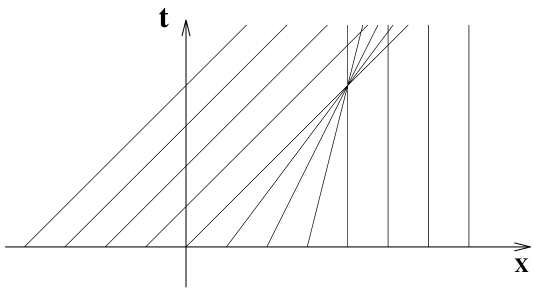
$$\dot{x} = u_0(x_0) \quad \Rightarrow \quad x(t) = x_0 + t u_0(x_0)$$

This seems to be harmless, but it is by no means!

With the given initial condition $u_0(x)$ we get

$$x(t) = \begin{cases} t + x_0 & : \quad x_0 \leq 0 \\ (1 - x_0)t + x_0 & : \quad 0 < x_0 < 1 \\ x_0 & : \quad x_0 \geq 1 \end{cases}$$

and the corresponding picture of the characteristics curves looks



The Burgers equations: Solution produces a singualrity.

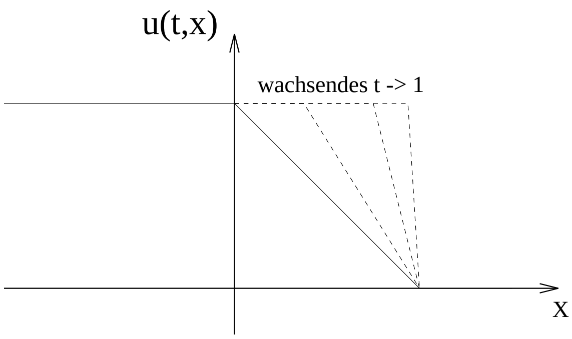
At time $t = 1$ we have infinitely many curves running through the point $x = 1$, i.e. the solution is not unique at the pot $(x, t) = (1, 1)$.

Indeed with the given initial condition a classical solution only exists local in time for $0 \leq t < 1$.

For $t \in [0, 1)$ the solution is given by

$$u(x, t) = \begin{cases} 1 & : \quad x < t \\ (1 - x)/(1 - t) & : \quad 0 \leq t \leq x < 1 \\ 0 & : \quad x > 1 \end{cases}$$

The corresponding picture of the solution at various times $t \in [0, 1)$ is:



2.3 Scalar conservation laws

The Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R} \times \{t = 0\} \end{cases}$$

has in general no global solution.

For the Burgers equation from the last section with initial condition

$$u_0(x) = \begin{cases} 1 & : \quad x \leq 0 \\ 1 - x & : \quad 0 < x < 1 \\ 0 & : \quad x \geq 1 \end{cases}$$

a classical solution only exists on the time interval $[0, 1)$:

$$u(x, t) = \begin{cases} 1 & : \quad x < t \\ (1 - x)/(1 - t) & : \quad 0 \leq t \leq x < 1 \\ 0 & : \quad x > 1 \end{cases}$$

Let $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a differentiable function with compact support.

We multiply $u_t + f(u)_x = 0$ with v and integrate over $\mathbb{R} \times [0, \infty)$, which yields

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x)\, v dx dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u v_t dx dt - \int_{-\infty}^\infty u_0(x) v(x, 0) dx - \int_0^\infty \int_{-\infty}^\infty f(u) v_x dx dt \end{aligned}$$

Together with the initial condition $u(x, 0) = u_0(x)$ we get

$$\int_0^\infty \int_{-\infty}^\infty (u v_t + f(u) v_x)\, dx dt + \int_{-\infty}^\infty u_0(x) v(x, 0) dx = 0$$

Weak solutions, integral solutions.

Definition: A differentiable function $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with compact support is called a test function.

Definition: A function $u \in L^\infty(\mathbb{R} \times [0, \infty))$ is called integral solution or weak solution, if the condition

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x)\, dxdt + \int_{-\infty}^\infty u_0(x)v(x,0)dx = 0$$

is satisfied for all test funktions v.

Remark: A integral solution might be not differentiable, the function rather may have discontinuities.

Riemann problems

Definition: The initial value problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R} \times \{t = 0\} \end{cases}$$

with discontinuous intial condition

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases}$$

is called a Riemann problem for scalar conservation laws.

Example: A Riemann problem for the Burgers equation reads

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R} \times \{t = 0\} \end{cases}$$

with discontinuous intial condition

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases}$$

Integral solutions for Riemann problems.

- **Shock wave solution** for the Burgers equation.

  For $u_l \neq u_r$ the so–called shock wave

$$u(x,t) = \begin{cases} u_l & : \quad x \leq s(t) \\ u_r & : \quad x > s(t) \end{cases}$$

  is an integral solution.
  Here the function $s(t)$ denotes the position of the shock front, i.e. the point of discontinuity or the jumpt point.
  The shock front is moving with the velocity $\dot{s}(t)$ where

$$\dot{s}(t) = \frac{[f]}{[u]} = \frac{f(u_l) - f(u_r)}{u_l - u_r}$$

  and $s(0) = 0$.

  This condition is called Rankine–Hugoniot condition.

Integral solutions for Riemann problems.

- **Rarefaction wave** for the Burgers equation.

  For $u_l < u_r$ the so–called rarefaction wave

$$u(x,t) = \begin{cases} u_l & : \quad x \leq u_l\, t \\ \dfrac{x}{t} & : \quad u_l\, t \leq x \leq u_r\, t \\ u_r & : \quad x \geq u_r\, t \end{cases}$$

  is an integral solution.
  Note that the solution $u(x,t)$ is a continuous function.

  Along the straight lines $x = u_l\, t$ and $x = u_r\, t$ the solution is not differentiable and therefore no classical solution.

  Remark: For $u_l < u_r$ the question arises, which of the two solutions (shock or rarefaction wave) is physical relevant. We will see, that the rarefaction wave only is physically relevant.

Description of the shock wave solution.

Definition: A shock wave solution u is an integral solution of a conservation law

$$u_t + f(u)_x = 0,$$

if there exists a so–called shock front $x = s(t)$, $s \in \mathcal{C}^1$, such that u is for $x < s(t)$ and $x > s(t)$, respectively, a smooth solution of the PDE and u has at $x = s(t)$ a jump with size

$$[u](t) = u(s(t)^+, t) - u(s(t)^-, t)$$

The quantity $\dot{s}(t)$ is called the shock speed.

Theorem: If $x = s(t)$ is the shock front of a shock wave solution for $u_t + f(u)_x = 0$, the corresponding shock speed $\dot{s}$ satisfies the Rankine–Hugoniot condition

$$\dot{s} = \frac{[f]}{[u]} = \frac{f(u(s(t)^-, t)) - f(u(s(t)^+, t)}{u(s(t)^-, t) - u(s(t)^+, t)}$$

Derivation of the Rankine–Hugoniot condition.

An integral solution saties the relation

$$\frac{d}{dt}\int_{x_1}^{x_2} u(\xi, t)d\xi = f(u(x_1, t)) - f(u(x_2, t))$$

If we choose $x_1 < s(t) < x_2$ it follows

$$\frac{d}{dt}\left(\int_{x_1}^{s(t)} u(\xi, t)d\xi + \int_{s(t)}^{x_2} u(\xi, t)d\xi\right) = f(u(x_1, t)) - f(u(x_2, t))$$

Because $u(x, t)$ is by definition differentiable for $x < s(t)$ and $x > s(t)$, respectively, we may differentiate in both integrals to get ableiten:

$$\int_{x_1}^{s(t)} \frac{\partial u}{\partial t}\mathrm{d}\xi + \dot{s}\, u(s(t)^-, t) + \int_{s(t)}^{x_2} \frac{\partial u}{\partial t}\mathrm{d}\xi - \dot{s}\, u(s(t)^+, t) + f_2 - f_1 = 0$$

Continuation of the derivation.

Hence

$$\int_{x_1}^{s(t)} \frac{\partial u}{\partial t} \mathrm{d}\xi + \dot{s}\, u(s(t)^-, t) + \int_{s(t)}^{x_2} \frac{\partial u}{\partial t} \mathrm{d}\xi - \dot{s}\, u(s(t)^+, t) + f_2 - f_1 = 0$$

with

$$f_1 := f(u(x_1, t)), \qquad f_2 := f(u(x_2, t))$$

In the limit $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$ both integrals vanish and we get

$$\dot{s}\, u(s(t)^-, t) - \dot{s}\, u(s(t)^+, t) = f(u(s(t)^-)) - f(u(s(t)^+))$$

But this is indeed the Rankine–Hugoniot condition given by

$$\dot{s} = \frac{[f]}{[u]}$$

Example.

We consider the Burgers equation with discontinuous initial condition

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases}$$

and $u_l > u_r$.

The Rankine–Hugoniot condition reads

$$\dot{s} = \frac{[f]}{[u]} = \frac{u_l^2/2 - u_r^2/2}{u_l - u_r} = \frac{(u_l - u_r)(u_l + u_r)}{2(u_l - u_r)} = \frac{1}{2}(u_l + u_r)$$

Therefore the shock wave solution for this problem is given by

$$u(x,t) = \begin{cases} u_l & : \quad x \leq \frac{1}{2}(u_l + u_r)\, t \\ u_r & : \quad x > \frac{1}{2}(u_l + u_r)\, t \end{cases}$$

Description of the rarefaction wave.

We consider the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{auf } \mathbb{R} \times \{t = 0\} \end{cases}$$

with discontinuous initial condition

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases}$$

where now $u_l < u_r$.

Additionally we asssume that $f \in \mathcal{C}^2(\mathbb{R})$ and $f'' > 0$, i.e. the flux function should be **strictly convex**.

Fiannly we define

$$g := (f')^{-1}$$

Description of the rarefaction wave.

By assumption the flux function f is strictly convex, i.e. f' is strictly monotonically increasing. Hence

$$u_l < u_r \quad \Rightarrow \quad f'(u_l) < f'(u_r)$$

Therefore there are **exactly two** types of characteristics, namely

$$x(t) = x_0 + f'(u_l)\,t \quad \text{and} \quad x(t) = x_0 + f'(u_r)\,t$$

But both families of curves **do not** cover the whole space $\mathbb{R} \times \mathbb{R}_+$, there is a region Ω without any characteristics,

$$\Omega := \{(x, t) \in \mathbb{R} \times \mathbb{R}_+ \ : \ f'(u_l) \cdot t < x < f'(u_r) \cdot t\}$$

In Ω the method of characteristic do not give any value and we may fill this region using an arbitrary **integral solution**.

Description of the rarefaction wave.

Theorem: For $u_l < u_r$ the rarefaction wave given by

$$u(x,t) := \begin{cases} u_l & : \quad x < f'(u_l)t \\ g(x/t) & : \quad f'(u_l)t < x < f'(u_r)t \\ u_r & : \quad x > f'(u_r)t \end{cases}$$

is an integral solution of the Riemann problem. The rarefaction wave is in particular a continuous function.

Proof: We first show that the function given above is continuous in both points

$$x = f'(u_l)\,t \quad \text{and} \quad x = f'(u_r)\,t$$

We have

$$g\left(\frac{f'(u_l)\,t}{t}\right) = g(f'(u_l)) = (f')^{-1}(f'(u_l)) = u_l$$

Description of the rarefaction wave.

as well as

$$g\left(\frac{f'(u_r)\,t}{t}\right) = g(f'(u_r)) = (f')^{-1}(f'(u_r)) = u_r$$

Furthermore the rarefaction wave is constant for $x < f'(u_l)\,t$ and $x > f'(u_r)\,t$ and is therefore a solution of the given conservation law.

For $f'(u_l)t < x < f'(u_r)t$ we compute

$$u_t = -\frac{x}{t^2} g'(x/t)$$

$$f(u)_x = f(g(x/t))_x = f'(g(x/t))\frac{g'(x/t)}{t} = \frac{x}{t^2} g'(x/t)$$

Hence it follows that even $g(x/t)$ is a solution of $u_t + f(u)_x = 0$.

From the contuniuity of the function it follows that the rarefaction is indeed an integral solution.

Problem: Integral solution are not unique!

Example: We again consider the Burgers equation with initial condition

$$u_0(x) = \begin{cases} 0 & : \quad x \leq 0 \\ 1 & : \quad x > 0 \end{cases}$$

Then we have, e.g., the two integral solutions

$$u_1(x,t) = \begin{cases} 0 & : \quad x \leq t/2 \\ 1 & : \quad x > t/2 \end{cases}$$

and

$$u_2(x,t) = \begin{cases} 0 & : \quad x < 0 \\ x/t & : \quad 0 \leq x \leq t \\ 1 & : \quad x > t \end{cases}$$

The first solution is a shock wave, the second one a rarefaction wave.

Question: Which of the two solutions is the physically correct one?

Entropy condition and entropy solutions.

Which of the two solutions is the physically correct one?

We need an additional condition that selects the physically correct integral solutions.

Definition: An integral solution is called entropy solution, if the solution satisfies the following entropy condition (Lax–Oleinik formula):

$$\exists\, C > 0, \text{ such that for all } x, z \in \mathbb{R},\ t > 0 \text{ with } z > 0 \text{ applies}$$

$$u(t, x+z) - u(t,x) < \frac{C}{t} z$$

Theorem: If an integral solution satisfies the entropy condition given above, then the solution is unique, i.e. entropy solutions are unique solutions.

Remark: In our last example actually the rarefaction wave satisfies the entropy condition.

Definition: A linear second–order PDE in n variables is given

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + fu = g$$

Here a_{ij}, b_i, f and g are functions of $\mathbf{x} = (x_1, \ldots, x_n)^T$.
The first term is called principal part of the PDE. Furthermore w.l.o.g.

$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), \quad i, j = 1, \ldots, n$$

Special case: If $a_{ij} = \mathrm{const.}$, $i, j = 1, \ldots, n$, we may write the PDE in matrix notation:

$$(\nabla^T \mathbf{A} \nabla) u + (\mathbf{b}^T \nabla) u + fu = g$$

with symmetric matrix $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,n}$.

Chapter 3. Second–order partial differential equations

3.1 Normal forms of linear second–order equations

Be given a differential equation in matrix notation

$$(\nabla^T \mathbf{A} \nabla) u + (\mathbf{b}^T \nabla) u + fu = g$$

with a constant and symmetric matrix $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,n}$.

Linear algebra: principal component analysis (PCA)

Theorem: Every real and symmetric matrix $\mathbf{A}$ is diagonalizable. Furthermore we have

$$\mathbf{D} = \mathbf{S}^{-1} \mathbf{A} \mathbf{S}$$

where $\mathbf{S}$ can be chosen as an orthogonal matrix.

Reminder: A real matrix $\mathbf{S}$ is orthogonal if

$$\mathbf{S}^{-1} = \mathbf{S}^T$$

Ansatz to derive normal forms.

Use the coordinate transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$ bzw. $\mathbf{y} = \mathbf{S}^T\,\mathbf{x}$ and define

$$\tilde{u}(\mathbf{y}) := u(\mathbf{S}\,\mathbf{y})$$

With $u(\mathbf{x}) = \tilde{u}(\mathbf{S}^T\,\mathbf{x})$ it follows

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial \tilde{u}}{\partial y_j}\frac{\partial y_j}{\partial x_i}$$

and because $\frac{\partial y_j}{\partial x_i} = s_{ij}$ we have

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} s_{ij}\frac{\partial \tilde{u}}{\partial y_j}$$

But the last relation just means that

$$\nabla_x\, u(\mathbf{x}) = \mathbf{S}\,\nabla_y \tilde{u}(\mathbf{S}^T\mathbf{x})$$

or in formal notation $\nabla_x = \mathbf{S}\,\nabla_y$. If we take the transpose we have

$$\nabla_x^T = (\mathbf{S}\,\nabla_y)^T = \nabla_y^T\, S^T$$

Diagonal form of a second–order PDE

Result: If u solves the equation $(\nabla^T \mathbf{A} \nabla)u + (\mathbf{b}^T\,\nabla)u + fu = g$, we obtain for $\tilde{u}$ the PDE

$$(\nabla^T \mathbf{S}^T \mathbf{A}\mathbf{S}\,\nabla)\tilde{u} + (\mathbf{b}^T \mathbf{S}\,\nabla)\tilde{u} + \tilde{f}\,\tilde{u} = \tilde{g}$$

Definition: Let the second–order partial differential equation

$$(\nabla^T \mathbf{A} \nabla)u + (\mathbf{b}^T\,\nabla)u + fu = g$$

be given where $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,n}$ is a constant and symmetric matrix.
Then the corresponding diagonal form of the PDE is given by

$$(\nabla^T \mathbf{D}\,\nabla)\tilde{u} + ((\mathbf{S}^T \tilde{\mathbf{b}})^T \nabla)\tilde{u} + \tilde{f}\,\tilde{u} = \tilde{g}$$

with diagonal matrix $\mathbf{D} = \mathbf{S}^T \mathbf{A}\mathbf{S}$ and $\mathbf{S}^T\mathbf{S} = \mathbf{I}$ as well as

$$\tilde{\mathbf{b}}(\mathbf{y}) = \mathbf{b}(\mathbf{S}\,\mathbf{y}), \quad \tilde{f}(\mathbf{y}) = f(\mathbf{S}\,\mathbf{y}) \quad \text{and} \quad \tilde{g}(\mathbf{y}) = g(\mathbf{S}\,\mathbf{y}).$$