Differential Equations II for Students of Engineering Partial Differential Equations

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## Chapter 1: Basics, Terminology, Examples

The study of partial differential equations is a wide field and encompasses various, entirely different theories and aspects.

Thus, this lecture can merely give a very basic introduction into the general topic and in fact focuses on treating some specific equations as model cases.

# 1.1 Terminology for PDEs

#### Terminology (for partial derivatives)

For a function  $u: \Omega \to \mathbb{R}^q$  in n variables  $x = (x_1, x_2 \dots, x_n) \in \Omega \subset \mathbb{R}^n$ with arbitrary  $n, q \in \mathbb{N}$  agree on notation for

• all first-order partial derivatives (Jacobi matrix; gradient if q = 1):

$$\mathrm{D}u := \mathrm{J}u := \left(\frac{\partial u}{\partial x_i}\right)_{i=1,2,\dots,n} = (\partial_i u)_{i=1,2,\dots,n} \,,$$

• all second-order partial derivatives (Hessian if q = 1):

$$\mathbf{D}^2 u := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n} = (\partial_i \partial_j u)_{i,j=1,\dots,n} \,,$$

• all *k*th-order partial derivatives with arbitrary  $k \in \mathbb{N}$ :

$$\mathbf{D}^{k}u := \left(\frac{\partial^{k}u}{\partial x_{i_{k}}\dots\partial x_{i_{1}}}\right)_{i_{1},\dots,i_{k}=1,\dots,n} = (\partial_{i_{k}}\dots\partial_{i_{1}}u)_{i_{1},\dots,i_{k}=1,\dots,n}.$$

## General form of partial differential equations

Definition (partial differential equation)

A partial differential equation (in brief: PDE or partial DE) is an equation with partial derivatives up to order  $m \ge 1$  in form

$$F(x,u(x),\mathrm{D} u(x),\mathrm{D}^2 u(x),\ldots,\mathrm{D}^m u(x))=0 \quad \text{ for all } x\in\Omega$$

or in brief functional notation

$$F(\,\cdot\,,u,\mathrm{D} u,\mathrm{D}^2 u,\ldots,\mathrm{D}^m u)\equiv 0\quad\text{in }\Omega$$

for an unknown function  $u: \Omega \to \mathbb{R}^q$  on an open set  $\Omega \subset \mathbb{R}^n$ ,  $n \ge 2$ . If u solves the equation, one calls u a solution to the PDE in  $\Omega$ .

The decisive difference to ODEs is that  $x = (x_1, x_2, ..., x_n)$  contains not only one, but multiple (in fact  $n \ge 2$ ) variables.

## Terminology in connection with PDEs

Terminology for PDE  $F(\cdot, u, Du, D^2u, \dots, D^mu) \equiv 0$  in  $\Omega \subset \mathbb{R}^n$ :

*m*: order of the PDE (provided  $D^m u$  indeed occurs),

- *n*: number of variables (recall  $n \ge 2$ ),
- q: number of (component) functions (of  $u: \Omega \to \mathbb{R}^q$ ),
- N: number of (component) equations (of PDE with  $_{n}\equiv$  " in  $\mathbb{R}^{N}$ ),
- *F*: given structure function of the PDE (from suitable domain to  $\mathbb{R}^N$ ).

In this lecture the focus is on the case N = q = 1 (scalar PDE for single function) with order  $m \in \{1, 2\}$ . Taking  $N = q \ge 2$  (PDE system for multiple functions) is also reasonable, but here mostly beyond the scope.

#### Boundary conditions

One expects unique solutions only for boundary value problems (BVPs) out of PDEs and additional boundary conditions (BCs) at  $\partial\Omega$ . As a rough rule of thumb a PDE system of order m for N = q functions needs  $\frac{mq}{2}$  BCs (where "half BCs" concern a part of the boundary only, similar to ICs for ODEs).

Common BCs are (variants of) Dirichlet BCs

$$u(x) = g(x)$$
 for  $x \in \partial \Omega$ 

with given function  $g: \partial \Omega \to \mathbb{R}^q$  and Neumann BCs

$$\partial_{\nu}u(x) = \psi(x) \qquad \text{for } x \in \partial\Omega$$

with outward unit normal field  $\nu: \partial\Omega \to \mathbb{R}^n$  to  $\partial\Omega$ , normal derivative  $\partial_{\nu}u(x) := Ju(x)\nu(x)$ , and given function  $\psi: \partial\Omega \to \mathbb{R}^q$  and beside these also initial conditions (ICs)/Cauchy conditions (soon more on these).

#### Classification of PDEs

Similar to ODEs one classifies PDEs of order m as follows:

- Autonomous PDEs take the form  $F_0(u, Du, D^2u, \dots, D^mu) \equiv 0$ .
- Linear PDEs exhibit an affine dependence on u, Du,  $D^2u$ , ...  $D^mu$ . The possibly *x*-dependent factors in front of u and its derivatives are then called coefficients, while terms independent of u and its derivatives are collected on the right-hand side as inhomogeneity.

Among non-linear PDEs one further distinguishes:

- Semilinear PDEs depend affinely on  $D^m u$  with coefficients which depend solely on x in front of the mth derivatives.
- Quasilinear PDEs depend affinely on  $D^m u$  (in general with coefficients which depend on  $(\cdot, u, Du, \dots, D^{m-1}u)$  in front of the *m*th derivatives).
- Fully non-linear PDEs are not quasilinear.

## Relevant types of PDEs

In this lecture, relevant types of scalar PDEs (for  $u: \Omega \to \mathbb{R}$ ) are:

• linear first-order PDEs (with coefficients  $a_i, b: \Omega \to \mathbb{R}$ ):

$$\sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}(x) + b(x)u(x) = f(x) \,.$$

• linear second-order PDEs (with coefficients  $a_{i,j}, b_i, c \colon \Omega \to \mathbb{R}$ ):

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_j \partial x_i}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) = f(x).$$

• semilinear first-order PDEs (with  $a_i \colon \Omega \to \mathbb{R}$  and  $b \colon \Omega \times \mathbb{R} \to \mathbb{R}$ ):

$$\sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}(x) = b(x, u(x)).$$

• quasilinear first-order PDEs (with  $a_i, b: \Omega \times \mathbb{R} \to \mathbb{R}$ ):

$$\sum_{i=1}^{n} a_i(x, u(x)) \frac{\partial u}{\partial x_i}(x) = b(x, u(x)).$$

# 1.2 Various examples of PDEs

In the sequel, various examples from the "zoo" of important PDEs are briefly discussed together with suitable BCs and interpretations. The fundamentally different interpretations and applications underline the extremely wide scope of PDE theory.

If no other indication is given, the examples are scalar PDEs for a single function.

#### Transport equation

Linear transport equation for  $u \colon [0,T) \times \mathbb{R}^n \to \mathbb{R}$ :

$$\frac{\partial u}{\partial t}(t,x) + a(t,x) \cdot \nabla_x u(t,x) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathbb{R}^n$$

with given T > 0 and  $a \colon (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$  ("•" is the inner product).

Typical feature: Occurrence of time variable  $t \in [0, T)$  and space variables  $x \in \mathbb{R}^n$ . Often one writes only  $\nabla u$ , but still with the meaning of  $\nabla_x u$ .

Classification: first-order, linear, homogeneous.

Reasonably complemented with IC ("half BC";  $u_0 \colon \mathbb{R}^n \to \mathbb{R}$  given):

$$u(0,x) = u_0(x)$$
 for  $x \in \mathbb{R}^n$ .

Interpretation: Solutions u model the density of mass or of electric charge, which is transported along the field a. Specifically, constant a gives rise to uniform drift  $u(t, x) = u_0(x-ta)$  with velocity  $a \in \mathbb{R}^n$ .

# Cauchy-Riemann equations

Cauchy-Riemann equations for  $f, g \colon \overline{\Omega} \to \mathbb{R}$  in variables (x, y):

$$\left. \begin{array}{l} \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \equiv 0 \,, \\ \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} \equiv 0 \end{array} \right\} \text{ in } \Omega \subset \mathbb{R}^2$$

Classification: system of 2 equations, first-order, linear, homogeneous.

Meaning: When identifying  $\mathbb{C} \ni x + \mathbf{i}y \cong (x, y) \in \mathbb{R}^2$  characterizes the holomorphic (i.e. complex differentiable) functions  $f + \mathbf{i}g \colon \Omega \to \mathbb{C}$  on  $\Omega \subset \mathbb{C}$ . More in lecture "Complex Functions"!

Reasonably complemented with Dirichlet BC for either f or g at  $\partial \Omega$  (though this leaves free an additive constant for the other function).

#### Laplace equation and Poisson equation

Laplace equation and Poisson or potential equation for  $u \colon \overline{\Omega} \to \mathbb{R}$ :

$$\Delta u(x) = 0$$
 rsp.  $\Delta u(x) = f(x)$  for  $x \in \Omega \subset \mathbb{R}^n$ 

with given  $f \colon \Omega \to \mathbb{R}$  and with the important Laplace operator

$$\Delta u(x) := \operatorname{div}(\nabla u)(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) = \operatorname{trace}(\mathrm{D}^2 u(x)).$$

Solutions of Laplace's equation are also known as harmonic functions.

Classification: second-order, linear, homogeneous rsp. inhomogeneous.

Reasonably complemented with Dirichlet BC or Neumann BC for u at  $\partial \Omega$ .

Meaning/interpretation: Characterizes real and imaginary parts of holomorphic functions. Solutions u model electric potential for charge density  $f/\varepsilon_0$  (with physical constant  $\varepsilon_0 > 0$ ).

## Diffusion or heat equation

Diffusion or heat equation for  $u: [0,T) \times \overline{\Omega} \to \mathbb{R}$ :

$$\frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) = 0 \qquad \text{for } (t,x) \in \Omega_T \subset \mathbb{R} \times \mathbb{R}^n \,,$$

again with time and space variables and with abbreviation  $S_T := (0,T) \times S$ . Classification: second-order, linear, homogeneous (has inhomogeneous variant). Complement e.g. with IC and Dirichlet BC ( $\rightsquigarrow 1$  BC at "parabolic boundary")  $u(0,x) = u_0(x)$  for  $x \in \Omega$ , u(t,x) = g(t,x) for  $(t,x) \in (\partial\Omega)_T$ 

for given  $u_0: \Omega \to \mathbb{R}$  and  $g: (\partial \Omega)_T \to \mathbb{R}$ .

Interpretation: Solutions u model the mass density/concentration in diffusion processes or the temperature in heat propagation.

In stationary case  $\frac{\partial u}{\partial t} \equiv 0$  get back Laplace equation.

## Navier-Stokes equations

Incompressible Navier-Stokes equations for  $(\vec{v}, p) \colon [0, T) \times \overline{\Omega} \to \mathbb{R}^n \times \mathbb{R}$ :

with constants  $\rho, \mu > 0$ .

Classification: system of n+1 equations, second-order, semilinear.

Reasonable BCs as for diffusion equation (also known as no-slip BCs).

Interpretation: Solutions  $(\vec{v}, p)$  model velocity and pressure in the flow of an incompressible fluid of constant density  $\rho$  and constant viscosity  $\mu$ . Foundational in fluid mechanics!

Specifically, for  $\mu = 0$ , reduces to Euler equations in fluid mechanics and in case  $\frac{\partial \vec{v}}{\partial t} \equiv 0$  gives stationary Navier-Stokes and Euler equations, respectively.

#### Wave equation

Wave equation for  $u \colon \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ :

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta_x u(t,x) = 0 \qquad \text{for } (t,x) \in \mathbb{R} \times \Omega \subset \mathbb{R} \times \mathbb{R}^n$$

Classification: second-order, linear, homogeneous (has inhomogeneous variant).

Complement e.g. with 2 ICs and Dirichlet BC (still to be seen as 1 BC overall)

$$\begin{split} u(0,x) &= u_0(x) \text{ for } x \in \Omega \,, \qquad \frac{\partial u}{\partial t}(0,x) = v_0(x) \text{ for } x \in \Omega \,, \\ u(t,x) &= g(t,x) \text{ for } (t,x) \in \mathbb{R} \times \partial \Omega \end{split}$$

for given  $u_0, v_0 \colon \Omega \to \mathbb{R}$  and  $g \colon \mathbb{R} \times \partial \Omega \to \mathbb{R}$ .

Interpretation: Solutions u model displacements in wave propagation and/or in oscillations.

In stationary case  $\frac{\partial u}{\partial t} \equiv 0$  get back Laplace equation.

# Schrödinger equation

Schrödinger equation for  $\psi \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ :

$$\mathbf{i}\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar}{2m}\Delta_x\psi - V\psi \equiv 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^n$$

with given  $V \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$  and constants  $\hbar, m > 0$ .

Classification: scalar/system over  $\mathbb{C}/\mathbb{R}$ , second-order, linear, homogeneous.

Reasonably complemented with IC ( $\psi_0 \colon \mathbb{R}^n \to \mathbb{C}$  given):

$$\psi(0, \cdot) = \psi_0 \quad \text{in } \mathbb{R}^n.$$

Interpretation: Solutions  $\psi$  are wavefunctions (quantum states) of particle of mass m in potential V (with reduced Planck constant  $\hbar$ ). Foundational for quantum mechanics!

Product-exponential ansatz sometimes yields eigenvalue problem for  $\Delta_x$ .

#### Maxwell equations

Vacuum Maxwell equations for 
$$(\vec{E}, \vec{B}) \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$$
:  
 $\varepsilon_0 \operatorname{div}_x \vec{E}(t, x) = \rho(t, x),$   
 $\frac{\partial \vec{B}}{\partial t}(t, x) + \operatorname{rot}_x \vec{E}(t, x) = 0,$   
 $\operatorname{div}_x \vec{B}(t, x) = 0,$   
 $\varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}(t, x) - \operatorname{rot}_x \vec{B}(t, x) = -\mu_0 \vec{j}(t, x)$ 
for  $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ 

with given  $(\rho, \vec{j}) \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$  and constants  $\varepsilon_0, \mu_0 > 0$ .

Classification: 8 component equations for 6 component functions (okay only since rot strongly degenerate;  $rot \circ \nabla \equiv 0$ ,  $div \circ rot \equiv 0$ ), linear, inhomogeneous.

Complement with ICs  $\vec{E}(0,x) = \vec{E}_0(x)$  and  $\vec{B}(0,x) = \vec{B}_0(x)$  for given  $(\vec{E}_0, \vec{B}_0) \colon \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$  s.t.  $\varepsilon_0 \operatorname{div} \vec{E}_0 = \rho(0, \cdot)$  and  $\operatorname{div} \vec{B}_0 \equiv 0$  in  $\mathbb{R}^3$ .

Interpretation: These four basic equations of electrodynamics determine the electric field  $\vec{E}$  and the magnetic field  $\vec{B}$  from given electric charge density  $\rho$  and electric current density  $\vec{j}$ .

## Minimal surface equation

Minimal surface equation for  $u \colon \overline{\Omega} \to \mathbb{R}$ :

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0\qquad\text{in }\Omega\subset\mathbb{R}^n\,.$$

Classification: second-order, quasilinear.

Usually complemented with Dirichlet BC for u at  $\partial \Omega$  or certain free BCs.

Interpretation: Graphs of solutions u are minimal surfaces, which have zero mean curvature at each of their points and are relevant objects in geometric analysis and differential geometry.

## Monge-Ampère equation

Monge-Ampère equation for  $u \colon \overline{\Omega} \to \mathbb{R}$ :

 $\det \left( \mathbf{D}^2 u(x) \right) = f(x) \qquad \text{for } x \in \Omega \subset \mathbb{R}^n$ 

with given (often everywhere positive)  $f: \Omega \to \mathbb{R}$ .

Classification: second-order, fully non-linear.

Reasonable with Dirichlet BC or Neumann BC or certain natural BC.

Applications: Solutions u are connected with optimal transport of mass distributions and with surfaces of prescribed Gauss curvature.

## Key aspects of the lecture

The focus is now on treating in more detail the following very illustrative model cases:

- general first-order PDEs including the transport equation,
- Laplace and Poisson equation (including eigenvalue problems),
- diffusion or heat equation,
- and wave equation.

# **Chapter 2: First-Order PDEs**

First-order PDEs occur in different applications, but mostly describe a time evolution, which starts at a certain IC. In general one has a better chance for explicitly solving or analyzing first-order PDEs than one has in case of second-order and higher-order PDEs.

In this chapter we first discuss different aspects of a central application context and only eventually approach a comparably general solution theory and some specific cases.

## 2.1 The continuity equation (and its background)

Here always use time/space variables (t, x) plus abbreviations  $u_t := \frac{\partial u}{\partial t}$ and  $\operatorname{div}(\ldots) := \operatorname{div}_x(\ldots)$ . The continuity equation is the linear PDE

 $u_t + \operatorname{div}(u\vec{v}) \equiv 0$  in open  $U \subset \mathbb{R} \times \mathbb{R}^n$ 

for an unknown function  $u: U \to \mathbb{R}$  and a given or *u*-dependent velocity field  $\vec{v}: U \to \mathbb{R}^n$  (both functions in variables  $(t, x) \in U$ ).

Interpretation: If u is the density of quantity (often mass), which moves according to  $\vec{v}$ , then at time t and in a point x the temporal rate of change  $u_t(t,x)$  equals the spatial in/outflow density  $-\operatorname{div}(u\vec{v})(t,x)$  $(\operatorname{div}(u\vec{v})>0 \rightsquigarrow \operatorname{source/outflow} \operatorname{density}; \operatorname{div}(u\vec{v})<0 \rightsquigarrow \operatorname{sink/inflow} \operatorname{density}).$ 

In 1d case n = 1, which is already of interest, get simply

$$u_t + (uv)_x \equiv 0$$
 in  $U \subset \mathbb{R} \times \mathbb{R}$ .

#### Continuity equation and conservation of mass

To underpin the interpretation consider the trajectory  $t \mapsto \Phi(t, x)$  of a particle, which starts at time t = 0 at  $x \in U_0$ . (Mathematically consider  $\Phi \in C^2(I \times U_0, \mathbb{R}^n)$ ,  $\Phi(0, x) = x$  for  $x \in U_0$ , with open  $0 \in I \subset \mathbb{R}$ ,  $U_0 \subset \mathbb{R}^n$ .) Then obtain the moving domain  $U := \{(t, \Phi(t, x)) : t \in I, x \in U_0\}$  and the velocity field  $\vec{v}$  of  $\Phi$  in U, given by

 $\vec{v}(t,\Phi(t,x))=\partial_t\Phi(t,x)\qquad \text{for }(t,x)\in I\times U_0\,.$ 

#### Theorem (Continuity equation and conservation of mass)

In the above setting, if  $x \mapsto \Phi(t, x)$  is a diffeomorphism for each  $t \in I$ , then, for  $u \in C^1(U)$ , the following are equivalent:

- (1) u solves the continuity equation  $u_t + \operatorname{div}(u\vec{v}) \equiv 0$  in U.
- (2) Conservation of mass in moving domains: There holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Phi(t,A_0)}u(t,x)\,\mathrm{d}x=0$$

for each compact and measurable subset  $A_0 \subset U_0$  and every  $t \in I$ .

## Continuity equation and conservation of mass (continued)

Complementary remarks:

- In the theorem and in the sequel, compact sets are closed and bounded, while measurable sets are (Jordan) measurable in the sense of Analysis III.
- The hypotheses of the theorem are satisfied for the trajectories of an ODE system in many (good) cases, but the derivation needs some more ODE theory.

**Proof of the theorem on continuity equation and mass conservation:** The Reynolds transport theorem for derivation on moving domains (see next slide) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Phi(t,A_0)} u(t,x) \,\mathrm{d}x = \int_{\Phi(t,A_0)} \left[ u_t(t,x) + \operatorname{div}(u\vec{v})(t,x) \right] \mathrm{d}x$$

for  $A_0$  as in (2) and  $t \in I$ . Thus, (1)  $\Longrightarrow$  (2) is evident. Now suppose that (2) holds. As every compact and measurable subset  $B_t \subset \Phi(t, U_0)$  has the form  $B_t = \Phi(t, A_0)$ , it is  $\int_{B_t} [\dots] dx = 0$  for each such  $B_t$ . So, one deduces  $[\dots] = 0$  for all  $x \in \Phi(t, U_0)$  and altogether for all  $(t, x) \in U$ .  $\Box$ 

# The Reynolds transport theorem

Theorem (Reynolds' transport theorem) Under the hypotheses of the previous theorem, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Phi(t,A_0)} u(t,x) \,\mathrm{d}x = \int_{\Phi(t,A_0)} \left[ u_t(t,x) + \mathrm{div}(u\vec{v})(t,x) \right] \mathrm{d}x$$

for each compact and measurable subset  $A_0 \subset U_0$  and every  $t \in I$ .

**Proof:** The change-of-variables rule from Analysis III asserts  $(D\Phi := D_x \Phi)$ 

$$\int_{\Phi(t,A_0)} u(t,x) \, \mathrm{d}x = \int_{A_0} u(t,\Phi(t,x)) \, \left| \det(\mathrm{D}\Phi(t,x)) \right| \, \mathrm{d}x \, .$$

Differentiating for the occurrences of t on the right — for third one the next lemma — then yields (in short-hand notation and with  $\partial_t \Phi = \vec{v}(\cdot, \Phi)$ )

$$\begin{split} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Phi(t,A_0)} u \,\mathrm{d}x &= \int_{A_0} \left[ u_t + \nabla u \cdot \vec{v} + (u \operatorname{div} \vec{v}) \right] (\,\cdot\,,\Phi) \,\left| \operatorname{det}(\mathrm{D}\Phi) \right| \,\mathrm{d}x \\ &= \int_{A_0} \left[ u_t + \operatorname{div}(u\vec{v}) \right] (\,\cdot\,,\Phi) \,\left| \operatorname{det}(\mathrm{D}\Phi) \right| \,\mathrm{d}x \\ &= \int_{\Phi(t,A_0)} \left[ u_t + \operatorname{div}(u\vec{v}) \right] \,\mathrm{d}x \,. \end{split}$$

## Lemma for proof of Reynolds transport theorem

Lemma (Euler's identity in fluid mechanics/derivative of the Jacobian) Under the hypotheses and in the notation of the previous theorems, it holds  $\partial_t |\det(D\Phi)| = (\operatorname{div} \vec{v})(\cdot, \Phi) |\det(D\Phi)|$  in  $I \times U_0$ .

**Proof:** By distinguishing between positive and negative sign of det(D $\Phi$ ) reduce to proving the claim without absolute values. By expanding the determinant det  $A = \sum_{k=1}^{n} a_{ik} (\operatorname{adj} A)_{ki}$  of  $A = (a_{ij})$  get  $\frac{\partial(\det A)}{\partial a_{ij}} = (\operatorname{adj} A)_{ji}$ . With this compute first  $\partial_t (\det(D\Phi)) = \sum_{i,j=1}^{n} (\operatorname{adj}(D\Phi))_{ji} \partial_t (D\Phi)_{ij} = \operatorname{trace}(\operatorname{adj}(D\Phi)D\partial_t\Phi)$ and then continue with  $\operatorname{adj} A = A^{-1} \det A$  and  $\partial_t \Phi = \vec{v}(\cdot, \Phi)$  to  $\dots = \operatorname{trace}[(D\Phi)^{-1}D(\vec{v}(\cdot, \Phi))] \det(D\Phi)$  $= \operatorname{trace}[(D\Phi)^{-1}D\vec{v}(\cdot, \Phi)D\Phi] \det(D\Phi)$ 

 $= \operatorname{trace}[\mathrm{D}\vec{v}(\,\cdot\,,\Phi)]\det(\mathrm{D}\Phi) = (\operatorname{div}\vec{v})(\,\cdot\,,\Phi)\det(\mathrm{D}\Phi)\,.$ 

So, the proof of the lemma (and the previous theorems) is complete.

### On interpretation and relevance of the continuity equation

Finally, we put on record the following observations on interpretation and relevance of the continuity equation  $u_t + \operatorname{div}(u\vec{v}) \equiv 0$ :

- The equation models conservation of mass or charge in physical systems. (For instance, conservation of charge  $\rho_t + \operatorname{div} \vec{j} \equiv 0$  is part of the Maxwell equations, as these imply  $\rho_t = \varepsilon_0 (\operatorname{div} \vec{E})_t = \varepsilon_0 \operatorname{div} (\vec{E}_t) = -\operatorname{div} \vec{j}$ .)
- In case of constant density u ≡ const the equation reduces to div v ≡ 0. (This occurs e.g. as incompressibility in Navier-Stokes/Euler equations.)
- In case of constant velocity  $\vec{v} \equiv a \in \mathbb{R}^n$  the equation reduces to the linear transport equation  $u_t + a \cdot \nabla u \equiv 0$ .
- In case of uv = -C∇u with constant C > 0 the equation reduces to the diffusion or heat equation ut C∆u ≡ 0. (Here, uv = -C∇u, for concentration or temperature u, has an interpretation as Fick's law of diffusion or Fourier's law of heat conduction, respectively. In the stationary case and for electric potential u, from div j ≡ 0 and Ohm's law of conductivity j = -C∇u in the same vein deduce the Laplace equation.)

## 2.2 The method of characteristics

The method of characteristics reduces scalar first-order PDE to certain underlying ODEs. This opens up a chance for explicitly determining solutions by methods of DE I.

#### Introductory example

In case of exemplary linear PDE (first-order, scalar, homogeneous)

$$2\frac{\partial u}{\partial x}(x,y) - \frac{\partial u}{\partial y}(x,y) - 4x u(x,y) = 0 \qquad \text{for } (x,y) \in \mathbb{R}^2$$
$$\underbrace{= (2,-1) \cdot \nabla u(x,y)}$$

only the derivative of u in direction of the vector (2, -1) does matter. Therefore, consider (parametrized) straight lines

$$\gamma_{(x_0,y_0)}(t) := (x_0,y_0) + t(2,-1) = (x_0+2t,y_0-t) \quad \text{for } t \in \mathbb{R}$$

with arbitrary base point  $(x_0, y_0) \in \mathbb{R}^2$  and direction vector  $(2, -1) \in \mathbb{R}^2$ . For  $\nu_{(x_0, y_0)}(t) := u(x_0 + 2t, y_0 - t)$  (i.e. u along the lines) observe

$$\left(\nu_{(x_0,y_0)}\right)'(t) = 2\frac{\partial u}{\partial x}(x_0+2t,y_0-t) - \frac{\partial u}{\partial y}(x_0+2t,y_0-t).$$

Thus, the PDE for u yields the following ODE for  $\nu$ :

$$u'(t) - 4(x_0+2t) \nu(t) = 0$$
 for  $t \in \mathbb{R}$ .

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#### Introductory example (continued)

If the PDE is complemented with a Cauchy condition (e.g. an IC) which prescribes  $u(x_0, y_0)$  for some  $(x_0, y_0) \in \mathbb{R}^2$ , one arrives at the IVP

$$u'(t) - 4(x_0+2t) \nu(t) = 0 \quad \text{with IC } \nu(0) = u(x_0, y_0).$$

A solution formula from DE I then gives the solution of the ODE IVP:

$$u(x_0+2t, y_0-t) = \nu_{(x_0, y_0)}(t) = u(x_0, y_0) e^{(x_0+2t)^2 - x_0^2}$$

If specifically one is concerned with an IC of simple form

$$u(x_0,0) = u_0(x_0) \qquad \text{for } x_0 \in \mathbb{R} \,,$$

one can use the preceding result for  $y_0 = 0$  and exploit  $x = x_0+2t$ ,  $y = y_0-t = -t$  to determine the solution of the PDE IVP:

$$u(x,y) = u_0(x+2y) e^{x^2 - (x+2y)^2} = u_0(x+2y) e^{-4y^2 - 4xy}.$$

## Flow lines/characteristics

The general scalar linear first-order PDE is

$$\sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}(x) + b(x) u(x) = f(x) \quad \text{for } x \in \Omega$$
$$= a(x) \cdot \nabla u(x)$$

(in open  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , with  $a \in C^1(\Omega, \mathbb{R}^n)$ ,  $b, f \in C^0(\Omega)$ ).

The straight lines in the example replace by flow lines/characteristic curves  $\gamma_{x_0}$  of the field a: By the Picard-Lindelöf theorem in DE I, for every  $x_0 \in \Omega$ , there exists a unique solution  $\gamma_{x_0} \in C^1(I_{x_0}, \Omega)$  to the nonlinear ODE IVP

$$\gamma'(t) = a(\gamma(t))$$
 for  $t \in I_{x_0}$  with IC  $\gamma(0) = x_0$ 

on a maximal existence interval  $I_{x_0}$  for a solution with values in  $\Omega$ . Flow lines never touch or intersect each other (but it is  $\gamma_{x_0}(t) = \gamma_{\gamma_{x_0}(s)}(t-s)$ ).

(By the way: The collection  $\Phi(t, x_0) := \gamma_{x_0}(t)$  of all  $\gamma_{x_0}$  is called the flow of a. In these terms the ODEs read  $\partial_t \Phi(t, x_0) = a(\Phi(t, x_0))$  and the ICs  $\Phi(0, x_0) = x_0$ .)

#### The method of characteristics in the linear case

Principle (method of characteristics for linear PDEs)

For  $\Omega$ , a, b, f,  $\gamma_{x_0}$ ,  $I_{x_0}$  as before,  $u \in C^1(\Omega)$ , the following are equivalent: (1) u solves the linear PDE  $a(x) \cdot \nabla u(x) + b(x)u(x) = f(x)$  for  $x \in \Omega$ . (2) For each  $x_0 \in \Omega$ ,  $\nu_{x_0}$  with  $\nu_{x_0}(t) := u(\gamma_{x_0}(t))$  solves the linear ODE  $\nu'(t) + b(\gamma_{x_0}(t))\nu(t) = f(\gamma_{x_0}(t))$  for  $t \in I_{x_0}$ .

This means: The PDE reduces to ODEs along the flow lines.

**Proof:** From  $\nu_{x_0}(t) = u(\gamma_{x_0}(t))$  deduce by chain rule and ODEs for the flow lines (compare with the introductory example)

$$\nu_{x_0}'(t) = \gamma_{x_0}'(t) \cdot \nabla u(\gamma_{x_0}(t)) = a(\gamma_{x_0}(t)) \cdot \nabla u(\gamma_{x_0}(t)).$$

Hence, the PDE evaluated at points  $x = \gamma_{x_0}(t)$  yields the ODEs, and vice versa the ODEs also yield the PDE, since each  $x \in \Omega$  can be written as  $x = \gamma_{x_0}(t)$  (in fact  $x = \gamma_x(0)$ , but can also use restricted  $x_0$  as on next slide).  $\Box$ 

### Cauchy conditions and general line of approach

The reasonable complement to scalar first-order PDEs is the Cauchy condition

$$u(x) = u_0(x)$$
 for  $x \in S$ .

Here given: curve (n=2), surface (n=3), and generally hypersurface  $S \subset \Omega$  such that S intersects each flow line exactly once; function  $u_0: S \to \mathbb{R}$  on S.

If the PDE is complemented this way, consider the ODEs for  $\gamma$  and  $\nu$  only for  $x_0\in S$  and complement the latter one with the corresponding IC

$$\nu(0) = u_0(x_0) \,.$$

General line of approach for method of characteristic then:

- Solve IVP for flow lines  $\gamma_{x_0}$  with  $x_0 \in S$ .
- Solve IVP for  $\nu_{x_0}$  with  $x_0 \in S$ , get  $u(\gamma_{x_0}(t)) = \nu_{x_0}(t)$  as term in t and  $x_0$ .
- Solve  $x = \gamma_{x_0}(t)$  for  $(t, x_0)$ , get solution u(x) as term in x.

## Example for method of characteristics (linear case)

As an example, consider the Cauchy problem for a linear PDE

$$\begin{split} y \, \frac{\partial u}{\partial x}(x,y) - x \, \frac{\partial u}{\partial y}(x,y) + u(x,y) &= 0 & \quad \text{for } (x,y) \in (0,\infty) \times \mathbb{R} \,, \\ u(x,0) &= \mathrm{e}^{-x^2} & \quad \text{for } x \in (0,\infty) \,. \end{split}$$

Solve successively (where, for  $x_0 > 0$ , we abbreviate  $\gamma_{x_0} = \gamma_{(x_0,0)}$ ,  $\nu_{x_0} = \nu_{(x_0,0)}$ ): read off IVP for flow lines:  $\gamma'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma(t)$  with IC  $\gamma(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ solve flow lines:  $\gamma_{x_0}(t) = \begin{pmatrix} x_0 \cos t \\ -x_0 \sin t \end{pmatrix}$  for  $|t| < \frac{\pi}{2}$ read off IVP for  $\nu_{x_0}$ :  $\nu'(t) + \nu(t) = 0$  with IC  $\nu(0) = e^{-x_0^2}$ solve solution:  $u(\gamma_{x_0}(t)) = \nu_{x_0}(t) = e^{-t-x_0^2}$ Finally, solve  $\begin{pmatrix} x \\ y \end{pmatrix} = \gamma_{x_0}(t)$  for  $t = -\arctan(y/x)$  and  $x_0 = \sqrt{x^2 + y^2}$ .

 $\stackrel{\text{plug in}}{\rightsquigarrow}$  solution Cauchy problem for PDE:  $u(x,y) = e^{\arctan(y/x) - x^2 - y^2}$ 

#### Method of characteristics for linear transport equation

As another example, consider the IVP for the linear transport equation

$$\begin{split} u_t(t,x) + a(t,x) \cdot \nabla u(t,x) &= 0 \qquad & \text{for } (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) &= u_0(x) \qquad & \text{for } x \in \mathbb{R}^n \end{split}$$

Approach then (where, for  $x_0 \in \mathbb{R}^n$ , we abbreviate  $\gamma_{x_0} = \gamma_{(0,x_0)}$ ,  $\nu_{x_0} = \nu_{(0,x_0)}$ ):  $\xrightarrow{\text{read off}}$  IVP for flow lines:  $\gamma'(t) = (1, a(\gamma(t)))$  with IC  $\gamma(0) = (0, x_0)$   $\xrightarrow{\text{leads to}} \gamma_{x_0}(t) = (t, \tilde{\gamma}_{x_0}(t)) \text{ and } \tilde{\gamma}'(t) = a(t, \tilde{\gamma}(t)) \text{ with IC } \tilde{\gamma}(0) = x_0$   $\xrightarrow{\text{read off}}$  IVP for  $\nu_{x_0}$ :  $\nu'(t) = 0$  with IC  $\nu(0) = u_0(x_0)$  $\xrightarrow{\text{solve}} (t, \tilde{\gamma}_{x_0}(t)) = u_0(t)$ 

 $\stackrel{\text{solve}}{\leadsto} u(t,\widetilde{\gamma}_{x_0}(t)) = \nu_{x_0}(t) = u_0(x_0) \text{, i.e. } u \text{ constant along flow lines}$ 

It remains to solve  $x = \tilde{\gamma}_{x_0}(t)$ . This works, for instance, for constant a with correspondingly  $\tilde{\gamma}_{x_0}(t) = x_0 + ta$ , and then transforms  $u(t, x_0 + ta) = u_0(x_0)$  into the solution formula  $u(t, x) = u_0(x-ta)$  already known from Chapter 1.
#### The method of characteristics in the quasilinear case

In the quasilinear case the method of characteristics works similarly:

Principle (method of characteristics for quasilinear PDEs)

Consider an open  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ ,  $a \in C^1(\Omega \times \mathbb{R}, \mathbb{R}^n)$ ,  $b \in C^1(\Omega \times \mathbb{R})$ . Then, for  $u \in C^1(\Omega)$ , the following are equivalent:

- (1) u solves the quasilinear PDE  $a(x, u(x)) \cdot \nabla u(x) = b(x, u(x))$  for  $x \in \Omega$ .
- (2) For each  $x_0 \in \Omega$ , the solution  $(\gamma_{x_0}, \nu_{x_0}) \in C^1(I_{x_0}, \Omega \times \mathbb{R})$  (on its maximal existence interval  $I_{x_0}$ ) to the nonlinear ODE system

$$\begin{array}{l} \gamma'(t) = a(\gamma(t),\nu(t)) \\ \nu'(t) = b(\gamma(t),\nu(t)) \end{array} \quad \text{with ICs} \begin{array}{l} \gamma(0) = x_0 \\ \nu(0) = u(x_0) \end{array}$$

satisfies  $u(\gamma_{x_0}(t)) = \nu_{x_0}(t)$  for all  $t \in I_{x_0}$ .

In the linear case we had specifically  $a(x, y) = a_0(x)$  and  $b(x, y) = -b_0(x)y + f(x)$ . Then the first ODE did involve solely  $\gamma$ , not  $\nu$ , and could be solved a priori, the second ODE was linear in  $\nu$ . These features do not extend to the quasilinear case.

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## General line of approach and proof (quasilinear case)

Also in the quasilinear case, the PDE is reasonably complemented with a Cauchy condition  $u(x) = u_0(x)$  for  $x \in S$  with suitable hypersurface  $S \subset \Omega$  and  $u_0: S \to \mathbb{R}$ , and the general line of approach does not change much:

- Solve coupled IVP for  $(\gamma_{x_0}, \nu_{x_0})$  with  $x_0 \in S$ , obtain  $\gamma_{x_0}(t)$  and  $u(\gamma_{x_0}(t)) = \nu_{x_0}(t)$  as terms in  $t, x_0$ .
- Solve  $x = \gamma_{x_0}(t)$  for  $(t, x_0)$ , determine solution u(x) as term in x.

**Proof of the general principle:** For fixed  $x_0$  abbreviate ODE solutions as  $\gamma = \gamma_{x_0}$  and  $y = y_{x_0}$  on  $I = I_{x_0}$ , the prove the two implications separately:

(2) $\Longrightarrow$ (1) (as in linear case): Use  $u(\gamma) = \nu$  to compute (at points  $t \in I$ ):

$$\begin{aligned} a(\gamma, u(\gamma)) \cdot \nabla u(\gamma) &= a(\gamma, \nu) \cdot \nabla u(\gamma) \stackrel{\mathsf{ODEs for } \gamma}{=} \gamma' \cdot \nabla u(\gamma) \\ &= \left[ u(\gamma) \right]' = \nu' \stackrel{\mathsf{ODE for } \nu}{=} b(\gamma, \nu) = b(\gamma, u(\gamma)) \,. \end{aligned}$$

Since each  $x \in \Omega$  has form  $x = \gamma_{x_0}(t)$  (in fact  $x = \gamma_x(0)$ ), deduce the PDE.

# Proof (quasilinear case; continued)

#### Proof (continued):

(1) $\Longrightarrow$ (2) (now with more subtle reasoning for  $u(\gamma) = \nu$ ): Set

$$\psi(t) := \nu(t) - u(\gamma(t)) \quad \text{for } t \in I$$

to obtain

$$\psi' = \nu' - \gamma' \cdot \nabla u(\gamma) = b(\gamma, \nu) - a(\gamma, \nu) \cdot \nabla u(\gamma)$$

Observe: If one had  $\nu = u(\gamma)$  and could replace  $\nu$  with  $u(\gamma)$ , then the right-hand side would vanish by the PDE. But  $u(\gamma) = \nu$  is only the aim. Anyway, it is  $\psi(0) = 0$  by the ICs. If  $\psi \neq 0$  holds, there is a "last point"  $t_* \in I$  s.t.  $\psi(t_*) = 0$ . Then use  $|\partial_y b(x, y) - \partial_y a(x, y) \cdot \nabla u(x)| \leq C$  for (x, y) close to  $(\gamma(t_*), \nu(t_*))$  with bound C, get for t close to  $t_*$  by estimation of the replacement error that  $|\psi'| \leq C |\nu - u(\gamma)| = C |\psi|$ . By subsequent lemma deduce  $\psi \equiv 0$  near  $t_*$ , which contradicts the choice of  $t_*$  as "last point". This leaves  $\psi \equiv 0$  and thus  $u(\gamma) = \nu$  as sole option.

## Lemma for the preceding proof (quasilinear case)

#### Lemma

If  $\psi \in C^1(I)$  satisfies a differential inequality  $|\psi'| \leq C|\psi|$  on an interval I with  $C \in [0, \infty)$ , then  $\psi(t_0) = 0$  at some  $t_0 \in I$  implies  $\psi \equiv 0$  on all of I.

**Proof:** Suppose  $\psi(b) \neq 0$  at some  $b \in I$  and denote by  $t_* \in I$  the point closest to b s.t.  $\psi(t_*) = 0$ . In case  $t_* < b$  work with  $t_* < a \leq b$ , compute

$$\begin{split} &\log \frac{|\psi(b)|}{|\psi(a)|} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} \log |\psi(t)| \, \mathrm{d}t = \int_a^b \frac{\psi'(t)}{\psi(t)} \, \mathrm{d}t \leq \int_a^b \frac{|\psi'(t)|}{|\psi(t)|} \, \mathrm{d}t \leq C(b-a) \,, \\ &\text{get } |\psi(b)| \leq \mathrm{e}^{C(b-a)} |\psi(a)|. \text{ For } a \to t_* \text{ deduce } |\psi(b)| \leq \mathrm{e}^{C(b-t_*)} |\psi(t_*)| = 0, \\ &\text{contradiction to } \psi(b) \neq 0! \text{ In case } t_* > b, \text{ similarly use } b \leq a < t_* \text{ and} \\ &-\int_b^a \dots \text{ to reach a contradiction. The conclusion is } \psi(b) = 0 \text{ for all } b \in I. \ \Box \end{split}$$

**Remark**: The lemma is a special case of Gronwall's lemma, which estimates solutions to differential inequalities by solutions of corresponding DEs. (Here the ODE is  $\psi' = \psi$  with IC  $\psi(t_0) = 0$ . This has solely  $\psi \equiv 0$  as solution.)

### Example for method of characteristics (quasilinear case)

As an example, consider the Cauchy problem for a PDE

$$\begin{split} x\,u(x,y)\,\frac{\partial u}{\partial x}(x,y)+2y\,u(x,y)\,\frac{\partial u}{\partial y}(x,y) &= -u(x,y)^2 \quad \text{for } (x,y)\in\Omega\,,\\ u(x,2x-1)&=1 \qquad \qquad \text{for } 0< x<1 \end{split}$$

in the reasonable domain  $\Omega := \{(x,y) \in \mathbb{R}^2 : x > 0 \,, y < x^2\}$  (cf. below).

Solve successively (with  $\gamma_{x_0} = \gamma_{(x_0,2x_0-1)}$ ,  $\nu_{x_0} = \nu_{(x_0,2x_0-1)}$  for  $0 < x_0 < 1$ ):  $\xrightarrow{\text{read off}}$  PDE is quasilinear with a(x, y, w) = (xw, 2yw),  $b(x, y, w) = -w^2$ .  $\xrightarrow{\text{read off}}$  The characteristic ODE system to the PDE reads

$$\begin{split} \gamma_1' &= \nu \, \gamma_1 & \text{with IC } \gamma_1(0) &= x_0 \,, \\ \gamma_2' &= 2\nu \, \gamma_2 & \text{with IC } \gamma_2(0) &= 2x_0 - 1 \,, \\ \nu' &= -\nu^2 & \text{with IC } \nu(0) &= 1 \,. \end{split}$$

solve for  $\nu$ 

$$\stackrel{\text{a for } \nu}{\to} u(\gamma_{x_0}(t)) = \nu_{x_0}(t) = (t+1)^{-1}$$

## Example for method of characteristics (continued)

simplify for 
$$\gamma$$
  $\gamma'_1(t) = (t+1)^{-1} \gamma_1(t)$  with IC  $\gamma_1(0) = x_0$ ,  
 $\gamma'_2(t) = 2(t+1)^{-1} \gamma_2(t)$  with IC  $\gamma_2(0) = 2x_0 - 1$ 
solve for  $\gamma$  characteristic curves:  $\gamma_{x_0}(t) = \begin{pmatrix} x_0(t+1) \\ (2x_0-1)(t+1)^2 \end{pmatrix}$ 

Now record  $I_{x_0} = (-1, \infty)$ , in  $\begin{pmatrix} x \\ y \end{pmatrix} = \gamma_{x_0}(t)$  first eliminate  $x_0$  by writing  $\frac{2x}{t+1} = 2x_0 = \frac{y}{(t+1)^2} + 1$ , then solve a quadratic equation in t to find  $t = x - 1 + \sqrt{x^2 - y}$  (positive sign in front of the root due to  $\frac{x}{t+1} = x_0 < 1$ ).  $\nu_{x_0}(t) = (t+1)^{-1}$  solution Cauchy problem for PDE:  $u(x,y) = (x + \sqrt{x^2 - y})^{-1}$ 

Geometric background:  $\gamma_{x_0}$  parametrizes branch  $\{(x, a_{x_0}x^2) : 0 < x < \infty\}$  of parabola with start near origin and  $a_{x_0} := (2x_0-1)/x_0^2$ , where all  $a_{x_0} \in (-\infty, 1)$  are realized by  $0 < x_0 < 1$ . The union of all branches is the above-defined domain  $\Omega$ , which is just right for solving the PDE of this example on it.

By the way: Dividing the PDE of this example by  $u \neq 0$  one gets back to the linear case. This changes only the parametrizations  $\gamma_{x_0}, \nu_{x_0}$ , but not the geometry.

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#### Concluding remarks on the method of characteristics

Concluding remarks on the method of characteristics for  $a(\cdot, u)\cdot \nabla u = b(\cdot, u)$ :

- For a(x, w) independent of w (semilinear case) first approach the ODEs for  $\gamma$ , which are then independent of  $\nu$ . For b(x, w) independent of x (as in preceding example) first approach the ODE for  $\nu$ , which is then independent of  $\gamma$ .
- In general, however, the ODEs may be fully coupled, and often there is no explicit formula for solutions  $(\gamma, \nu)$  of the ODE system.
- Also solving  $x = \gamma_{x_0}(t)$  for  $t, x_0$  may not work by an explicit formula or may fail at all (e.g. due to non-uniqueness if different  $\gamma_{x_0}$  intersect). In good cases one can solve at least locally and can apply the method of characteristics in proving local existence results, but no details on this!
- Further quasilinear cases and examples follow in the next section.

In principle the method covers even the fully nonlinear case. But then the characteristic ODE systems gets still more complicated and additionally involves placeholders for derivatives of u. No details on this either!

## 2.3 Scalar conservation laws

Scalar conservation laws are first-order PDE of form

$$u_t + \operatorname{div}_x \left( \vec{F}(u) \right) \equiv 0 \quad \text{in } (0,T) \times \Omega$$

for a scalar function u of variables  $(t, x) \in (0, T) \times \Omega$ , where as usual  $\Omega \subset \mathbb{R}^n$ , T > 0, and  $\vec{F} \colon \mathbb{R} \to \mathbb{R}^n$  are given.

The equation is quasilinear (chain rule!) and in case  $u \neq 0$  is nothing but a version the continuity equation from 2.1 with u-dependent velocity field  $\vec{v}(t,x) := \frac{\vec{F}(u(t,x))}{u(t,x)}$ . Thus, by 2.1 we have conservation of mass in moving domains (end e.g. with BC  $\vec{v} \equiv 0$  at  $(0,T) \times \partial \Omega$  in all of  $(0,T) \times \Omega$  as well).

Here we treat space dimension n = 1 only, for simplicity with  $\Omega = \mathbb{R}$  and  $T = \infty$ . Thus, we are concerned with (in two equivalent formulations)

$$\boxed{u_t + (F(u))_x \equiv 0} \qquad \text{rsp.} \qquad \boxed{u_t + f(u) \cdot u_x \equiv 0} \qquad \text{in } (0, \infty) \times \mathbb{R}$$

where  $F, f \colon \mathbb{R} \to \mathbb{R}$  are given and correspond to each other by F' = f.

### Burgers' equation

For F(w) := aw and  $f \equiv a$  we get back the linear transport equation.

However, the focus is now more on the nonlinear model case  $F(w) := \frac{1}{2}w^2$ and correspondingly f(w) := w of Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x \equiv 0$$
 rsp.  $u_t + u \cdot u_x \equiv 0$  in  $(0, \infty) \times \mathbb{R}$ 

Burgers' equation serves as basic nonlinear 1d model for time evolution of a mass density u with potential emergence of shock waves, e.g. ultrasonic waves in air. (A nonlinearity of similar type in 3d is the convective term in the Navier-Stokes and Euler equations.)

#### Method of characteristics for conservation laws

Now consider the Cauchy-Problem for a conservation law with IC

$$u_t + f(u) \cdot u_x \equiv 0$$
 in  $(0, \infty) \times \mathbb{R}$ ,  $u(0, \cdot) = u_0$  on  $\mathbb{R}$ 

and apply the method of characteristics (again index  $x_0$  short for  $(0,x_0)$ ):

• characteristic ODEs: 
$$\gamma' = (1, f(\nu))$$
 with IC  $\gamma(0) = (0, x_0)$ ,  
 $\nu' \equiv 0$  with IC  $\nu(0) = u_0(x_0)$ ,

• solution:  $\nu_{x_0} \equiv u_0(x_0)$  constant, thus u constant along  $\gamma_{x_0}$ ,  $\gamma_{x_0}(t) = (t, x_0 + t f(u_0(x_0)))$  (characteristic lines).

Thus, deduce the implicit solution formula  $u(t, x_0+tf(u_0(x_0))) = u_0(x_0)$ . If one can indeed solve and in fact obtain a solution u, depends on  $f, u_0$ .

(By the way, in the nonlinear case the "velocity"  $f(u_0(x_0))$  of  $\gamma_{x_0}$  differs from the particle velocity  $\vec{v}(\gamma_{x_0}) \equiv \frac{F(u_0(x_0))}{u_0(x_0)}$  modeled in 2.1.)

# Exemplary solutions of Burgers' equation

In the model case of Burgers' equation  $u_t + u \cdot u_x \equiv 0$  with f(w) = w one has the following examples (with u constant on shown characteristic lines):

• For  $u_0(x) := x$ , from  $u(t, (1+t)x_0) = x_0$  pass to  $u(t, x) = \frac{x}{1+t}$  and obtain a solution defined for all  $t \ge 0$  and  $x \in \mathbb{R}$ .



• For  $u_0(x) = -x$ , from  $u(t, (1-t)x_0) = -x_0$  pass to  $u(t, x) = \frac{-x}{1-t}$  and obtain a solution with singularity at t = 1 (and opposite sign for t > 1).



The problem in this example is that all characteristic lines  $\gamma_{x_0}(t) = (t, (1-t)x_0)$  intersect at (t, x) = (1, 0).

# Exemplary solutions of Burgers' equation (continued)

Similar to the preceding example, but more relevant (as  $u_0 \ge 0$  bounded) is:

• For 
$$u_0(x) = \begin{cases} 1 & , x \leq 0 \\ 1-x & , 0 < x < 1 \\ 0 & , x \geq 1 \end{cases}$$
 obtain by  $u(t,x) = \begin{cases} 1 & , x \leq t \\ \frac{1-x}{1-t} & , t < x < 1 \\ 0 & , x \geq 1 \end{cases}$ 

a solution only for t < 1 (for  $t \ge 1$  ambiguity in case distinction).



All those  $\gamma_{x_0}$  with  $x_0 \in [0,1]$  intersect at (t,x) = (1,1).

(The kinks of  $u_0$  at x = 0 and x = 1 and of u at x = t and x = 1 are not essential and are not the root of the problem here.)

In the sequel we introduce a new kind of solution, which stays defined even for  $t \ge 1$  and resolves the problem of this example and similar ones.

#### Test functions and integration by parts

In the sequel call  $\varphi \in C^1([0,\infty) \times \mathbb{R})$  with  $\varphi \equiv 0$  <u>outside</u>  $[0,M] \times [-M,M]$  for some  $M \in [0,\infty)$  (roughly speaking with "zero boundary values at  $\infty$ ") a test function. For such  $\varphi$  and a solution u to  $u_t + (F(u))_x \equiv 0$  with  $u(0, \cdot) = u_0$  the compute (arguments (t, x) dropped for ease of notation)

$$\begin{split} 0 &= \int_0^\infty \int_{-\infty}^\infty \left[ u_t + \left( F(u) \right)_x \right] \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{-\infty}^\infty \left[ \int_0^\infty u_t \varphi \, \mathrm{d}t \right] \mathrm{d}x + \int_0^\infty \left[ \int_{-\infty}^\infty \left( F(u) \right)_x \varphi \, \mathrm{d}x \right] \mathrm{d}t \\ &\stackrel{\mathsf{ibp}}{=} \int_{-\infty}^\infty \left[ -u(0,x) \, \varphi(0,x) - \int_0^\infty u \, \varphi_t \, \mathrm{d}t \right] \mathrm{d}x + \int_0^\infty \left[ - \int_{-\infty}^\infty F(u) \, \varphi_x \, \mathrm{d}x \right] \mathrm{d}t \\ &= - \int_0^\infty \int_{-\infty}^\infty \left[ u \, \varphi_t + F(u) \, \varphi_x \right] \mathrm{d}x \, \mathrm{d}t - \int_{-\infty}^\infty u_0(x) \, \varphi(0,x) \, \mathrm{d}x \, . \end{split}$$

(One may also write  $\pm M$  instead of  $\pm\infty$ ; does not change anything here.)

The decisive point is that all derivatives are shifted from u to  $\varphi$ .

## Weak solutions

On the basis of the preceding computation one defines:

#### Definition (weak solutions)

Consider  $F \in C^0(\mathbb{R})$  and  $^{\ddagger} u_0 \in L^{\infty}_{loc}(\mathbb{R})$ . A weak solution or integral solution to the Cauchy problem for a conservation law

$$u_t + \left(F(u)\right)_x \equiv 0 \text{ in } (0,\infty) \times \mathbb{R}, \qquad u(0,\,\cdot\,) = u_0 \text{ on } \mathbb{R}$$

is a function  $^{\ddagger}$   $u\in L^{\infty}_{loc}\big([0,\infty){\times}\mathbb{R}\big)$  such that

$$\int_0^\infty \int_{-\infty}^\infty \left[ u \,\varphi_t + F(u) \,\varphi_x \right] \mathrm{d}x \,\mathrm{d}t + \int_{-\infty}^\infty u_0(x) \,\varphi(0,x) \,\mathrm{d}x = 0$$

holds for all test functions  $\varphi$  (in the sense of the previous slide).

Indeed, the preceding computation proves that each "ordinary" solution is a weak solution as well. But there are further weak solutions:

 $\begin{array}{l} \overline{^{\ddagger} u_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R})} \text{ and } u \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R}) \text{ mean essentially: } u_0 \colon \mathbb{R} \to \mathbb{R} \text{ bounded on each interval} \\ [-M,M] \text{ and } u \colon [0,\infty) \times \mathbb{R} \to \mathbb{R} \text{ bounded on each rectangle } [0,M] \times [-M,M] \text{ with } M \in [0,\infty). \end{array}$ 

#### Riemann problems and shock-wave solutions

A Riemann problem for a conservation law is a Cauchy problem with initial datum discontinuous at x = 0 ( $w_l, w_r \in \mathbb{R}$  constants,  $w_r \neq w_l$ ) of form

$$u_t + (F(u))_x \equiv 0 \text{ in } (0,\infty) \times \mathbb{R}, \qquad u(0,x) = \begin{cases} w_l & \text{for } x < 0 \\ w_r & \text{for } x > 0 \end{cases}$$

We will show that a weak solution of this problem is given by the correspondingly discontinuous function

$$u(t,x) = \begin{cases} w_l & \text{for } x < m \cdot t \\ w_r & \text{for } x > m \cdot t \end{cases},$$

where  $m \in \mathbb{R}$  is determined by the Rankine–Hugoniot condition

$$m := \frac{F(w_r) - F(w_l)}{w_r - w_l}$$

One takes this solution as a shock wave whose wave front is described by  $x = m \cdot t$  and moves at speed m.

# Shock-wave solutions to Burgers' equation

In the specific case  $F(w) = \frac{1}{2}w^2$  the Rankine–Hugoniot condition reads  $m = \frac{\frac{1}{2}w_r^2 - \frac{1}{2}w_l^2}{w_r - w_l} = \frac{1}{2}(w_l + w_r).$ 

Therefore, the Riemann-Problem for Burgers' equation (with  $w_l, w_r \in \mathbb{R}$ )

$$u_t + \left(\frac{1}{2}u^2\right)_x \equiv 0 \text{ in } (0,\infty) \times \mathbb{R}, \qquad u(0,x) = \begin{cases} w_l & \text{for } x < 0\\ w_r & \text{for } x > 0 \end{cases}$$

has a weak solution given by the shock wave

$$u(t,x) = \begin{cases} w_l & \text{for } x < \frac{1}{2}(w_l + w_r) \cdot t \\ w_r & \text{for } x > \frac{1}{2}(w_l + w_r) \cdot t \end{cases}$$

In case  $w_l = 1$ ,  $w_r = 0$  get  $m = \frac{1}{2}$  (left picture). In the example of slide 48 an analogous shock wave (t, x shifted; right picture) is developed at t = 1.

$$u \equiv 1$$

$$u \equiv 0$$

$$u \equiv 1$$

$$u \equiv 0$$

$$u \equiv 1$$

#### General shock-wave solutions

General wave fronts describe by a function  $s \in C^1([0,\infty))$ . Then partition  $D := (0,\infty) \times \mathbb{R}$  into the realms left and right of the wave front x = s(t)

 $D_l := \{(t,x) \in D : x < s(t)\} \qquad \text{and} \qquad D_r := \{(t,x) \in D : x > s(t)\}$ 

with corresponding functions  $u_l \in C^1(\overline{D_l})$  and  $u_r \in C^1(\overline{D_r})$  and with initial data  $u_{0,l} \in C^0((-\infty, s(0)])$  and  $u_{0,r} \in C^0([s(0), \infty))$ .



With this terminology the preceding cases can be generalized as follows.

## General shock-wave solutions (continued)

Theorem (on general shock-wave solutions to conservations laws)

For  $s, D_l, D_r, u_l, u_r, u_{0,l}, u_{0,r}$  as before and  $F \in C^1(\mathbb{R})$ , it is equivalent: (1) The (potentially discontinuous) function

$$u(t,x) := \begin{cases} u_l & \text{for } x < s(t) \\ u_r & \text{for } x > s(t) \end{cases}$$

is a weak solution to the Cauchy problem

$$u_t + \left(F(u)\right)_x \equiv 0 \text{ in } D, \quad u(0,x) = \begin{cases} u_{0,l}(x) & \text{for } x < s(0) \\ u_{0,r}(x) & \text{for } x > s(0) \end{cases}$$

(2)  $u_l$  and  $u_r$  solve the "ordinary" Cauchy problems in  $D_l$  and  $D_r$  with  $u_{0,l}$  and  $u_{0,r}$ , and the general Rankine–Hugoniot condition

$$s'(t) = \frac{F(u_r(t,x)) - F(u_l(t,x))}{u_r(t,x) - u_l(t,x)}$$

is satisfied for all (t,x) such that x = s(t) and  $u_r(t,x) \neq u_l(t,x)$ .

#### Proof of the theorem on shock-wave solutions

**Proof of the theorem:** For test function  $\varphi$ , look at *l*- and *r*-terms of

$$\int_{D} \left[ u \varphi_t + F(u) \varphi_x \right] \mathrm{d}(t, x) + \int_{-\infty}^{\infty} u_0 \varphi(0, \cdot) \, \mathrm{d}x$$

(cf. definition weak solution). If  $u_r$  solves, rewrite r-terms first as

$$\int_{D_r} \left[ u \varphi_t + F(u) \varphi_x \right] \mathrm{d}(t, x) + \int_{s(0)}^{\infty} u_0 \varphi(0, \cdot) \mathrm{d}x$$
$$= \int_{D_r} \left[ \operatorname{rot} \left( \begin{array}{c} -F(u) \varphi \\ u \varphi \end{array} \right) - \operatorname{rot} \left( \begin{array}{c} -F(u) \\ u \end{array} \right) \varphi \right] \mathrm{d}(t, x) + \int_{s(0)}^{\infty} u_0 \varphi(0, \cdot) \mathrm{d}x \,,$$

then by Green's theorem  $(\partial D_r \text{ union of } \{0\} \times [s(0), \infty) \text{ and } \{(t, x) \in D : x = s(t)\};$ parametrize as curve c with tangent vectors (0, -1) and (1, s'), respectively) find

$$= \oint_{c} \left( -F(u_{r})\varphi \right) \cdot d(t,x) - \int_{D_{r}} \left[ u_{t} + \left(F(u)\right)_{x} \right] \varphi d(t,x) + \int_{s(0)}^{\infty} u_{0} \varphi(0,\cdot) dx$$
$$= \int_{0}^{\infty} \left[ -F(u_{r}(t,s(t))) + s'(t)u_{r}(t,s(t)) \right] \varphi(t,s(t)) dt \,.$$

## Proof of the theorem on shock-wave solutions (continued)

If  $u_l$  solves, rewrite l-terms in similar way as

$$\int_0^\infty \left[ F(u_l(t,s(t))) - s'(t)u_l(t,s(t)) \right] \varphi(t,s(t)) \,\mathrm{d}t$$

In combination, for a weak solution need to have (arguments t and (t, s(t)) omitted for better readability)

$$\int_0^\infty \left[ -(F(u_r) - F(u_l)) + s'(u_r - u_l) \right] \varphi \, \mathrm{d}t = 0$$

for all test functions  $\varphi$ . This is equivalent with

$$s'(t)(u_r(t,x) - u_l(t,x)) = F(u_r(t,x)) - F(u_l(t,x))$$
 for  $x = s(t)$ 

and by rearranging terms also with

$$s'(t) = \frac{F(u_r(t,x)) - F(u_l(t,x))}{u_r(t,x) - u_l(t,x)} \quad \text{ for } x = s(t) \text{ s.t. } u_r(t,x) \neq u_l(t,x) \, .$$

So,  $(2) \Longrightarrow (1)$  is proved. By keeping track which (other) integrals in the computation need to vanish, one deduces  $(1) \Longrightarrow (2)$  as well.

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# Rarefaction waves (case of Burgers' equation)

The Riemann problem for Burgers' equation

$$u_t + u \cdot u_x \equiv 0 \text{ in } (0, \infty) \times \mathbb{R}, \quad u(0, x) = \begin{cases} w_l & \text{for } x < 0 \\ w_r & \text{for } x > 0 \end{cases}$$

with constants  $w_l, w_r \in \mathbb{R}$  has in case  $w_r > w_l$  as one weak solution the shock wave discussed before (picture for  $w_l = 0$ ,  $w_r = 1$ )

$$u(t,x) = \begin{cases} w_l & \text{for } x < \frac{1}{2}(w_l + w_r) \cdot t \\ w_r & \text{for } x > \frac{1}{2}(w_l + w_r) \cdot t \end{cases} \quad u \equiv 0$$

and as another weak solution the rarefaction wave (picture for  $w_l = 0$ ,  $w_r = 1$ )

$$u(t,x) = \begin{cases} w_l & \text{for } x < w_l \cdot t \\ \frac{x}{t} & \text{for } w_l \cdot t < x < w_r \cdot t \\ w_r & \text{for } x > w_r \cdot t \end{cases} \quad u \equiv 0$$

Thus, in this case the solution is not unique!!!. For yet other solutions the shock (front) turns into a rarefaction wave at an arbitrary time  $t_* > 0$ .

 $t \ 0 < u < 1$ 

# Rarefaction waves (case of a general conservation law)

For a general conservation law, there are analogous rarefaction waves:

Theorem (on rarefaction-wave solutions to a general conservation law)

For  $w_l, w_r \in \mathbb{R}$  s.t.  $w_r > w_l$  and  $f \in C^1(\mathbb{R})$  s.t.  $\inf_{\mathbb{R}} f' > 0$  (in particular f strictly increasing, inverse function  $f^{-1}$  exists), the formula

$$u(t,x) := \begin{cases} w_l & \text{for } x < f(w_l) \cdot t \\ f^{-1}\left(\frac{x}{t}\right) & \text{for } f(w_l) \cdot t < x < f(w_r) \cdot t \\ w_r & \text{for } x > f(w_r) \cdot t \end{cases}$$

yields a weak solution to the Riemann problem for a conservation law

$$u_t + f(u) \cdot u_x \equiv 0 \text{ in } (0,\infty) \times \mathbb{R}, \quad u(0,x) = \begin{cases} w_l & \text{for } x < 0 \\ w_r & \text{for } x > 0 \end{cases}$$

The model case of Burgers' equation is simply f(w) = w and  $f^{-1}(\chi) = \chi$ .

## Proof of the theorem on rarefaction waves

**On proving the theorem:** For u defined in the theorem read off/check: • u satisfies the IC at t = 0 (evident).

- u solves  $u_t + f(u) \cdot u_x \equiv 0$  where  $x < f(w_l) \cdot t$  or  $x > f(w_r) \cdot t$  (there  $u \equiv \text{const}$ ).
- Where  $f(w_l) \cdot t < x < f(w_r) \cdot t$  holds, deduce from  $u(t, x) = f^{-1}(\frac{x}{t})$  by chain rule and definition of inverse function the solution property:

$$[u_t + f(u) \cdot u_x](t, x) = (f^{-1})'(\frac{x}{t}) \cdot \left(-\frac{x}{t^2}\right) + f(f^{-1}(\frac{x}{t})) \cdot (f^{-1})'(\frac{x}{t}) \cdot \frac{1}{t}$$
  
=  $(f^{-1})'(\frac{x}{t}) \cdot \left(-\frac{x}{t^2} + \frac{x}{t} \cdot \frac{1}{t}\right) = 0.$ 

u continuously extends at x = f(w<sub>l</sub>)·t and x = f(w<sub>r</sub>)·t with t > 0 (as e.g., for x = f(w<sub>l</sub>)·t, one has f<sup>-1</sup>(<sup>x</sup>/<sub>t</sub>) = f<sup>-1</sup>(f(w<sub>l</sub>)) = w<sub>l</sub>).
On this basis then proceed similar to the proof of the theorem on shocks. (More precisely: Use Green or int. by parts on portions of {(t, x) : |(t, x)| ≥ ε}. Show that integrals on {(t, x) : |(t, x)| < ε} are small, since u is bounded and |(u<sub>t</sub>, u<sub>x</sub>)| is finitely integrable near (0,0). We omit all further details here.)

## (Non-)Uniqueness and "physical" solutions

For  $u_t + f(u) \cdot u_x \equiv 0$  (with f as in the last theorem), we record:

- We saw that the solution to the Riemann problem with w<sub>r</sub> > w<sub>l</sub> is not unique. Among the weak solutions the rarefaction wave (continuous!) is physically more plausible than the shock wave (discontinuous!). (In addition, the rarefaction wave is unique "from time t<sub>\*</sub> > 0 onward", the shock may or may not turn into a rarefaction wave at each time t<sub>\*</sub> > 0.)
- In the Riemann problem with  $w_r < w_l$ , however, the shock wave has no alternative. Thus, in full generality one cannot rule out discontinuities.
- "Physical" solutions include shocks with  $u_r(t, s(t)) \le u_l(t, s(t))$  along the wave front s and in general are weak solutions u such that

$$u_r(t,x) \le u_l(t,x)$$
 for all  $(t,x) \in (0,\infty) \times \mathbb{R}$ , (\*)

where  $u_l(t,x) := \lim_{h \searrow 0} u(t,x-h)$  and  $u_r(t,x) := \lim_{h \searrow 0} u(t,x+h)$ . A technical variant of (\*) characterizes a class of solutions known as entropy solutions and, for these, gains full mathematical uniqueness.

## Final example for combined phenomena

The Cauchy problem for Burgers' equation

$$u_t + u \cdot u_x \equiv 0 \text{ in } (0,\infty) \times \mathbb{R} \,, \qquad u(0,x) = \begin{cases} 0 & \text{for } x < 0 \text{ or } x > 2 \\ 2 - \frac{1}{2}x & \text{for } 0 < x < 2 \end{cases}$$

has a "weak solution for t < 2" (singularity at (t, x) = (2, 4)) given by

$$u(t,x) = \begin{cases} 0 & \text{for } x < s_0(t) \text{ or } x > s_2(t) \\ \frac{4-x}{2-t} & \text{for } s_0(t) < x < s_2(t) \end{cases}$$

with two shock-wave fronts  $s_0(t) = 4 - \sqrt{8(2-t)}$  and  $s_2(t) = 4 - \sqrt{2(2-t)}$ .



Formula  $\frac{4-x}{2-t}$  obtained via characteristics, then  $s_0$  and  $s_2$  determined from Rankine-Hugenoit condition by solving ODE! Shock at  $s_0$  is "unphysical".

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# Final example for combined phenomena (continued)

The "physical" weak solution of the previous Cauchy problem is rather

$$u(t,x) = \begin{cases} 0 & \text{for } x < 0 \text{ or } x > s_2(t) \\ \frac{x}{t} & \text{for } 0 < x < \min\{2t, s_2(t)\} \\ \frac{4-x}{2-t} & \text{for } 2t < x < s_2(t) \end{cases}$$

with rarefaction wave from (t, x) = (0, 0) onward and with single shock-wave front  $s_2(t) = \begin{cases} 4-\sqrt{2(2-t)} & \text{for } t \leq \frac{3}{2} \\ \sqrt{6t} & \text{for } t > \frac{3}{2} \end{cases}$  (computed via Rankine-Hugenoit).



For other solutions the shock-wave front  $s_0$  of the previous slide turns into a rarefaction wave only at a time  $t_* \in (0, 2)$ . In fact, there are yet others ...

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# **Chapter 3: Second-Order PDEs**

This chapter deals with three model cases of second-order PDEs, as previously announced:

- the Laplace and Poisson equation (including eigenvalue problems),
- the diffusion or heat equation,
- and the wave equation.

In comparison with first-order PDEs, fully elementary formulas for solutions are now more rare and usually can be found for specific cases only. Still, there is a multifaceted solution theory, which comes with (more or less explicit) integral formulas for solutions in quite some generality.

## 3.1 The Laplace and Poisson equation

This section treats the Laplace equation and the Poisson equation

$$\Delta u \equiv 0$$
 and  $\Delta u = f$  in open  $\Omega \subset \mathbb{R}^n$ ,

where until further notice we understand  $n \ge 2$ . The equations are often complemented with the earlier-mentioned Dirichlet BC u = g at  $\partial \Omega$ .

It is standard to coin a terms of its own for solutions to Laplace's equation:

#### Definition (harmonic functions)

A function  $u \in C^2(\Omega)$  is called harmonic in open  $\Omega \subset \mathbb{R}^n$  if it solves the Laplace equation  $\Delta u \equiv 0$  in  $\Omega$ .

Since the Laplace equation is linear, one readily checks (for  $r, s \in \mathbb{R}$ ):

u and v harmonic in  $\Omega \implies ru + sv$  harmonic in  $\Omega$ .

Specifically for n = 2, one can obtain (cf. next slide) harmonic functions as real and imaginary parts of holomorphic functions.

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#### Entire harmonic functions, harmonic polynomials

Examples for harmonic functions u on all of  $\mathbb{R}^n$ , called entire harmonic functions, are harmonic polynomials:

- degree 0: all constant functions, e.g.  $u \equiv 1$ ,
- degree 1: all linear functions, e.g.  $u(x) = x_i$ ,
- degree 2: e.g.  $u(x) = x_i^2 x_j^2$  and  $u(x) = x_i x_j$  with  $i \neq j$  $(n = 2 \rightsquigarrow x^2 - y^2$  and xy real and imaginary parts of holomorphic  $z^2$ ),
- degree 3: e.g.  $u(x) = x_i^3 3x_i x_j^2$  and  $u(x) = x_i x_j x_k$  with  $i \neq j \neq k \neq i$  $(n = 2 \rightsquigarrow x^3 - 3xy^2$  real part of holomorphic  $z^3$ ),
- degree 4 and higher: . . .

Also combinations, e.g.  $x_1^3 - 3x_1x_2^2 - 7x_2x_3 + 4$  for n = 3, are harmonic!

Other entire harmonic u, which are not polynomials:

• e.g. 
$$u(x) = e^{x_i} \cos x_j$$
 and  $u(x) = e^{x_i} \sin x_j$  with  $i \neq j$   
 $(n = 2 \rightsquigarrow e^x \cos y$  and  $e^x \sin y$  real and imaginary parts of holomorphic  $e^z$ ).

# The fundamental solution to Laplace's equation

Definition/Proposition (fundamental solution to Laplace's equation) The function  $\Phi$  given by

$$\begin{split} \Phi(x) &:= \frac{1}{2\pi} \log |x| \quad \text{for } n = 2 \,, \qquad \Phi(x) := -\frac{1}{n(n-2)\alpha_n} |x|^{2-n} \quad \text{for } n \geq 3 \\ (\text{with measure } \alpha_n \text{ of unit ball } \mathcal{B}_1 &:= \{x \in \mathbb{R}^n : |x| < 1\} \text{ in } \mathbb{R}^n; \, \alpha_2 = \pi, \, \alpha_3 = \frac{4}{3}\pi) \\ \text{is harmonic in } \mathbb{R}^n \setminus \{0\} \text{ and is called the fundamental solution to Laplace's equation. All rotationally symmetric harmonic functions } u(x) = h(|x|) \text{ in } \\ \mathbb{R}^n \setminus \{0\} \text{ take the form } u(x) = a \, \Phi(x) + b \text{ with } a, b \in \mathbb{R}. \end{split}$$

- The fundamental solution  $\Phi$  is singular at x = 0 and does not extend to all of  $\mathbb{R}^n$ . Its singularity is prototypical for harmonic functions.
- The choice of the prefactors  $\frac{1}{2\pi}$  rsp.  $-\frac{1}{n(n-2)\alpha_n}$  may seem peculiar, but has its advantages: It avoids the explicit occurrence of such factors in the next theorem and normalizes to  $\int_{\partial B_1} \partial_{\nu} \Phi(x) \, dS(x) = 1$ .
- $\Phi$  has a physical interpretation as the electrical potential of an electrical unit charge placed in the origin.

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#### Derivation of the fundamental solution

Derivation of the fundamental solution and the proposition: First compute the auxiliary derivatives  $\frac{\partial}{\partial x_i}|x| = \frac{x_i}{|x|}$  and  $\frac{\partial}{\partial x_i}\frac{x_i}{|x|} = \frac{|x|^2 - x_i^2}{|x|^3}$ . Then, for rotationally symmetric u(x) = h(|x|), derive successively:

$$\begin{split} \partial_{i}u(x) &= h'(|x|)\frac{x_{i}}{|x|}, \qquad \partial_{i}^{2}u(x) = h''(|x|)\frac{x_{i}^{2}}{|x|^{2}} + h'(|x|)\frac{|x|^{2} - x_{i}^{2}}{|x|^{3}}, \\ \Delta u(x) &= h''(|x|)\frac{|x|^{2}}{|x|^{2}} + h'(|x|)\frac{n|x|^{2} - |x|^{2}}{|x|^{3}} = h''(|x|) + \frac{n-1}{|x|}h'(|x|). \end{split}$$

Thus,  $\boldsymbol{u}$  is harmonic if and only if  $\boldsymbol{h}$  solves the ODE

$$h''(r) + \frac{n-1}{r}h'(r) = 0.$$

The ODE solutions satisfy  $h'(r) = c r^{1-n}$  and thus are exactly  $(b, c \in \mathbb{R})$ 

$$h(r) = c \, \log r + b \quad \text{for} \, \, n = 2 \,, \qquad h(r) = - \frac{c}{n-2} \, r^{2-n} + b \quad \text{for} \, \, n \geq 3 \,.$$

By using u(x) = h(|x|) get form of  $\Phi$  and proposition (with  $a = n\alpha_n c$ ).  $\Box$ 

## Solution formula for Poisson's equation

Solutions to Poisson's equation with general right-hand side f can be obtained from an integral formula, which involves the fundamental solution:

Theorem (solving Poisson's equation by convolution with  $\Phi$ ) For  $f \in C^1(\mathbb{R}^n)$  with  $\{x \in \mathbb{R}^n : f(x) \neq 0\}$  bounded, by setting  $u(x) := \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, dy$  for  $x \in \mathbb{R}^n$ , one obtains a solution  $u \in C^2(\mathbb{R}^n)$  to Poisson's equation  $\Delta u = f$  in  $\mathbb{R}^n$ .

The proof of the theorem is more intricate and is not discussed here. The formula of the theorem does <u>not</u> allow for prescribing a BC (on bounded open  $\Omega \subset \mathbb{R}^n$ ) and in this aspect remains unsatisfactory.

### The maximum principle for harmonic functions

Theorem (maximum principle for harmonic functions) If  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  is harmonic in bounded open  $\Omega \subset \mathbb{R}^n$ , then one has: (1) weak maximum principle: There holds  $\min_{\partial \Omega} u \leq u(x) \leq \max_{\partial \Omega} u$  for all  $x \in \Omega$ . (2) strong maximum principle: If  $\Omega$  is connected and u is <u>non-constant</u>, there holds even

$$\min_{\partial \Omega} u < u(x) < \max_{\partial \Omega} u \qquad \text{for all } x \in \Omega \,.$$

**Corollary 1:** For harmonic u, as in the theorem, one further has  $|u(x)| \leq \max_{\partial \Omega} |u|$  for all  $x \in \Omega$ . **Proof:** For  $x \in \Omega$ , it is either  $|u(x)| = u(x) \leq \max_{\partial \Omega} u \leq \max_{\partial \Omega} |u|$  or, since -uis harmonic as well,  $|u(x)| = -u(x) \leq \max_{\partial \Omega}(-u) \leq \max_{\partial \Omega} |u|$ .

#### Conclusions for the Dirichlet problem

Corollary 2 (uniqueness in the Dirichlet problem for Poisson's equation) For bounded open  $\Omega \subset \mathbb{R}^n$ ,  $f \in C^0(\Omega)$ ,  $g \in C^0(\partial\Omega)$ , the Dirichlet problem for Poisson's equation

 $\Delta u = f \text{ in } \Omega, \qquad \qquad u = g \text{ at } \partial \Omega$ 

has at most one solution  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ .

**Proof:** If  $u_1, u_2$  are two solutions, then  $u_2-u_1$  is harmonic with  $u_2-u_1 \equiv 0$  at  $\partial\Omega$ . Corollary 1 then implies  $|u_2(x)-u_1(x)| \leq 0$  for all  $x \in \Omega$  and thus  $u_2 = u_1$ .

**Corollary 3:** For solutions  $u_1$  and  $u_2$  to

 $\Delta u_i = f \text{ in } \Omega \,, \qquad \qquad u_i = g_i \text{ at } \partial \Omega$ 

( $\Omega$ , f,  $g_1$ ,  $g_2$  as before), one has continuous dependence on boundary data:

$$\max_{\overline{\Omega}} |u_2 - u_1| \le \max_{\partial \Omega} |g_2 - g_1|.$$

**Proof:** Corollary 1 for harmonic function  $u_2-u_1$  with  $u_2-u_1 = g_2-g_1$  at  $\partial\Omega$ .  $\Box$ 

#### Proof of the weak maximum principle

Proof of the weak maximum principle: For arbitrary  $\varepsilon>0$  set  $u_\varepsilon(x):=u(x)+\varepsilon|x|^2$  and observe

$$\operatorname{trace}(\mathrm{D}^2 u_\varepsilon(x)) = \Delta u_\varepsilon(x) = \Delta u(x) + 2n\varepsilon = 2n\varepsilon > 0 \qquad \text{for } x \in \Omega \,.$$

Thus,  $D^2 u_{\varepsilon}(x)$  is <u>not</u> a negative-semidefinite matrix,  $u_{\varepsilon}$  does <u>not</u> have a max in  $\Omega$  (necessary criterion) and reaches  $\max_{\overline{\Omega}} u_{\varepsilon}$  only at  $\partial\Omega$ . This gives

$$u(x) \leq u_{\varepsilon}(x) < \max_{\partial \Omega} u_{\varepsilon} \leq \max_{\partial \Omega} u + \varepsilon \max_{y \in \partial \Omega} |y|^2 \qquad \text{for } x \in \Omega \,.$$

In the limit  $\varepsilon \to 0$  infer  $u(x) \le \max_{\partial \Omega} u$ . The reverse inequality  $u(x) \ge \min_{\partial \Omega} u$  follows similarly (e.g. with help of  $u_{\varepsilon}(x) := u(x) - \varepsilon |x|^2$ ).  $\Box$ 

Warning! Though the first part of the proof works with '<', in the limit  $\varepsilon \to 0$  merely ' $\leq$ ' remains. Thus, the argument confirms the weak maximum principle, but not the strong one (which is obtained later via the mean value property).

## Poisson kernel and Poisson integral formula

An integral formula solves the Dirichlet problem for Laplace's equation on the unit ball  $B_1 := B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$  in  $\mathbb{R}^n$ :

Definition (Poisson kernel)

The (n-dimensional) Poisson kernel  $K_P$  is defined by

 $\mathcal{K}_{\mathcal{P}}(x,y) := \frac{1}{n\alpha_n} \frac{1-|x|^2}{|y-x|^n} \qquad \text{for } x \in \mathcal{B}_1 \,, \, y \in \partial \mathcal{B}_1 \,.$ 

#### Theorem (Poisson integral formula)

For every  $g \in C^0(\partial B_1)$ , the unique solution  $u \in C^2(B_1) \cap C^0(\overline{B_1})$  to

 $\Delta u \equiv 0 \text{ in } B_1, \qquad \qquad u = g \text{ at } \partial B_1$ 

is given by the Poisson integral formula (PIF)

$$u(x) = \int_{\partial B_1} K_P(x, y) g(y) dS(y)$$
 for  $x \in B_1$ 

(The integral with dS is in case n = 2 a path integral of first kind, in case n = 3 a surface integral of first kind, for  $n \ge 3$  an (n-1)-dimensional analog of these.)
## Remarks on the Poisson integral formula

- The PIF expresses the solution u at an arbitrary point  $x \in B_1$  as integral of the boundary values g at  $\partial B_1$  only.
- In particular the PIF ensures existence of the solution, (whose uniqueness we already know from the maximum principle).
- A closely related variant is the PIF for the Dirichlet problem on an arbitrary ball  $B_r(x_0) := \{x \in \mathbb{R}^n : |x-x_0| < r\}$  with center  $x_0 \in \mathbb{R}^n$  and radius r > 0:

$$u(x) = \frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} K_P\left(\frac{x-x_0}{r}, \frac{y-x_0}{r}\right) g(y) \, \mathrm{dS}(y) \quad \text{for } x \in B_r(x_0) \,.$$

For the Dirichlet problem to Laplace's equation on an arbitrary smooth domain Ω ⊂ ℝ<sup>n</sup>, one has a variant of the PIF (Green function representation) with a certain Ω-dependent kernel instead of K<sub>P</sub>. However, one can explicitly compute this kernel only for particularly simple Ω, beside balls e.g. for the half-space Ω = {x ∈ ℝ<sup>n</sup> : x<sub>n</sub> > 0}.

### Proof of the theorem on the Poisson integral formula

On proving the theorem: On needs three decisive properties of  $\mathrm{K}_{\mathrm{P}}$ :

- (1)  $K_P(x, y)$  is harmonic in  $x \in B_1$  in the sense of  $\Delta_x(K_P(x, y)) = 0$ (check by direct computation).
- (2) For fixed  $x_* \in \partial B_1$ , it is  $\lim_{x \to x_*} K_P(x, y) = 0$  for  $y \in \partial B_1 \setminus \{x_*\}$  (convergence uniform away from  $x_*$ ; check by direct computation as well).

(3) It holds  $\int_{\partial B_1} K_P(x, y) dS(y) = 1$  for all  $x \in B_1$  (proof more tricky).

From (1) one deduces that the PIF defines a harmonic function u with  $\Delta u \equiv 0$  in  $B_1$ . Moreover, for  $x_* \in \partial B_1$ , the computation

$$\lim_{x \to x_*} \int_{\partial \mathcal{B}_1} \mathcal{K}_{\mathcal{P}}(x, y) g(y) \, \mathrm{dS}(y) \stackrel{(2)}{=} \lim_{x \to x_*} \int_{\partial \mathcal{B}_1} \mathcal{K}_{\mathcal{P}}(x, y) g(x_*) \, \mathrm{dS}(y) \stackrel{(3)}{=} g(x_*) \,,$$

shows  $u(x_*) = g(x_*)$  and confirms also the Dirichlet BC u = g at  $\partial B_1$ .

## Concluding remarks on the Poisson integral formula

• Alternatively one can derive the PIF and its variants by a (more) constructive approach.

• In principle, one can solve the Dirichlet problem for Poisson's equation

$$\Delta u = f \text{ in } \Omega, \qquad \qquad u = g \text{ at } \partial \Omega$$

as follows: A solution u is obtained as  $u = u_* + u_0$ , where  $u_*$  solves  $\Delta u_* = f$  in  $\Omega$  with arbitrary boundary values (thm on convolution with  $\Phi$ ) and  $u_0$  solves  $\Delta u_0 \equiv 0$  in  $\Omega$  with  $u_0 = g - u_*$  at  $\partial \Omega$  (PIF or variant).

## The mean value property

Another remarkable property of harmonic functions is:

#### Corollary (mean value property)

For  $u \in C^2(B_r(x_0)) \cap C^0(\overline{B_r(x_0)})$  harmonic in  $B_r(x_0)$ , one has the mean value properties . . .

• ... on the solid ball: 
$$u(x_0) = \frac{1}{\alpha_n r^n} \int_{B_r(x_0)} u(x) dx$$
,

• ... on the surface of the ball:  $u(x_0) = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B_r(x_0)} u(y) dS(y)$ .

Here, the division by the volume  $\alpha_n r^n$  and the surface area  $n\alpha_n r^{n-1}$  of the ball  $B_r(x_0)$  turns the right-hand sides into mean values of u.

## Proofs of mean value property and strong max. principle

**Proof of the mean value property:** The version for the surface is the PIF for  $B_r(x_0)$  with  $x = x_0$  (taking into account  $K_P(0, \cdot) = \frac{1}{n\alpha_n}$ ). With this version and polar coordinates compute

$$\int_{B_r(x_0)} u(x) \, \mathrm{d}x = \int_0^r \int_{\partial B_\varrho(x_0)} u(y) \, \mathrm{d}S(y) \, \mathrm{d}\varrho$$
$$= \int_0^r n\alpha_n \varrho^{n-1} u(x_0) \, \mathrm{d}\varrho = \alpha_n r^n u(x_0) \, \mathrm{d}\varrho$$

which confirms the version for the solid ball.

**Proof of the strong max. principle:** For  $M := \max_{\partial\Omega} u$ , it is  $u \leq M$  in  $\Omega$  (weak max. principle). Now show:  $u(x_0) = M$  at  $x_0 \in \Omega \implies u$  constant. For  $B_r(x_0) \subset \Omega$ , from  $u \leq M$  and  $\frac{1}{\alpha_n r^n} \int_{B_r(x_0)} u(x) dx = u(x_0) = M$  (by mean value property) deduce that  $u \equiv M$  is constant on  $\overline{B_r(x_0)}$ . The same arguments works at every point of  $\overline{B_r(x_0)}$ , and altogether  $u \equiv M$  is constant in all of  $\Omega$  as claimed (since  $\Omega$  is connected by assumption).

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#### Euclidean product ansatz for $\Delta u \equiv 0$

Specific PDE solutions in  $\mathbb{R}^2$  (or  $\Omega \subset \mathbb{R}^2$ ) can be found by product ansatz

 $u(x,y) = arphi(x) \, \psi(y)$  with factors  $arphi, \psi \in \mathrm{C}^2(\mathbb{R})$  .

Plugging this into the Laplace equation  $\Delta u \equiv 0$  in  $\mathbb{R}^2$  yields (for  $\varphi \neq 0 \neq \psi$ )

$$arphi''(x)\,\psi(y)+arphi(x)\,\psi''(y)=0$$
 and equivalently  $rac{arphi''(x)}{arphi(x)}=-rac{\psi''(y)}{\psi(y)}\,.$ 

Since the last equation involves x only on the left and y only on the right, one gets  $\frac{\varphi''(x)}{\varphi(x)} = -\frac{\psi''(y)}{\psi(y)} = K$  for a constant  $K \in \mathbb{R}$  and infers the ODEs

$$\varphi''(x) = K\varphi(x)$$
 and  $\psi''(y) = -K\psi(y)$ .

(very basic) case K = 0: φ and ψ are affine functions.
 → Solution u(x, y) is linear combination of xy, x, y, and 1.

#### Euclidean product ansatz for $\Delta u \equiv 0$ (continued)

• case K > 0: For  $K = \omega^2$ ,  $0 \neq \omega \in \mathbb{R}$ , the general solutions of the ODEs are (with constants  $A_1, A_2, B_1, B_2 \in \mathbb{R}$ )

$$\varphi(x) = A_1 e^{\omega x} + A_2 e^{-\omega x}, \qquad \psi(y) = B_1 \cos(\omega y) + B_2 \sin(\omega y),$$

and the solutions of Laplace's equation is (with constants  $C_{ij} = A_i B_j$ )

$$u(x,y) = \varphi(x) \psi(y) = C_{11} e^{\omega x} \cos(\omega y) + C_{21} e^{-\omega x} \cos(\omega y) + C_{12} e^{\omega x} \sin(\omega y) + C_{22} e^{-\omega x} \sin(\omega y)$$

• case K < 0: For  $K = -\omega^2$ ,  $0 \neq \omega \in \mathbb{R}$ , one finds analogously

$$u(x,y) = \varphi(x) \psi(y) = C_{11} \cos(\omega x) e^{\omega y} + C_{21} \sin(\omega x) e^{\omega y} + C_{12} \cos(\omega x) e^{-\omega y} + C_{22} \sin(\omega x) e^{-\omega y}.$$

A similar type of solutions has been mentioned already on slide 65.

#### Euclidean product ansatz for $-\Delta u = \lambda u$

The (Dirichlet) eigenvalue problem for the Laplace operator on  $\Omega \subset \mathbb{R}^2$  is about solutions  $\neq 0$  to the Dirichlet problem for the Helmholtz equation

$$-\Delta u = \lambda u$$
 in  $\Omega$ ,  $u \equiv 0$  at  $\partial \Omega$ 

with parameter  $\lambda \in \mathbb{R}$ . The product ansatz  $u(x,y) = \varphi(x) \psi(y)$  gives constancy of  $-\frac{\varphi''(x)}{\varphi(x)} = \lambda + \frac{\psi''(y)}{\psi(y)}$  and then induces the ODEs

 $\varphi''(x) = -K_1 \varphi(x) \qquad \quad \text{and} \qquad \quad \psi''(y) = -K_2 \psi(y) \,,$ 

with constants  $K_1, K_2 \in \mathbb{R}$  s.t.  $K_1 + K_2 = \lambda$ . In contrast to the preceding considerations, in case  $\lambda > 0$  one may have *two positive* constants  $K_1 = \omega_1^2$  and  $K_2 = \omega_2^2$  and thus may have periodic solutions to *both* ODEs:

 $\varphi(x) = A_1 \cos(\omega_1 x) + A_2 \sin(\omega_1 x), \qquad \psi(y) = B_1 \cos(\omega_2 y) + B_2 \sin(\omega_2 y).$ 

## Euclidean product ansatz for $-\Delta u = \lambda u$ (continued)

In case  $\lambda = \omega_1^2 + \omega_2^2$  with non-zero integers  $\omega_1, \omega_2$ , this yields sine solutions  $u(x,y) = C\sin(\omega_1 x)\sin(\omega_2 y)$ 

(with constant  $C \in \mathbb{R}$ ) to the (Dirichlet) eigenvalue problem on the square

$$-\Delta u = \lambda u ext{ in } (0,\pi)^2, \qquad u \equiv 0 ext{ at } \partial ig((0,\pi)^2ig).$$

Fourier methods (compare the following) show that these solutions (and linear combinations) are the only solutions  $\neq 0$ . Hence, one puts on record:

- The positive integers  $\lambda = \omega_1^2 + \omega_2^2$  with integers  $\omega_1, \omega_2$  are the (Dirichlet) eigenvalues of the Laplace operator  $-\Delta$  on  $(0, \pi)^2$ . The sequence of these eigenvalues starts  $2, 5, 8, 10, 13, 17, 18, 20, 25, 26, \ldots$
- The solutions  $\neq 0$  for a given eigenvalue  $\lambda$  (here the above sine solutions and linear combinations of these with same  $\lambda$ ) are eigenfunctions of  $-\Delta$ .

Similarly, also on other domains  $\Omega$ , the eigenvalues of  $-\Delta$  are a sequence of positive numbers (but usually are not integers).

### Remarks on the Euclidean product ansatz

Remarks on the Euclidean product ansatz:

- In general, the ansatz  $u(x,y) = \varphi(x) \psi(y)$  yields specific solutions in  $\mathbb{R}^2$  or in  $\Omega \subset \mathbb{R}^2$  only and correspondingly can produce solutions with specific boundary conditions at  $\partial\Omega$  only.
- On  $\mathbb{R}^n$  (or on  $\Omega \subset \mathbb{R}^n$ ) one can use the analogous product ansatz  $u(x) = \psi_1(x_1) \psi_2(x_2) \dots \psi_n(x_n)$ . Then similar computations yield specific solutions to the Laplace equation in  $\mathbb{R}^n$  and the Dirichlet eigenvalues and eigenfunctions of  $-\Delta$  on the cube  $(0, \pi)^n$  (and more generally also on cuboids in  $\mathbb{R}^n$ ).

## Fourier (sine) ansatz for $\Delta u \equiv 0$

Alternatively, for PDE solutions u in  $\mathbb{R}^2$  such that u(x, y) is  $2\pi$ -periodic in x and odd in x, one may use the ansatz

$$u(x,y) = \sum_{k=1}^{\infty} a_k(y) \sin(kx)$$

of a Fourier (sine) series in x with y-dependent coefficients  $a_k \in C^2(\mathbb{R})$ (no cosine terms here, since u(x, y) is assumed to be odd in x!).

Plugging the ansatz into the Laplace equation  $\Delta u \equiv 0$  in  $\mathbb{R}^2$  gives

$$\sum_{k=1}^{\infty} \left[ a_k''(y) - k^2 a_k(y) \right] \sin(kx) = 0 \,.$$

Comparing coefficients, one deduces ODEs  $a_k''(y) = k^2 a_k(y)$  with solutions

$$\begin{split} a_k(y) &= b_k \mathrm{e}^{ky} + \widetilde{b}_k \mathrm{e}^{-ky} \quad \text{or} \quad a_k(y) = c_k \sinh(ky) + \widetilde{c}_k \sinh(k(y-L)) \\ \text{(for free constants } b_k, \widetilde{b}_k, c_k, \widetilde{c}_k \in \mathbb{R} \text{ and arbitrary given } L \in (0,\infty)\text{)}. \end{split}$$

# Fourier (sine) ansatz for $\Delta u \equiv 0$ (continued)

The solutions formula just derived can be cast in form

$$u(x,y) = \sum_{k=1}^{\infty} \left[ c_k \sinh(ky) + \tilde{c}_k \sinh(k(y-L)) \right] \sin(kx)$$

and is useful for the Dirichlet Problem on a rectangle

with L > 0 and  $g_0, g_L \in C^1(\mathbb{R})$  odd and  $2\pi$ -periodic (after extension).

Here,  $c_k, \tilde{c}_k$  are determined by the Fourier coefficients (FCs) of  $g_0, g_L$ :

• 
$$u(x,0) = \sum_{k=1}^{\infty} \tilde{c}_k \sinh(-kL) \sin(kx) \rightsquigarrow \tilde{c}_k \sinh(-kL)$$
 FCs of  $g_0$ 

•  $u(x,L) = \sum_{k=1}^{\infty} c_k \sinh(kL) \sin(kx) \quad \rightsquigarrow c_k \sinh(kL) \text{ FCs of } g_L$ 

In this way, for this type of Dirichlet problem, always find a solution u (in form of an infinite series).

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## Example for a Fourier ansatz

As an example consider the Dirichlet problem for the Laplace equation

$$\begin{split} \Delta u &\equiv 0 \text{ in } (0,\pi) \times (0,1) \,, \\ u(0,y) &\equiv 0 \,, \qquad u(\pi,y) \equiv 0 \, & \text{ for } 0 < y < 1 \,, \\ u(x,0) &= 0 \,, \qquad u(x,1) = x(\pi - x) \, & \text{ for } 0 < x < \pi \,. \end{split}$$

The BC u(x,0) = 0 means  $g_0 \equiv 0$  and implies  $\tilde{c}_k = 0$  for all k. Hence, the solution formula (with L = 1) reduces to

$$u(x,y) = \sum_{k=1}^{\infty} c_k \sinh(ky) \sin(kx)$$

with the Fourier coefficients  $c_k \sinh(k)$  of the *odd* function  $g_1(x) = x(\pi - |x|)$  determined by the following coefficient formula:

$$c_k \sinh(k) = \frac{1}{\pi} \int_{-\pi}^{\pi} g_1(x) \sin(kx) \, \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} g_1(x) \sin(kx) \, \mathrm{d}x.$$

# Example for a Fourier ansatz (continued)

With multiple integrations (by parts) compute, for the examplary case:

$$c_k \sinh(k) = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(kx) \, dx = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin(kx) \, dx$$
$$= \frac{2}{\pi k} \left[ \underbrace{-(\pi x - x^2) \cos(kx) \Big|_{x=0}^{\pi}}_{=0} + \int_0^{\pi} (\pi - 2x) \cos(kx) \, dx \right]$$
$$= \frac{2}{\pi k^2} \left[ \underbrace{(\pi - 2x) \sin(kx) \Big|_{x=0}^{\pi}}_{=0} + 2 \int_0^{\pi} \sin(kx) \, dx \right]$$
$$= -\frac{4}{\pi k^3} \cos(kx) \Big|_{x=0}^{\pi} = \begin{cases} 0, & k \text{ even} \\ \frac{8}{\pi k^3}, & k \text{ odd} \end{cases}.$$

This yields the solution u of the Dirichlet problem as the infinite series

$$u(x,y) = \sum_{\substack{k=1\\k \text{ odd}}}^{\infty} \frac{8}{\pi k^3 \sinh(k)} \sinh(ky) \sin(kx) \,.$$

## Remarks on the Fourier ansatz

Remarks on the Fourier ansatz:

- Clearly, one may transform from an arbitrary rectangle  $(x_1, x_2) \times (y_1, y_2)$  with  $x_1 < x_2$  and  $y_1 < y_2$  to  $(0, \pi) \times (0, L)$ . (Translate, scale with factor  $\frac{\pi}{x_2 x_1}$ , take  $L := \frac{\pi(y_2 y_1)}{x_2 x_1}$ !)
- For the eigenvalue problem on a square/rectangle/cube/cuboid, a similar computation confirms that <u>all</u> eigenvalues have been found.
- The Poisson equation  $\Delta u = f$  can be treated in similar manner if one expands also f(x, y) as a Fourier series in x
- If the roles of x and y are switched, the ansatz applies analogously.
- In contrast to the simple product ansatz discussed before, the Fourier ansatz can partially deal with general boundary data (at least on two sides on a rectangle if one has zero data on the other two).

#### Polar-coordinates product ansatz for $\Delta u \equiv 0$

One can express the 2d Laplace operator in planar polar coordinates as

$$\Delta u(r\cos\vartheta, r\sin\vartheta) = \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2}\right]u(r\cos\vartheta, r\sin\vartheta)$$

for r > 0 and  $\vartheta \in \mathbb{R}$  (checking formula "from right to left" straightforward).

#### Thus, the polar-coordinates product ansatz

 $u(r\cos\vartheta, r\sin\vartheta) = \eta(r) \kappa(\vartheta)$  with factors  $\eta \in C^2((0,\infty))$ ,  $\kappa \in C^2(\mathbb{R})$ for solutions u to Laplace's equation  $\Delta u \equiv 0$  in  $\mathbb{R}^2$  yields

$$\eta''(r)\kappa(\vartheta) + \frac{1}{r}\eta'(r)\kappa(\vartheta) + \frac{1}{r^2}\eta(r)\kappa''(\vartheta) = 0.$$

The rewriting  $\frac{r^2\eta''(r)}{\eta(r)} + \frac{r\eta'(r)}{\eta(r)} = -\frac{\kappa''(\vartheta)}{\kappa(\vartheta)}$  leads to the ODEs ( $\lambda \in \mathbb{R}$  constant)  $r^2\eta''(r) + r\eta'(r) = \lambda \eta(r)$  and  $\kappa''(\vartheta) = -\lambda \kappa(\vartheta)$ .

## Polar-coordinates product ansatz for $\Delta u \equiv 0$ (continued)

The (relevant) solutions of these two ODEs can be made fully explicit:

- ODE  $\kappa'' = -\lambda \kappa$ : polar-coordinates structure  $\rightsquigarrow$  only  $2\pi$ -periodic solutions  $\kappa$  relevant, and these exist only for  $\lambda = \omega^2$  in the following cases:
  - case  $\omega = 0$ :  $\kappa$  constant,
  - case  $\omega \in \mathbb{Z} \setminus \{0\}$ :  $\kappa(\vartheta) = B_1 \cos(\omega \vartheta) + B_2 \sin(\omega \vartheta)$ .
- ODE  $r^2 \eta''(r) + r \eta'(r) = \omega^2 \eta(r)$ : is Eulerian, by DE I equivalent with  $\widetilde{\eta}''(s) = \omega^2 \widetilde{\eta}(s)$  via trafos  $\widetilde{\eta}(s) = \eta(e^s)$  and  $\eta(r) = \widetilde{\eta}(\log r)$ . Therefore: • case  $\omega = 0$ :  $\widetilde{\eta}(s) = A_1 s + A_2 \longrightarrow \eta(r) = A_1 \log r + A_2$ • case  $\omega \in \mathbb{Z} \setminus \{0\}$ :  $\widetilde{\eta}(s) = A_1 e^{\omega s} + A_2 e^{-\omega s} \rightsquigarrow \eta(r) = A_1 r^{\omega} + A_2 r^{-\omega}$ .
- All in all,  $u(r\cos\vartheta, r\sin\vartheta) = \eta(r) \,\kappa(\vartheta)$  is a linear combination of:
  - case ω = 0: log r (fundamental sol. in polar coords) and 1 (constant),
    case ω ∈ Z \ {0}: r<sup>ω</sup> cos(ωϑ), r<sup>ω</sup> sin(ωϑ) (harmonic poly.s in pol. coords) and r<sup>-ω</sup> cos(ωϑ), r<sup>-ω</sup> sin(ωϑ) (new types of solutions with singularity; correspond to partial derivatives of the fundamental solution).

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DE II for Engineering

## 3.2 The diffusion or heat equation

The homogeneous and inhomogeneous diffusion or heat equation (HE) for a scalar function u of time-space variables (t, x) read

 $u_t - \Delta u \equiv 0$  in  $\Omega_T$  and  $u_t - \Delta u = f$  in  $\Omega_T$ ,

respectively, with an open spatial domain  $\Omega \subset \mathbb{R}^n$  and a time horizon  $T \in (0, \infty]$ , and with abbreviations  $S_T := (0, T) \times S$  and  $\Delta := \Delta_x$ .

Basic remarks on the heat equation:

- The typical Cauchy-Dirichlet problem complements the heat equation with an IC  $u(0, \cdot) = u_0$  in  $\Omega$  and a Dirichlet BC u = g at  $(\partial \Omega)_T$ .
- The heat equation is linear. Its theory often resembles the one of the Laplace and Poisson equation (with modified effects in the variable *t*).
- If not stated otherwise, the space dimension  $n \in \mathbb{N}$  is arbitrary. However, partially we will restrict our treatment to the 1d case n=1.

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# Exemplary solutions to the 1d HE, and parabolic scaling

Exemplary solutions to the 1d homogeneous HE (n=1) in all of  $\mathbb{R} \times \mathbb{R}$  are:

- solutions in product(-ansatz) form  $u(t,x)=\varphi(t)\,\psi(x),$ 
  - e.g.  $u(t,x) = e^{t\pm x}$  or  $u(t,x) = e^t \cosh x$ ,  $u(t,x) = e^t \sinh x$ , e.g.  $u(t,x) = e^{-t} \cos x$ ,  $u(t,x) = e^{-t} \sin x$ ,
- polynomial solutions,

e.g. 
$$u(t,x) = 2t + x^2$$
,  $u(t,x) = 6tx + x^3$ ,  $u(t,x) = 12t^2 + 12tx^2 + x^4$ .

The polynomial examples are homogeneous in  $(t, x^2)$ . Corresponding analogies between t and  $x^2$  are typical for the heat equation and manifest also in the parabolic scaling of the heat equation (valid for arbitrary n):

u solves the homogeneous heat equation ,  $\,\lambda>0$ 

 $\implies \widetilde{u}(t,x):=u(\lambda^2 t,\lambda x)$  solves the homogeneous heat equation .

## The fundamental solution of the heat equation

The fundamental solution  $\Phi$  of the heat equation is given by

$$\Phi(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n$$

and has the following properties:

- $\Phi$  is positive and solves the heat equation in  $(0,\infty) imes \mathbb{R}^n$ ,
- $\int_{\mathbb{R}^n} \Phi(t, x) \, \mathrm{d}x = 1$  for all t > 0 (normalization),
- $\Phi(\lambda^2 t, \lambda x) = \lambda^{-n} \Phi(t, x)$  for all  $t, \lambda > 0$  and  $x \in \mathbb{R}^n$  (scaling),
- $\Phi(t, x)$  is rotationally symmetric in x,

• 
$$\lim_{t \searrow 0} \Phi(t, x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$
 for  $x \in \mathbb{R}^n$  ("initial condition").

One can derive the formula for  $\Phi$  from (e.g.) the first four properties. A physics interpretation is that  $\Phi$  describes the evolution of a unit amount of heat or a unit mass concentrated at time t = 0 in the point x = 0.

# Solution formula for the Cauchy problem for the HE

In the full-space case  $\Omega = \mathbb{R}^n$ , one considers the Cauchy problem out of HE and IC (BC dropped) and can solve by the following integral formula:

Theorem (solving the HE Cauchy problem by partial convolution with  $\Phi$ ) For bounded  $u_0 \in C^0(\mathbb{R}^n)$ , by setting

$$u(t,x) := \int_{\mathbb{R}^n} \Phi(t,x{-}y) \, u_0(y) \, \mathrm{d}y$$

one obtains a bounded solution  $u \in C^2((0,T) \times \mathbb{R}^n) \cap C^0([0,T) \times \mathbb{R}^n)$  to the Cauchy problem for the homogeneous heat equation

 $u_t - \Delta u \equiv 0$  in  $(0, T) \times \mathbb{R}^n$ ,  $u(0, \cdot) = u_0$  in  $\mathbb{R}^n$ .

The theorem resembles the solution of Poisson's equation by convolution, but has in common even more with the PIF. In fact,  $\Phi$  has properties very much analogous to those of the Poisson kernel  $\mathrm{K}_{\mathrm{P}}$  used in the proof of the PIF, and the current theorem can derived in essentially the same way.  $\Box$ 

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#### Conservation principle for heat and mass

#### Corollary (conservation principle for heat and mass)

If  $u_0 \in C^0(\mathbb{R}^n)$  is bounded with  $\int_{\mathbb{R}^n} |u_0(x)| \, dx < \infty$ , the solution u of the theorem satisfies

$$\int_{\mathbb{R}^n} u(t,x) \, \mathrm{d}x = \int_{\mathbb{R}^n} u_0(x) \, \mathrm{d}x \qquad \text{for all } t > 0 \, .$$

Physical meaning: Total amount of heat and total mass, respectively, (if finite) are preserved from initial time t = 0 up to any time t > 0.

Proof of the corollary: Use the formula of the theorem to compute

$$\int_{\mathbb{R}^n} u(t,x) \, \mathrm{d}x = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(t,x-y) \, u_0(y) \, \mathrm{d}y \, \mathrm{d}x$$
$$= \int_{\mathbb{R}^n} \underbrace{\int_{\mathbb{R}^n} \Phi(t,x-y) \, \mathrm{d}x}_{=1} u_0(y) \, \mathrm{d}y = \int_{\mathbb{R}^n} u_0(y) \, \mathrm{d}y \, . \qquad \Box$$

#### Fourier (sine) ansatz for the 1d heat equation $u_t - u_{xx} \equiv 0$

The Cauchy-Dirichlet problem for the 1d HE on  $(0,\pi)$ 

can be approached, as done earlier for Laplace's equation, by a Fourier (sine) series ansatz with t-dependent coefficients  $a_k \in C^1((0,T))$ :

$$u(t,x) = \sum_{k=1}^{\infty} a_k(t) \sin(kx)$$

This ansatz in the heat equation gives the ODE  $a'_k(t) = -k^2 a_k(t)$  with solution  $a_k(t) = c_k \exp(-k^2 t)$ . All in all, this results in the solution formula

$$u(t,x) = \sum_{k=1}^{\infty} c_k \exp(-k^2 t) \sin(kx) ,$$

where evaluation at t = 0 reveals that  $c_k \in \mathbb{R}$  are the Fourier coefficients of the (odd and  $2\pi$ -periodic extension of the) initial datum  $u_0 \in C^1(\mathbb{R})$ .

 $u_0$ 

### Example for the Fourier series ansatz for the 1d HE

As an example consider the Cauchy-Dirichlet problem for the HE (with  $T=\infty$ )

$$\begin{split} u_t - u_{xx} &\equiv 0 \, \operatorname{in} \, (0,\infty) \!\times\! (0,\pi) \,, \\ u(0,x) &= x(\pi\!-\!x) & \quad \text{for} \, 0 \!<\! x \!<\! \pi \,, \\ u(t,0) &= 0 \,, \qquad u(t,\pi) = 0 & \quad \text{for} \, 0 \!<\! t \!<\! \infty \,. \end{split}$$

The Fourier coefficients of  $u_0(x)=x(\pi-|x|)$  are

$$c_k = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(kx) \, \mathrm{d}x = \dots = \begin{cases} 0, & k \text{ even} \\ \frac{8}{\pi k^3}, & k \text{ odd} \end{cases},$$

where the actual computation has been carried out already on slide 86. Plugging the results into the solution formula for the current situation gives

$$u(t,x) = \sum_{\substack{k=1\\k \text{ odd}}}^{\infty} \frac{8}{\pi k^3} \exp(-k^2 t) \sin(kx) \,.$$

### Remarks on the Fourier series ansatz for the 1d HE

Remarks on the Fourier series ansatz for the 1d HE:

- From  $(x_1, x_2)$  with arbitrary  $x_1 < x_2$  one may transform to  $(0, \pi)$ . (Translate, use parabolic scaling with factor  $\lambda = \frac{\pi}{x_2 - x_1}!$ )
- The inhomogeneous 1d HE  $u_t u_{xx} = f$  can be treated in similar manner if one expands also f(t, x) as a Fourier series in x.
- In contrast to a simple product ansatz, the Fourier ansatz can deal with general initial data (at least in case of zero Dirichlet boundary data, and in some other cases as well).

## The maximum principle for the heat equation

The maximum principle for the homogeneous HE resembles the one for Laplace's equation, but uses only the parabolic boundary

$$\partial_{\mathrm{par}}\Omega_T := (\{0\} \times \overline{\Omega}) \cup (\partial \Omega)_T$$

of the time-space cylinder  $\Omega_T = (0, T) \times \Omega$ :

Theorem (maximum principle for the homogeneous heat equation) For a solution  $u \in C^2(\Omega_T) \cap C^0([0,T) \times \overline{\Omega})$  to the homogeneous heat equation in  $\Omega_T$ , with bounded open  $\Omega \subset \mathbb{R}^n$  and  $T \in (0,\infty]$ , one has: (1) weak maximum principle: There holds

$$\inf_{\partial_{\mathrm{par}}\Omega_T} u \le u(t,x) \le \sup_{\partial_{\mathrm{par}}\Omega_T} u \quad \text{for all } (t,x) \in \Omega_T \,.$$

(2) strong maximum principle: If  $\Omega$  is connected and u is <u>non-constant</u> in  $\Omega_t$  with  $t \in (0,T)$ , there holds even

$$\inf_{\partial_{\mathrm{par}}\Omega_t} u < u(t,x) < \sup_{\partial_{\mathrm{par}}\Omega_t} u \qquad \text{for all } x\in\Omega\,.$$

 $\partial_{\mathrm{par}}\Omega_T$ 

### Conclusions from the maximum principle for the HE

**Corollary 1:** For a solution u to the homogeneous HE, as in the theorem:

$$|u(t,x)| \leq \sup_{\partial_{\mathrm{par}}\Omega_T} |u| \qquad \text{for all } (t,x) \in \Omega_T \,.$$

Corollary 2 (uniqueness in the Cauchy-Dirichlet problem for the HE) For bounded open  $\Omega \subset \mathbb{R}^n$ ,  $T \in (0, \infty]$ ,  $f \in C^0(\Omega_T)$ ,  $u_0 \in C^0(\Omega)$ ,  $g \in C^0((\partial \Omega)_T)$ , the Cauchy-Dirichlet problem for the heat equation  $u_t - \Delta u = f \text{ in } \Omega_T$ ,  $u(0, \cdot) = u_0 \text{ in } \Omega$ ,  $u = g \text{ at } (\partial \Omega)_T$ has at most one solution  $u \in C^2(\Omega_T) \cap C^0([0, T) \times \overline{\Omega})$ .

The corollaries are proved the same way as for the Laplace/Poisson eqn.  $\Box$ Also the earlier Corollary 3 (cont. dependence on boundary data) has an analog.

#### Proof of the weak maximum principle for the heat equation

In analogy with the case of Laplace's equation one implements:

**Proof of the weak maximum principle:** It is enough to prove the claim in  $\Omega_{\Theta}$  with  $0 < \Theta < T$ . For  $u_{\varepsilon}(t, x) := u(t, x) - \varepsilon t$  with  $\varepsilon > 0$  observe

$$(u_{\varepsilon})_t(t,x) - \Delta u_{\varepsilon}(t,x) < 0 \qquad \text{for all } (t,x) \in (0,\Theta] \times \Omega$$

Since a maximum point (t, x) of  $u_{\varepsilon}$  in  $(0, \Theta] \times \Omega$  necessarily satisfies  $(u_{\varepsilon})_t(t, x) \geq 0$  and  $\Delta u_{\varepsilon}(t, x) \leq 0$ , such a point does <u>not</u> exist. This implies  $\max_{[0,\Theta] \times \overline{\Omega}} u_{\varepsilon} = \sup_{\partial_{\mathrm{par}} \Omega_{\Theta}} u_{\varepsilon}$  and

$$u(t,x) - \varepsilon \Theta < u_\varepsilon(t,x) < \sup_{\partial_{\mathrm{par}}\Omega_\Theta} u_\varepsilon \leq \sup_{\partial_{\mathrm{par}}\Omega_\Theta} u \qquad \text{for } (t,x) \in \Omega_\Theta \,.$$

In the limit  $\varepsilon \to 0$  one infers  $u(t, x) \leq \sup_{\partial_{par}\Omega_{\Theta}} u$ . The reverse inequality  $u(t, x) \geq \inf_{\partial_{par}\Omega_{\Theta}} u$  follows similarly.

The proof of the *strong* maximum principle requires an HE-adapted mean value property and is not discussed here any further.

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# 3.3 The wave equation

The homogeneous and inhomogeneous wave equation (WE) for a scalar function u of time-space variables (t, x) read

 $u_{tt} - \Delta u \equiv 0$  in  $\Omega_T$  and  $u_{tt} - \Delta u = f$  in  $\Omega_T$ , respectively, with open  $\Omega \subset \mathbb{R}^n$  and  $T \in (0, \infty]$  (other terminology as before).

Basic remarks on the wave equation:

- The typical Cauchy-Dirichlet problem complements the wave equation with <u>two</u> ICs  $u(0, \cdot) = u_0$  and  $u_t(0, \cdot) = v_0$  in  $\Omega$  and with a Dirichlet BC u = g at  $(\partial \Omega)_T$ .
- Also the wave equation is linear, but its theory strongly differs from the ones of the Laplace/Poisson equation and the heat equation.
- We mostly work in space dimension n = 1 and turn briefly to  $n \in \{2, 3\}$ . Theory for arbitrary  $n \in \mathbb{N}$  is beyond the scope of this lecture.

#### Exemplary solutions to the 1d wave equation

Exemplary solutions u to the 1d wave equation (n=1) in all of  $\mathbb{R} \times \mathbb{R}$  are:

- solutions in product(-ansatz) form u(t, x) = φ(t) ψ(x),
  e.g. e<sup>±t</sup>e<sup>±x</sup> (all 4 combinations of signs; alternatively write with cosh, sinh)
  e.g. cost cos x, cost sin x, sin t cos x, sin t sin x
- polynomial solutions,

e.g. 
$$u(t,x) = t^2 + x^2$$
,  $u(t,x) = t^3 + 3tx^2$ ,  $u(t,x) = t^4 + 6t^2x^2 + x^4$ .

For comparison with the following theory, we put on record that all exemplary solutions can be rewritten as functions of x+t and x-t, e.g.

$$e^{t}e^{-x} = e^{-(x-t)}, \qquad \cos t \cos x = \frac{1}{2}\cos(x+t) + \frac{1}{2}\cos(x-t),$$
  
in  $t\sin x = -\frac{1}{2}\sin(x+t) + \frac{1}{2}\sin(x-t), \qquad t^{2} + x^{2} = \frac{1}{2}(x+t)^{2} + \frac{1}{2}(x-t)^{2}.$ 

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## Solving the 1d wave equation by change of variables

The homogeneous 1d WE  $u_{tt}-u_{xx} \equiv 0$  in  $(t_1, t_2) \times (x_1, x_2)$  can be solved by change of variables  $r = \frac{x+t}{2}$ ,  $s = \frac{x-t}{2}$  or equivalently x = r+s, t = r-s:

If u solves, for w(r,s) := u(r-s, r+s), one finds the simple PDE  $w_{sr} \equiv 0$ , and then one can solve by two integrations. The resulting general solution is w(r,s) = h(2r) + k(2s) and in terms of the original u reads

 $\left| u(t,x) = h(x+t) + k(x-t) \right|$  with arbitrary functions  $h,k \in C^2(\mathbb{R})$ .

Interpretation: h(x+t) leftward-moving wave packet (speed -1), k(x-t) rightward-moving wave packet (speed 1).

 $\begin{array}{l} \mbox{Calculation of the PDE } w_{sr} \equiv 0 \mbox{ for } w : \mbox{From } w(r,s) := u(r-s,r+s) \mbox{ deduce by } \\ \mbox{the chain rule first } w_s(r,s) = -u_t(r-s,r+s) + u_x(r-s,r+s) \mbox{ and then } \\ w_{sr}(r,s) = -u_{tt}(r-s,r+s) - u_{tx}(r-s,r+s) + u_{xt}(r-s,r+s) + u_{xx}(r-s,r+s) \\ = -u_{tt}(r-s,r+s) + u_{xx}(r-s,r+s) \overset{\mbox{WE}}{=} 0 \ . \end{array}$ 

# Cauchy problem for the 1d WE, d'Alembert's formula

The preceding solution formula can be adapted to the Cauchy problem for the 1d WE with the two ICs for  $u(0, \cdot)$  and  $u_t(0, \cdot)$ :

Theorem (d'Alembert's solution formula for Cauchy problem for 1d WE) Consider  $u_0 \in C^2(\mathbb{R})$  and  $v_0 \in C^1(\mathbb{R})$ . The unique solution  $u \in C^2(\mathbb{R} \times \mathbb{R})$ to the Cauchy problem for the homogeneous 1d wave equation

 $u_{tt} - u_{xx} \equiv 0$  in  $\mathbb{R} \times \mathbb{R}$ ,

$$u(0, \cdot) = u_0 \text{ in } \mathbb{R}, \qquad u_t(0, \cdot) = v_0 \text{ in } \mathbb{R}$$

is given by d'Alembert's solution formula

$$u(t,x) := \frac{1}{2}u_0(x+t) + \frac{1}{2}u_0(x-t) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,\mathrm{d}y \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}\,.$$

- Interpretation: wave propagation (as before) with initial displacement  $u_0$  and initial velocity  $v_0$  at time t = 0.
- The WE  $u_{tt}-c^2u_{xx} \equiv 0$  with speed of propagation c > 0 can be solved analogously by  $u(t,x) := \frac{1}{2}u_0(x+ct) + \frac{1}{2}u_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y) \,\mathrm{d}y.$

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## Deduction of d'Alembert's formula

#### Proof/deduction of d'Alembert's solution formula:

On the basis of the solution formula u(t,x) = h(x+t) + k(x-t), compute:

$$\underset{\longrightarrow}{\operatorname{ICs}} \begin{array}{c} h+k=u_0 \\ h'-k'=v_0 \end{array} \xrightarrow{\operatorname{derivative}} \begin{array}{c} h'+k'=u_0' \\ \swarrow \\ h'-k'=v_0 \end{array} \xrightarrow{\operatorname{solve}} \begin{array}{c} h'=\frac{1}{2}u_0'+\frac{1}{2}v_0 \\ \swarrow \\ k'=\frac{1}{2}u_0'-\frac{1}{2}v_0 \end{array}$$

antiderivative 
$$h(x) = \frac{1}{2}u_0(x) + \frac{1}{2}\int_0^x v_0(y) \, dy + C$$
  
 $k(x) = \frac{1}{2}u_0(x) - \frac{1}{2}\int_0^x v_0(y) \, dy - C$  (same *C*, as  $h+k = u_0$ )

$$\underset{\longrightarrow}{\text{plug in }} u(t,x) = h(x+t) + k(x-t)$$

$$= \frac{1}{2}u_0(x+t) + \frac{1}{2}u_0(x-t) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y) \, \mathrm{d}y$$

So, d'Alembert's formula is obtained.

# Cauchy-Dirichlet problem for the 1d WE on the half-line

In strong analogy with d'Alembert's formula one may solve the Cauchy-Dirichlet problem for the homogeneous 1d wave equation on  $(0, \infty)$ :

$$\begin{split} u_{tt} - u_{xx} &\equiv 0 \text{ in } (0, \infty) \times (0, \infty) ,\\ u(0, \cdot) &= u_0 \text{ in } (0, \infty) , \qquad u_t(0, \cdot) = v_0 \text{ in } (0, \infty) ,\\ u(\cdot, 0) &= g \text{ in } (0, \infty) . \end{split}$$

$$\begin{split} & \text{For } u_0, g \in \mathcal{C}^2([0,\infty)), \, v_0 \in \mathcal{C}^1([0,\infty)) \text{ with } g(0) = u_0(0), \, g'(0) = v_0(0), \\ & g''(0) = u_0''(0) \text{ (compatibility of data at } (t,x) = (0,0)), \text{ obtain the solution as} \\ & u(t,x) = \begin{cases} \frac{1}{2}u_0(x+t) + \frac{1}{2}u_0(x-t) + \frac{1}{2}\int_{x-t}^{x+t}v_0(y)\,\mathrm{d}y & \text{for } x \geq t \\ \frac{1}{2}u_0(t+x) - \frac{1}{2}u_0(t-x) + g(t-x) + \frac{1}{2}\int_{t-x}^{t+x}v_0(y)\,\mathrm{d}y & \text{for } x \leq t \end{cases}. \end{split}$$

Specifically for  $g \equiv 0$ , the last formula models the odd reflection of wave packets on a string or rope which is clamped at x = 0.

For bounded intervals  $\Omega$ , in principle one can derive similar formulas, but possibly will need lots of case distinctions (since multiple reflection may occur).

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Fourier (sine) ansatz for the 1d wave equation  $u_{tt}-u_{xx} \equiv 0$ 

The Cauchy-Dirichlet problem for the 1d WE on  $(0, \pi)$ 

$$\begin{aligned} u_{tt} - u_{xx} &\equiv 0 \text{ in } (-T, T) \times (0, \pi) , \\ u(0, \cdot) &= u_0 , \qquad u_t(0, \cdot) = v_0 , \\ u(\cdot, 0) &\equiv 0 , \qquad u(\cdot, \pi) \equiv 0 \end{aligned} \qquad 0$$

can also be treated by the previous method. However, more favorable is usually the Fourier (sine) series ansatz with  $a_k \in C^2((-T,T))$ :

$$u(t,x) = \sum_{k=1}^{\infty} a_k(t) \sin(kx) \,.$$

This ansatz in the WE leads to the ODE  $a_k''(t) = -k^2 a_k(t)$  with general solution  $a_k(t) = c_k \cos(kt) + \tilde{c}_k \sin(kt)$ . This results in the solution formula

$$u(t,x) = \sum_{k=1} \left[ c_k \cos(kt) + \tilde{c}_k \sin(kt) \right] \sin(kx) ,$$

where  $c_k$  and  $k\tilde{c}_k$  are the Fourier coefficients of the (odd and  $2\pi$ -periodic extensions of the) initial data  $u_0 \in C^2(\mathbb{R})$  and  $v_0 \in C^1(\mathbb{R})$ .

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# Cauchy problem for the 3d and 2d wave equation

Theorem (solution formulas for the Cauchy problem for the 3d and 2d WE) Consider  $u_0 \in C^3(\mathbb{R})$  and  $v_0 \in C^2(\mathbb{R})$ . The unique solution  $u \in C^2(\mathbb{R} \times \mathbb{R}^n)$ to the Cauchy problem for the homogeneous wave equation

$$u_{tt} - \Delta u \equiv 0$$
 in  $\mathbb{R} \times \mathbb{R}^n$ ,

 $u(0, \cdot) = u_0 \text{ in } \mathbb{R}^n, \qquad u_t(0, \cdot) = v_0 \text{ in } \mathbb{R}^n$ 

with  $n \in \{2,3\}$  is given as follows:

• case n = 3: Kirchhoff's solution formula for  $(t, x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3$ :

$$u(t,x) := \frac{1}{4\pi t^2} \int_{\partial B_{|t|}(x)} \left[ u_0(y) + (y-x) \cdot \nabla u_0(y) + t \, v_0(y) \right] dS(y) \,,$$

• case n = 2: Poisson's solution formula for  $(t, x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^2$ :

$$u(t,x) := \frac{1}{2\pi |t|} \int_{\mathcal{B}_{|t|}(x)} \frac{u_0(y) + (y-x) \cdot \nabla u_0(y) + t \, v_0(y)}{\sqrt{t^2 - |y-x|^2}} \, \mathrm{d}y \, .$$

The Kirchhoff formula involves a surface integral over the surface  $\partial B_{|t|}(x)$  of the ball  $B_{|t|}(x)$ , the Poisson formula involves an integral over the circular disc  $B_{|t|}(x)$ .

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## On the verification of Kirchhoff's formula

**Partial verification:** For n = 3, split integral as  $u(t, x) = u_{u_0}(t, x) + u_{v_0}(t, x)$  with  $u_{v_0}(t,x) := \frac{1}{4\pi t^2} \int_{\partial \mathbf{B}_{tru}(x)} t \, v_0(y) \, \mathrm{dS}(y) = \frac{t}{4\pi} \int_{\partial \mathbf{B}_{tr}} v_0(x+t\nu) \, \mathrm{dS}(\nu) \, .$ For t > 0, then verify solution property of  $u_{v_0}$  (via  $(tf(t))_{\iota\iota} = (2+t\frac{d}{d\iota})f'(t)$ ):  $\left(u_{v_0}\right)_{tt}(t,x) = \frac{1}{4\pi} \left(2 + t\frac{\mathrm{d}}{\mathrm{d}t}\right) \int_{\mathrm{OP}} \nu \cdot \nabla v_0(x + t\nu) \,\mathrm{dS}(\nu)$  $= \frac{1}{4\pi} \left( 2 + t \frac{\mathrm{d}}{\mathrm{d}t} \right) \frac{1}{t^2} \int_{\partial \mathbf{B}_{\mathbf{c}}(x)} \frac{y - x}{t} \cdot \nabla v_0(y) \,\mathrm{dS}(y)$  $=\frac{2}{t^2}-\frac{2t}{t^3}+\frac{1}{t}\frac{d}{t}=\frac{1}{t}\frac{d}{t}$  $= \frac{1}{4\pi t} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathrm{B}_{-}(x)} \Delta v_0(y) \,\mathrm{d}y = \frac{1}{4\pi t} \int_{\partial \mathrm{B}_{-}(x)} \Delta v_0(y) \,\mathrm{dS}(y)$  $= \frac{t}{4\pi} \int_{\Omega \mathcal{D}} \Delta v_0(x+t\nu) \,\mathrm{dS}(\nu) = \Delta_x \left[ u_{v_0}(t,x) \right].$ 

Solution properties for t < 0 and for  $u_{u_0}$  follow similarly. The proof that Kirchhoff's formula yields <u>all</u> solutions, however, is more elaborate. Poisson's formula for n = 2 can be deduced by Hadamard's method of (dimension) descent.

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## Concluding remark on the wave equation

Concluding remark on the wave equation:

- In case of the Dirichlet problem for Laplace's equation or the Cauchy problem for the heat equation, solution formulas use the boundary or initial datum *on its full domain*. In contrast, for computing a solution u of the wave equation  $u_{tt} \Delta u \equiv 0$  at (t, x) one merely needs ...
  - case n = 1 (d'Alembert):  $u_0$  at  $\{x-t, x+t\}$  and  $v_0$  in (x-|t|, x+|t|) only,
  - case n=3 (Kirchhoff):  $u_0$  and  $v_0$  at  $\partial \mathrm{B}_{|t|}(x)$  only,
  - case n = 2 (Poisson):  $u_0$  and  $v_0$  in  $B_{|t|}(x)$  only.

This highlights a central point of the theory: The wave equation models wave propagation with (scalar) speed 1 (and in form  $u_{tt}-c^2\Delta u \equiv 0$  with arbitrary speed c > 0). This contrasts with *infinite* speed of propagation in the previously considered case of the heat equation.

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