Differential Equations II for Students of Engineering

Partial Differential Equations

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Chapter 1: Basics, Terminology, Examples

The study of partial differential equations is a wide field and encompasses various, entirely different theories and aspects.

Thus, this lecture can merely give a very basic introduction into the general topic and in fact focuses on treating some specific equations as model cases.

1.1 Terminology for PDEs

Terminology (for partial derivatives)

For a function $u\colon \Omega \to \mathbb{R}^q$ *in* n *variables* $x = (x_1, x_2 \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ *with arbitrary* $n, q \in \mathbb{N}$ *agree on notation for*

- *all first-order partial derivatives* (*Jacobi matrix*; *gradient if* $q = 1$):

$$\mathrm{D}u := \mathrm{J}u := \left(\tfrac{\partial u}{\partial x_i}\right)_{i=1,2,\ldots,n} = (\partial_i u)_{i=1,2,\ldots,n}\,,$$

- *all second-order partial derivatives* (*Hessian if* $q = 1$):

$$\mathrm{D}^2 u := \left(\tfrac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n} = (\partial_i \partial_j u)_{i,j=1,\ldots,n}\,,$$

- *all* k*th-order partial derivatives with arbitrary* $k \in \mathbb{N}$:

$$\mathrm{D}^k u := \left(\tfrac{\partial^k u}{\partial x_{i_k} \ldots \partial x_{i_1}}\right)_{i_1,\ldots,i_k=1,\ldots,n} = (\partial_{i_k} \ldots \partial_{i_1} u)_{i_1,\ldots,i_k=1,\ldots,n}\,.$$

General form of partial differential equations

Definition (partial differential equation)

A partial differential equation (in brief: PDE or partial DE) is an equation with partial derivatives up to order $m \geq 1$ *in form*

$$F(x, u(x), \mathrm{D}u(x), \mathrm{D}^2u(x), \ldots, \mathrm{D}^m u(x)) = 0 \quad \text{for all } x \in \Omega$$

or in brief functional notation

$$F(\,\cdot\,, u, \mathrm{D}u, \mathrm{D}^2u, \ldots, \mathrm{D}^m u) \equiv 0 \quad \text{in } \Omega$$

for an unknown function $u\colon \Omega \to \mathbb{R}^q$ *on an open set* $\Omega \subset \mathbb{R}^n$, $n \geq 2$.
If u *solves the equation, one calls* u *a solution to the PDE in* Ω.

The decisive difference to ODEs is that $x = (x_1, x_2, \ldots, x_n)$ contains not only one, but multiple (in fact $n \geq 2$) variables.

Terminology in connection with PDEs

Terminology for PDE $F(\,\cdot\,, u, \mathrm{D}u, \mathrm{D}^2u, \ldots, \mathrm{D}^m u) \equiv 0$ in $\Omega \subset \mathbb{R}^n$:

- m: order of the PDE (provided $\mathrm{D}^m u$ indeed occurs),
- n: number of variables (recall $n \geq 2$),
- q: number of (component) functions (of $u\colon \Omega \to \mathbb{R}^q$),
- N: number of (component) equations (of PDE with „$\equiv$“ in $\mathbb{R}^N$),
- F: given structure function of the PDE
  (from suitable domain to $\mathbb{R}^N$).

In this lecture the focus is on the case $N = q = 1$ (scalar PDE for single function) with order $m \in \{1, 2\}$. Taking $N = q \geq 2$ (PDE system for multiple functions) is also reasonable, but here mostly beyond the scope.

Boundary conditions

One expects unique solutions only for boundary value problems (BVPs) out of PDEs and additional boundary conditions (BCs) at $\partial\Omega$. As a rough rule of thumb a PDE system of order m for $N = q$ functions needs $\overline{\frac{mq}{2}\text{ BCs}}$ (where „half BCs" concern a part of the boundary only, similar to ICs for ODEs).

Common BCs are (variants of) Dirichlet BCs

$$u(x) = g(x) \qquad \text{for } x \in \partial\Omega$$

with given function $g\colon \partial\Omega \to \mathbb{R}^q$ and Neumann BCs

$$\partial_\nu u(x) = \psi(x) \qquad \text{for } x \in \partial\Omega$$

with outward unit normal field $\nu\colon \partial\Omega \to \mathbb{R}^n$ to $\partial\Omega$, normal derivative $\partial_\nu u(x) := \mathrm{J}u(x)\nu(x)$, and given function $\psi\colon \partial\Omega \to \mathbb{R}^q$ and beside these also initial conditions (ICs)/Cauchy conditions (soon more on these).

Classification of PDEs

Similar to ODEs one classifies PDEs of order m as follows:

- Autonomous PDEs take the form $F_0(u, \mathrm{D}u, \mathrm{D}^2u, \ldots, \mathrm{D}^m u) \equiv 0$.
- Linear PDEs exhibit an affine dependence on u, $\mathrm{D}u$, D^2u, $\ldots \mathrm{D}^m u$. The possibly x-dependent factors in front of u and its derivatives are then called coefficients, while terms independent of u and its derivatives are collected on the right-hand side as inhomogeneity.

Among non-linear PDEs one further distinguishes:

- Semilinear PDEs depend affinely on $\mathrm{D}^m u$ with coefficients which depend solely on x in front of the mth derivatives.
- Quasilinear PDEs depend affinely on $\mathrm{D}^m u$ (in general with coefficients which depend on $(\,\cdot\,, u, \mathrm{D}u, \ldots, \mathrm{D}^{m-1}u)$ in front of the mth derivatives).
- Fully non-linear PDEs are not quasilinear.

Relevant types of PDEs

In this lecture, relevant types of scalar PDEs (for $u\colon \Omega \to \mathbb{R}$) are:

- linear first-order PDEs (with coefficients $a_i, b\colon \Omega \to \mathbb{R}$):

$$\sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}(x) + b(x)u(x) = f(x)\,.$$

- linear second-order PDEs (with coefficients $a_{i,j}, b_i, c\colon \Omega \to \mathbb{R}$):

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_j \partial x_i}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) = f(x)\,.$$

- semilinear first-order PDEs (with $a_i\colon \Omega \to \mathbb{R}$ and $b\colon \Omega \times \mathbb{R} \to \mathbb{R}$):

$$\sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}(x) = b(x, u(x))\,.$$

- quasilinear first-order PDEs (with $a_i, b\colon \Omega \times \mathbb{R} \to \mathbb{R}$):

$$\sum_{i=1}^{n} a_i(x, u(x)) \frac{\partial u}{\partial x_i}(x) = b(x, u(x))\,.$$

1.2 Various examples of PDEs

In the sequel, various examples from the „zoo“ of important PDEs are briefly discussed together with suitable BCs and interpretations. The fundamentally different interpretations and applications underline the extremely wide scope of PDE theory.

If no other indication is given, the examples are scalar PDEs for a single function.

Transport equation

Linear transport equation for $u\colon [0,T) \times \mathbb{R}^n \to \mathbb{R}$:

$$\frac{\partial u}{\partial t}(t,x) + a(t,x) \cdot \nabla_x u(t,x) = 0 \qquad \text{for } (t,x) \in (0,T) \times \mathbb{R}^n$$

with given $T > 0$ and $a\colon (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ („$\cdot$“ is the inner product).

Typical feature: Occurrence of time variable $t \in [0,T)$ and space variables $x \in \mathbb{R}^n$. Often one writes only ∇u, but still with the meaning of $\nabla_x u$.

Classification: first-order, linear, homogeneous.

Reasonably complemented with IC („half BC“; $u_0\colon \mathbb{R}^n \to \mathbb{R}$ given):

$$u(0,x) = u_0(x) \qquad \text{for } x \in \mathbb{R}^n\,.$$

Interpretation: Solutions u model the density of mass or of electric charge, which is transported along the field a. Specifically, constant a gives rise to uniform drift $u(t,x) = u_0(x - ta)$ with velocity $a \in \mathbb{R}^n$.

Cauchy-Riemann equations

Cauchy-Riemann equations for $f, g\colon \overline{\Omega} \to \mathbb{R}$ in variables (x, y):

$$\left.\begin{aligned} \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} &\equiv 0\,, \\ \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} &\equiv 0 \end{aligned}\right\} \text{ in } \Omega \subset \mathbb{R}^2$$

Classification: system of 2 equations, first-order, linear, homogeneous.

Meaning: When identifying $\mathbb{C} \ni x+\mathbf{i}y \mathrel{\widehat{=}} (x, y) \in \mathbb{R}^2$ characterizes the holomorphic (i.e. complex differentiable) functions $f+\mathbf{i}g\colon \Omega \to \mathbb{C}$ on $\Omega \subset \mathbb{C}$. More in lecture „Complex Functions“!

Reasonably complemented with Dirichlet BC for either f or g at $\partial\Omega$ (though this leaves free an additive constant for the other function).

Laplace equation and Poisson equation

Laplace equation and Poisson or potential equation for $u\colon \overline{\Omega} \to \mathbb{R}$:

$$\Delta u(x) = 0 \qquad \text{rsp.} \qquad \Delta u(x) = f(x) \qquad \text{for } x \in \Omega \subset \mathbb{R}^n$$

with given $f\colon \Omega \to \mathbb{R}$ and with the important Laplace operator

$$\Delta u(x) := \operatorname{div}(\nabla u)(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) = \operatorname{trace}(\mathrm{D}^2 u(x))\,.$$

Solutions of Laplace's equation are also known as harmonic functions.

Classification: second-order, linear, homogeneous rsp. inhomogeneous.

Reasonably complemented with Dirichlet BC or Neumann BC for u at $\partial\Omega$.

Meaning/interpretation: Characterizes real and imaginary parts of holomorphic functions. Solutions u model electric potential for charge density f/ε_0 (with physical constant $\varepsilon_0 > 0$).

Diffusion or heat equation

Diffusion or heat equation for $u\colon [0,T) \times \overline{\Omega} \to \mathbb{R}$:

$$\frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) = 0 \qquad \text{for } (t,x) \in \Omega_T \subset \mathbb{R} \times \mathbb{R}^n\,,$$

again with time and space variables and with abbreviation $S_T := (0,T)\times S$.

Classification: second-order, linear, homogeneous (has inhomogeneous variant).

Complement e.g. with IC and Dirichlet BC ($\rightsquigarrow 1$ BC at „parabolic boundary“)

$$u(0,x) = u_0(x) \text{ for } x \in \Omega\,, \qquad u(t,x) = g(t,x) \text{ for } (t,x) \in (\partial\Omega)_T$$

for given $u_0\colon \Omega \to \mathbb{R}$ and $g\colon (\partial\Omega)_T \to \mathbb{R}$.

Interpretation: Solutions u model the mass density/concentration in diffusion processes or the temperature in heat propagation.

In stationary case $\frac{\partial u}{\partial t} \equiv 0$ get back Laplace equation.

Navier-Stokes equations

Incompressible Navier-Stokes equations for $(\vec{v}, p)\colon [0,T) \times \overline{\Omega} \to \mathbb{R}^n \times \mathbb{R}$:

$$\left.\begin{aligned} \rho \frac{\partial \vec{v}}{\partial t} - \mu \, \Delta_x \vec{v} + \rho \sum_{i=1}^{n} v_i \frac{\partial \vec{v}}{\partial x_i} = -\nabla_x p \,, \\ \operatorname{div}_x \vec{v} = 0 \end{aligned}\right\} \text{ in } \Omega_T \subset \mathbb{R} \times \mathbb{R}^n$$

with constants $\rho, \mu > 0$.

Classification: system of $n{+}1$ equations, second-order, semilinear.

Reasonable BCs as for diffusion equation (also known as no-slip BCs).

Interpretation: Solutions $(\vec{v}, p)$ model velocity and pressure in the flow of an incompressible fluid of constant density ρ and constant viscosity μ.
Foundational in fluid mechanics!

Specifically, for $\mu = 0$, reduces to Euler equations in fluid mechanics and in case $\frac{\partial \vec{v}}{\partial t} \equiv 0$ gives stationary Navier-Stokes and Euler equations, respectively.

Wave equation

Wave equation for $u\colon \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$:

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta_x u(t,x) = 0 \qquad \text{for } (t,x) \in \mathbb{R} \times \Omega \subset \mathbb{R} \times \mathbb{R}^n$$

Classification: second-order, linear, homogeneous (has inhomogeneous variant).

Complement e.g. with 2 ICs and Dirichlet BC (still to be seen as 1 BC overall)

$$u(0,x) = u_0(x) \text{ for } x \in \Omega\,, \qquad \frac{\partial u}{\partial t}(0,x) = v_0(x) \text{ for } x \in \Omega\,,$$

$$u(t,x) = g(t,x) \text{ for } (t,x) \in \mathbb{R} \times \partial\Omega$$

for given $u_0, v_0\colon \Omega \to \mathbb{R}$ and $g\colon \mathbb{R} \times \partial\Omega \to \mathbb{R}$.

Interpretation: Solutions u model displacements in wave propagation and/or in oscillations.

In stationary case $\frac{\partial u}{\partial t} \equiv 0$ get back Laplace equation.

Schrödinger equation

Schrödinger equation for $\psi \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$:

$$\mathbf{i}\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar}{2m} \Delta_x \psi - V\psi \equiv 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^n$$

with given $V \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and constants $\hbar, m > 0$.

Classification: scalar/system over $\mathbb{C}/\mathbb{R}$, second-order, linear, homogeneous.

Reasonably complemented with IC ($\psi_0 \colon \mathbb{R}^n \to \mathbb{C}$ given):

$$\psi(0, \cdot\,) = \psi_0 \qquad \text{in } \mathbb{R}^n\,.$$

Interpretation: Solutions ψ are wavefunctions (quantum states) of particle of mass m in potential V (with reduced Planck constant $\hbar$). Foundational for quantum mechanics!

Product-exponential ansatz sometimes yields eigenvalue problem for Δ_x.

Maxwell equations

Vacuum Maxwell equations for $(\vec{E},\vec{B})\colon \mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^3\times\mathbb{R}^3$:

$$\left.\begin{aligned} \varepsilon_0\,\mathrm{div}_x\vec{E}(t,x) &= \rho(t,x)\,,\\ \frac{\partial\vec{B}}{\partial t}(t,x)+\mathrm{rot}_x\vec{E}(t,x) &= 0\,,\\ \mathrm{div}_x\vec{B}(t,x) &= 0\,,\\ \varepsilon_0\mu_0\,\frac{\partial\vec{E}}{\partial t}(t,x)-\mathrm{rot}_x\vec{B}(t,x) &= -\mu_0\,\vec{\jmath}\,(t,x) \end{aligned}\right\}\text{ for } (t,x)\in\mathbb{R}\times\mathbb{R}^3$$

with given $(\rho,\vec{\jmath}\,)\colon \mathbb{R}\times\mathbb{R}^3\to\mathbb{R}\times\mathbb{R}^3$ and constants $\varepsilon_0,\mu_0>0$.

Classification: 8 component equations for 6 component functions (okay only since rot strongly degenerate; $\mathrm{rot}\circ\nabla\equiv 0$, $\mathrm{div}\circ\mathrm{rot}\equiv 0$), linear, inhomogeneous.

Complement with ICs $\vec{E}(0,x)=\vec{E}_0(x)$ and $\vec{B}(0,x)=\vec{B}_0(x)$ for given $(\vec{E}_0,\vec{B}_0)\colon \mathbb{R}^3\to\mathbb{R}^3\times\mathbb{R}^3$ s.t. $\varepsilon_0\,\mathrm{div}\,\vec{E}_0=\rho(0,\,\cdot\,)$ and $\mathrm{div}\,\vec{B}_0\equiv 0$ in $\mathbb{R}^3$.

Interpretation: These four basic equations of electrodynamics determine the electric field $\vec{E}$ and the magnetic field $\vec{B}$ from given electric charge density ρ and electric current density $\vec{\jmath}$.

Minimal surface equation

Minimal surface equation for $u\colon \overline{\Omega} \to \mathbb{R}$:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n\,.$$

Classification: second-order, quasilinear.

Usually complemented with Dirichlet BC for u at $\partial\Omega$ or certain free BCs.

Interpretation: Graphs of solutions u are minimal surfaces, which have zero mean curvature at each of their points and are relevant objects in geometric analysis and differential geometry.

Monge-Ampère equation

Monge-Ampère equation for $u\colon \overline{\Omega} \to \mathbb{R}$:

$$\det\big(\mathrm{D}^2 u(x)\big) = f(x) \qquad \text{for } x \in \Omega \subset \mathbb{R}^n$$

with given (often everywhere positive) $f\colon \Omega \to \mathbb{R}$.

Classification: second-order, fully non-linear.

Reasonable with Dirichlet BC or Neumann BC or certain natural BC.

Applications: Solutions u are connected with optimal transport of mass distributions and with surfaces of prescribed Gauss curvature.

Key aspects of the lecture

The focus is now on treating in more detail the following very illustrative model cases among the preceding examples:

- transport equation and more general first-order PDEs,
- Laplace and Poisson equation (including eigenvalue problems),
- diffusion or heat equation,
- and wave equation.

Chapter 2: First-Order PDEs

First-order PDEs occur in different applications, but mostly describe a time evolution, which starts at a certain IC. In general one has a better chance for explicitly solving or analyzing first-order PDEs than one has in case of second-order and higher-order PDEs.

In this chapter we first discuss different aspects of a central application context and only eventually approach a comparably general solution theory and some model cases.

2.1 The continuity equation (and its background)

Here always use time/space variables (t, x) plus abbreviations $u_t := \frac{\partial u}{\partial t}$ and $\mathrm{div}(\ldots) := \mathrm{div}_x(\ldots)$. The continuity equation is the linear PDE

$$\boxed{u_t + \mathrm{div}(u\vec{v}) \equiv 0} \qquad \text{in open } U \subset \mathbb{R} \times \mathbb{R}^n$$

for an unknown function $u \colon U \to \mathbb{R}$ and a given or u-dependent velocity field $\vec{v} \colon U \to \mathbb{R}^n$ (both functions in variables $(t, x) \in U$).

Interpretation: If u is the density of quantity (often mass), which moves according to $\vec{v}$, then at time t and in a point x the temporal rate of change $u_t(t, x)$ equals the spatial in/outflow density $-\mathrm{div}(u\vec{v})(t, x)$ $\big(\mathrm{div}(u\vec{v}){>}0 \rightsquigarrow$ source/outflow density; $\mathrm{div}(u\vec{v}){<}0 \rightsquigarrow$ sink/inflow density$\big)$.

In 1d case $n = 1$, which is already of interest, get simply

$$u_t + (uv)_x \equiv 0 \qquad \text{in } U \subset \mathbb{R} \times \mathbb{R}\,.$$

Continuity equation and conservation of mass

To underpin the interpretation consider the trajectory $t \mapsto \Phi(t,x)$ of a particle, which starts at time $t=0$ at $x \in U_0$. (Mathematically consider $\Phi \in \mathrm{C}^2(I \times U_0, \mathbb{R}^n)$, $\Phi(0,x) = x$ for $x \in U_0$, with open $0 \in I \subset \mathbb{R}$, $U_0 \subset \mathbb{R}^n$.) Then obtain the moving domain $U := \{(t, \Phi(t,x)) : t \in I\,,\ x \in U_0\}$ and the velocity field $\vec{v}$ of Φ in U, given by

$$\vec{v}(t, \Phi(t,x)) = \partial_t \Phi(t,x) \qquad \text{for } (t,x) \in I \times U_0\,.$$

Theorem (Continuity equation and conservation of mass)

In the above setting, if $x \mapsto \Phi(t,x)$ is a diffeomorphism for each $t \in I$, then, for $u \in \mathrm{C}^1(U)$, the following are **equivalent**:

(1) *u solves the continuity equation $u_t + \operatorname{div}(u\vec{v}) \equiv 0$ in U.*

(2) *Conservation of mass in moving domains: There holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Phi(t,A_0)} u(t,x)\,\mathrm{d}x = 0$$

for each compact and measurable subset $A_0 \subset U_0$ and every $t \in I$.

Continuity equation and conservation of mass (continued)

Complementary remarks:

- In the theorem and in the sequel, compact sets are closed and bounded, while measurable sets are (Jordan) measurable in the sense of Analysis III.
- The hypotheses of the theorem are satisfied for the trajectories of an ODE system in many (good) cases, but the derivation needs some more ODE theory.

Proof of the theorem on continuity equation and mass conservation:
The Reynolds transport theorem for derivation on moving domains (see next slide) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Phi(t,A_0)} u(t,x)\,\mathrm{d}x = \int_{\Phi(t,A_0)} \big[u_t(t,x) + \mathrm{div}(u\vec{v})(t,x)\big]\,\mathrm{d}x$$

for A_0 as in (2) and $t \in I$. Thus, (1) $\Longrightarrow$ (2) is evident. Now suppose that (2) holds. As every compact and measurable subset $B_t \subset \Phi(t, U_0)$ has the form $B_t = \Phi(t, A_0)$, it is $\int_{B_t} \big[\ldots\big]\,\mathrm{d}x = 0$ for <u>each</u> such B_t. So, one deduces $\big[\ldots\big] = 0$ for all $x \in \Phi(t, U_0)$ and altogether for all $(t,x) \in U$. □

The Reynolds transport theorem

Theorem (Reynolds' transport theorem)

Under the hypotheses of the previous theorem, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Phi(t,A_0)} u(t,x)\,\mathrm{d}x = \int_{\Phi(t,A_0)} \big[u_t(t,x) + \mathrm{div}(u\vec{v})(t,x)\big]\,\mathrm{d}x$$

for each compact and measurable subset $A_0 \subset U_0$ and every $t \in I$.

Proof: The change-of-variables rule from Analysis III asserts $(\mathrm{D}\Phi := \mathrm{D}_x\Phi)$

$$\textstyle\int_{\Phi(t,A_0)} u(t,x)\,\mathrm{d}x = \int_{A_0} u(t,\Phi(t,x))\;|\det(\mathrm{D}\Phi(t,x))|\,\mathrm{d}x\,.$$

Differentiating for the occurrences of t on the right — for third one the next lemma — then yields (in short-hand notation and with $\partial_t\Phi = \vec{v}(\,\cdot\,,\Phi)$)

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\textstyle\int_{\Phi(t,A_0)} u\,\mathrm{d}x &= \textstyle\int_{A_0} \big[u_t + \nabla u\cdot\vec{v} + (u\,\mathrm{div}\,\vec{v})\big](\,\cdot\,,\Phi)\;|\det(\mathrm{D}\Phi)|\,\mathrm{d}x\\ &= \textstyle\int_{A_0} \big[u_t + \mathrm{div}(u\vec{v})\big](\,\cdot\,,\Phi)\;|\det(\mathrm{D}\Phi)|\,\mathrm{d}x\\ &= \textstyle\int_{\Phi(t,A_0)} \big[u_t + \mathrm{div}(u\vec{v})\big]\,\mathrm{d}x\,.\end{aligned}$$

□

Lemma for proof of Reynolds transport theorem

Lemma (Euler's identity in fluid mechanics/derivative of the Jacobian)

Under the hypotheses and in the notation of the previous theorems, it holds

$$\partial_t|\det(\mathrm{D}\Phi)| = (\operatorname{div}\vec{v})(\,\cdot\,,\Phi)\,|\det(\mathrm{D}\Phi)| \qquad \textit{in } I\times U_0\,.$$

Proof: By distinguishing between positive and negative sign of $\det(\mathrm{D}\Phi)$ reduce to proving the claim without absolute values. By expanding the determinant $\det A = \sum_{k=1}^n a_{ik}(\mathrm{adj}A)_{ki}$ of $A=(a_{ij})$ get $\frac{\partial(\det A)}{\partial a_{ij}} = (\mathrm{adj}A)_{ji}$. With this compute first

$$\partial_t(\det(\mathrm{D}\Phi)) = \sum_{i,j=1}^n(\mathrm{adj}(\mathrm{D}\Phi))_{ji}\partial_t(\mathrm{D}\Phi)_{ij} = \mathrm{Spur}(\mathrm{adj}(\mathrm{D}\Phi)\mathrm{D}\partial_t\Phi)$$

and then continue with $\mathrm{adj}A = A^{-1}\det A$ and $\partial_t\Phi = \vec{v}(\,\cdot\,,\Phi)$ to

$$\begin{aligned}\ldots &= \mathrm{Spur}[(\mathrm{D}\Phi)^{-1}\mathrm{D}(\vec{v}(\,\cdot\,,\Phi))]\det(\mathrm{D}\Phi)\\ &= \mathrm{Spur}[(\mathrm{D}\Phi)^{-1}\mathrm{D}\vec{v}(\,\cdot\,,\Phi)\mathrm{D}\Phi]\det(\mathrm{D}\Phi)\\ &= \mathrm{Spur}[\mathrm{D}\vec{v}(\,\cdot\,,\Phi)]\det(\mathrm{D}\Phi) = (\operatorname{div}\vec{v})(\,\cdot\,,\Phi)\det(\mathrm{D}\Phi)\,.\end{aligned}$$

So, the proof of the lemma (and the previous theorems) is complete. □

On interpretation and relevance of the continuity equation

Finally, we put on record the following observations on interpretation and relevance of the continuity equation $\boxed{u_t + \operatorname{div}(u\vec{v}) \equiv 0}$:

- The equation models conservation of mass or charge in physical systems. (For instance, conservation of charge $\rho_t + \operatorname{div}\vec{\jmath} \equiv 0$ is part of the Maxwell equations, as these imply $\rho_t = \varepsilon_0(\operatorname{div}\vec{E})_t = \varepsilon_0\operatorname{div}(\vec{E}_t) = -\operatorname{div}\vec{\jmath}$.)
- In case of constant density $u \equiv \text{const}$ the equation reduces to $\operatorname{div}\vec{v} \equiv 0$. (This occurs e.g. as incompressibility in Navier-Stokes/Euler equations.)
- In case of constant velocity $\vec{v} \equiv a \in \mathbb{R}^n$ the equation reduces to the linear transport equation $u_t + a \cdot \nabla u \equiv 0$.
- In case of $u\vec{v} = -C\,\nabla u$ with constant $C > 0$ the equation reduces to the diffusion or heat equation $u_t - C\Delta u \equiv 0$.
  (Here, $u\vec{v} = -C\,\nabla u$, for concentration or temperature u, has an interpretation as Fick's law of diffusion or Fourier's law of heat conduction, respectively.
  In the stationary case and for electric potential u, from $\operatorname{div}\vec{\jmath} \equiv 0$ and Ohm's law of conductivity $\vec{\jmath} = -C\nabla u$ in the same vein deduce the Laplace equation.)

2.2 The method of characteristics

The method of characteristics reduces scalar first-order PDE to certain underlying ODEs. This opens up a chance for explicitly determining solutions by methods of DE I.

Introductory example

In case of exemplary linear PDE (first-order, scalar, homogeneous)

$$\boxed{\underbrace{2\,\frac{\partial u}{\partial x}(x,y) - \frac{\partial u}{\partial y}(x,y)}_{=(2,-1)\cdot\nabla u(x,y)} -\ 4x\,u(x,y) = 0} \qquad \text{for } (x,y)\in\mathbb{R}^2$$

only the derivative of u in direction of the vector $(2,-1)$ does matter. Therefore, consider (parametrized) straight lines

$$\gamma_{(x_0,y_0)}(t) := (x_0,y_0) + t(2,-1) = (x_0{+}2t, y_0{-}t) \qquad \text{for } t\in\mathbb{R}$$

with arbitrary base point $(x_0,y_0)\in\mathbb{R}^2$ and direction vector $(2,-1)\in\mathbb{R}^2$. For $\nu_{(x_0,y_0)}(t) := u(x_0{+}2t, y_0{-}t)$ (i.e. u along the lines) observe

$$\big(\nu_{(x_0,y_0)}\big)'(t) = 2\,\frac{\partial u}{\partial x}(x_0{+}2t, y_0{-}t) - \frac{\partial u}{\partial y}(x_0{+}2t, y_0{-}t)\,.$$

Thus, the PDE for u yields the following ODE for ν:

$$\boxed{\nu'(t) - 4(x_0{+}2t)\,\nu(t) = 0} \qquad \text{for } t\in\mathbb{R}\,.$$

Introductory example (continued)

If the PDE is complemented with a Cauchy condition (e.g. an IC) which prescribes $u(x_0, y_0)$ for some $(x_0, y_0) \in \mathbb{R}^2$, one arrives at the IVP

$$\nu'(t) - 4(x_0{+}2t)\,\nu(t) = 0 \qquad \text{with IC } \nu(0) = u(x_0, y_0)\,.$$

A solution formula from DE I then gives the solution of the ODE IVP:

$$u(x_0{+}2t, y_0{-}t) = \nu_{(x_0,y_0)}(t) = u(x_0, y_0)\,\mathrm{e}^{(x_0+2t)^2 - x_0^2}\,.$$

If specifically one is concerned with an IC of simple form

$$u(x, 0) = u_0(x)\,,$$

one can use the preceding result for $y_0 = 0$ and exploit $x = x_0{+}2t$, $y = y_0{-}t = -t$ to determine the solution of the PDE IVP:

$$u(x, y) = u_0(x{+}2y)\,\mathrm{e}^{x^2-(x+2y)^2} = u_0(x{+}2y)\,\mathrm{e}^{-4y^2-4xy}\,.$$

Flow lines/characteristics

The general scalar linear first-order PDE is

$$\underbrace{\sum_{i=1}^{n} a_i(x)\,\frac{\partial u}{\partial x_i}(x)}_{=\,a(x)\cdot\nabla u(x)} + b(x)\,u(x) = f(x) \qquad \text{for } x \in \Omega$$

(in open $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with $a \in \mathrm{C}^1(\Omega, \mathbb{R}^n)$, $b, f \in \mathrm{C}^0(\Omega)$).

The straight lines in the example replace by flow lines/characteristic curves γ_{x_0} of the field a: By the Picard-Lindelöf theorem in DE I, for every $x_0 \in \Omega$, there exists a unique solution $\gamma_{x_0} \in \mathrm{C}^1(I_{x_0}, \Omega)$ to the nonlinear ODE IVP

$$\gamma'(t) = a(\gamma(t)) \text{ for } t \in I_{x_0} \qquad \text{with IC } \gamma(0) = x_0$$

on a maximal existence interval I_{x_0} for a solution with values in Ω. Flow lines never touch or intersect each other (but it is $\gamma_{x_0}(t) = \gamma_{\gamma_{x_0}(s)}(t-s)$).

(By the way: The collection $\Phi(t, x_0) := \gamma_{x_0}(t)$ of all γ_{x_0} is called the flow of a. In these terms the ODEs read $\partial_t \Phi(t, x_0) = a(\Phi(t, x_0))$ and the ICs $\Phi(0, x_0) = x_0$.)

The method of characteristics in the linear case

Principle (method of characteristics for linear PDEs)

For Ω, a, b, f, γ_{x_0}, I_{x_0} *as before,* $u \in \mathrm{C}^1(\Omega)$, *the following are* **equivalent**:

(1) u *solves the linear PDE* $a(x) \cdot \nabla u(x) + b(x)u(x) = f(x)$ *for* $x \in \Omega$.

(2) *For each* $x_0 \in \Omega$, ν_{x_0} *with* $\boxed{\nu_{x_0}(t) := u(\gamma_{x_0}(t))}$ *solves the linear ODE*

$$\nu'(t) + b(\gamma_{x_0}(t))\,\nu(t) = f(\gamma_{x_0}(t)) \qquad \text{for } t \in I_{x_0}\,.$$

This means: The PDE reduces to ODEs along the flow lines.

Proof: From $\nu_{x_0}(t) = u(\gamma_{x_0}(t))$ deduce by chain rule and ODEs for the flow lines (compare with the introductory example)

$$\nu_{x_0}'(t) = \gamma_{x_0}'(t) \cdot \nabla u(\gamma_{x_0}(t)) = a(\gamma_{x_0}(t)) \cdot \nabla u(\gamma_{x_0}(t))\,.$$

Hence, the PDE evaluated at points $x = \gamma_{x_0}(t)$ yields the ODEs, and vice versa the ODEs also yield the PDE, since each $x \in \Omega$ can be written as $x = \gamma_{x_0}(t)$ (in fact $x = \gamma_x(0)$, but can also use restricted x_0 as on next slide). $\square$

Cauchy conditions and general line of approach

The reasonable complement to scalar first-order PDEs is the Cauchy condition

$$u(x) = u_0(x) \qquad \text{for } x \in S\,.$$

Here given: curve ($n{=}2$), surface ($n{=}3$), and generally hypersurface $S \subset \Omega$ such that S intersects each flow line exactly once; function $u_0\colon S \to \mathbb{R}$ on S.

If the PDE is complemented this way, consider the ODEs for γ and ν only for $x_0 \in S$ and complement the latter one with the corresponding IC

$$\nu(0) = u_0(x_0)\,.$$

General line of approach for method of characteristic then:

- Solve IVP for flow lines γ_{x_0} with $x_0 \in S$.
- Solve IVP for ν_{x_0} with $x_0 \in S$, get $u(\gamma_{x_0}(t)) {=} \nu_{x_0}(t)$ as term in t and x_0.
- Solve $x = \gamma_{x_0}(t)$ for (t, x_0), get solution $u(x)$ as term in x.

Example for method of characteristics (linear case)

As an example, consider the Cauchy problem for a linear PDE

$$y\,\frac{\partial u}{\partial x}(x,y) - x\,\frac{\partial u}{\partial y}(x,y) + u(x,y) = 0 \qquad \text{for } (x,y) \in (0,\infty)\times\mathbb{R}\,,$$
$$u(x,0) = \mathrm{e}^{-x^2} \qquad \text{for } x \in (0,\infty)\,.$$

Solve successively (where, for $x_0 > 0$, we abbreviate $\gamma_{x_0} = \gamma_{(x_0,0)}$, $\nu_{x_0} = \nu_{(x_0,0)}$):

$\overset{\text{read off}}{\rightsquigarrow}$ IVP for flow lines: $\gamma'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\gamma(t)$ with IC $\gamma(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$

$\overset{\text{solve}}{\rightsquigarrow}$ flow lines: $\gamma_{x_0}(t) = \begin{pmatrix} x_0 \cos t \\ -x_0 \sin t \end{pmatrix}$ for $|t| < \frac{\pi}{2}$

$\overset{\text{read off}}{\rightsquigarrow}$ IVP for ν_{x_0}: $\nu'(t) + \nu(t) = 0$ with IC $\nu(0) = \mathrm{e}^{-x_0^2}$

$\overset{\text{solve}}{\rightsquigarrow}$ solution: $u(\gamma_{x_0}(t)) = \nu_{x_0}(t) = \mathrm{e}^{-t-x_0^2}$

Finally, solve $\begin{pmatrix} x \\ y \end{pmatrix} = \gamma_{x_0}(t)$ for $t = -\arctan(y/x)$ and $x_0 = \sqrt{x^2+y^2}$.

$\overset{\text{plug in}}{\rightsquigarrow}$ solution Cauchy problem for PDE: $u(x,y) = \mathrm{e}^{\arctan(y/x) - x^2 - y^2}$

Method of characteristics for linear transport equation

As another example, consider the IVP for the linear transport equation

$$\begin{aligned} u_t(t,x) + a(t,x)\cdot\nabla u(t,x) &= 0 && \text{for } (t,x)\in(0,T)\times\mathbb{R}^n \\ u(0,x) &= u_0(x) && \text{for } x\in\mathbb{R}^n \end{aligned}$$

Approach then (where, for $x_0 \in \mathbb{R}^n$, we abbreviate $\gamma_{x_0} = \gamma_{(0,x_0)}$, $\nu_{x_0} = \nu_{(0,x_0)}$):

$\overset{\text{read off}}{\rightsquigarrow}$ IVP for flow lines: $\gamma'(t) = (1, a(\gamma(t)))$ with IC $\gamma(0) = (0, x_0)$

$\overset{\text{leads to}}{\rightsquigarrow}$ $\gamma_{x_0}(t) = (t, \widetilde{\gamma}_{x_0}(t))$ and $\widetilde{\gamma}'(t) = a(t, \widetilde{\gamma}(t))$ with IC $\widetilde{\gamma}(0) = x_0$

$\overset{\text{read off}}{\rightsquigarrow}$ IVP for ν_{x_0}: $\nu'(t) = 0$ with IC $\nu(0) = u_0(x_0)$

$\overset{\text{solve}}{\rightsquigarrow}$ $u(t, \widetilde{\gamma}_{x_0}(t)) = \nu_{x_0}(t) = u_0(x_0)$, i.e. u constant along flow lines

It remains to solve $x = \widetilde{\gamma}_{x_0}(t)$. This works, for instance, for constant a with correspondingly $\widetilde{\gamma}_{x_0}(t) = x_0+ta$, and then transforms $u(t, x_0+ta) = u_0(x_0)$ into the solution formula $u(t,x) = u_0(x-ta)$ already known from Chapter 1.

References (preliminary; to be extended)

Lecture notes:

- beamer slides by H.J. Oberle and J. Struckmeier.
- own notes „Partial Differential Equations I“.

English Books:

- L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.