# Partielle Differentialgleichungen
# TUHH
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## Modellierung mit PDEs

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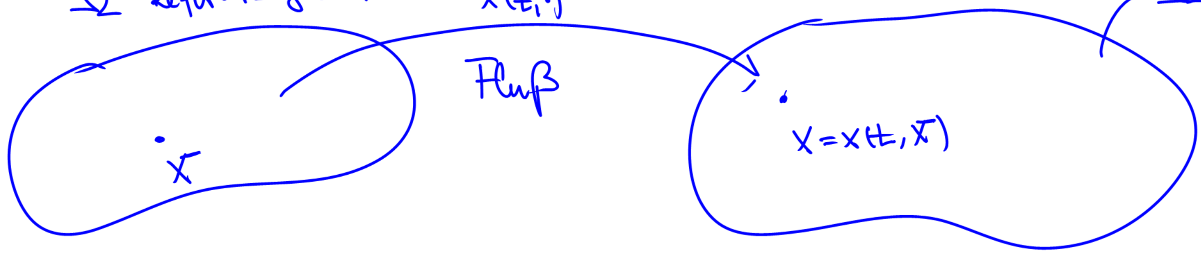
Reynolds Transport Theorem $\qquad$ $t$ Zeit

$\Omega$ Referenzgebiet $\qquad$ $x(t,\cdot)$ $\qquad$ $\Omega(t) = \{x(t,\bar{X});\ \bar{X} \in \Omega\}$




$\subset \mathbb{R}^d$

Sei $f(t,x)$ physikalische Variable (z.Bsp. eine Dichte)

„Masse" in $\Omega(t)$ : $\quad \frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx \overset{!}{=} 0 \qquad$ (Dichte × Volumen)

meint Masseerhaltung

A: $\quad$ A1| $\quad x(t_0,\bar{X}) = \bar{X} \qquad$ ($t_0$ Anfangszeit)

A2| $\quad (t,\bar{X}) \mapsto x(t,\bar{X})$ stetig diffbar

A3| $\quad \forall_{t \geq t_0}: \ \Omega \ni \bar{X} \mapsto x(t,\bar{X}) \in \Omega(t)$ invertierbar

A4| $\quad J(t,\bar{X}) := \det\left(\frac{\partial x_j}{\partial \bar{X}_k}(t,\bar{X})\right)_{j,k=1}^{d} > 0 \quad \forall_{t \geq t_0},\ \bar{X} \in \Omega$

<u>Satz</u>: $$\frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx = \int_{\Omega(t)} f_t(t,x) + \operatorname{div}_x\big(f(t,x)\,v(t,x)\big)\,dx$$

wobei $v(t,x) = v(t,x(t,\bar{x})) = \frac{\partial}{\partial t} x(t,\bar{x})$ Flußgeschwindigkeit

<u>Nachweis</u>: Es gilt die Transformationsformel

$$\int_{\Omega(t)} f(t,x)\,dx = \int_{\Omega} f(t,x(t,\bar{x}))\underbrace{\left|\det\left(\frac{\partial x_j}{\partial \bar{x}_k}(t,\bar{x})\right)_{j,k=1}^{d}\right|}_{J(t,\bar{x})}\,d\bar{x}$$

$$\Longrightarrow \frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx = \int_{\Omega}\frac{d}{dt}\left[f(t,x(t,\bar{x}))\,J(t,\bar{x})\right]d\bar{x}$$

$$= \int_{\Omega}\left[f_t(t,x(t,\bar{x})) + \sum_{k=1}^{d} f_{x_k}(t,x(t,\bar{x}))\,\overbrace{\frac{\partial x_k}{\partial t}(t,\bar{x})}^{v_k(t,\bar{x})}\right]J(t,\bar{x}) +$$

$$+ f(t,x(t,\bar{x}))\,\frac{d}{dt}J(t,\bar{x})\,d\bar{x} = (*)$$

$$\frac{d}{dt}\underbrace{J(t,\bar{x})}_{\det A(t)} = \frac{d}{dt}\det A(t) = \underbrace{\operatorname{spur}}_{\text{Summe der Diagonalelemente}}\left(A^{-1}(t)\,\frac{d}{dt}A(t)\right)\det A(t)$$

Nachweis SgL
Eck/Garcke/Knabner

$$\frac{\partial a_{ij}(t)}{\partial t} = \frac{d}{dt}\big(A(t)\big)_{ij} = \frac{d}{dt}\frac{\partial x_i}{\partial \bar{x}_j}(t,\bar{x}) = \frac{\partial}{\partial \bar{x}_j}\frac{\partial x_i}{\partial t}(t,\bar{x}) = \frac{\partial}{\partial \bar{x}_j} v_i(t,\bar{x})$$

$$= \frac{\partial}{\partial \bar{x}_j} v_i(t,x(t,\bar{x})) = \sum_{k=1}^{d}\frac{\partial v_i}{\partial x_k}(t,x(t,\bar{x}))\,\underbrace{\frac{\partial x_k}{\partial \bar{x}_j}(t,\bar{x})}_{a_{kj}(t)}$$

einsetzen

Damit

$$\operatorname{Spur}\left(A^{-1}(t)\,\frac{d}{dt}A(t)\right) = \sum_{i,j=1}^{d} A^{-1}(t)_{ji}\,\frac{\partial a_{ij}(t)}{\partial t}$$

$$= \sum_{i,j,k=1}^{d} A^{-1}(t)_{ji}\,\frac{\partial v_i}{\partial x_k}(t,x(t,X))\,a_{kj}(t) \qquad A^{-1}(t)A(t) = I$$

$$= \sum_{i,k=1}^{d} \delta_{ki}\,\frac{\partial v_i}{\partial x_k}(t,x(t,X)) = \operatorname{div}_x v(t,x)\big|_{x=x(t,X)}$$

Also: $\frac{d}{dt}J(t,X) = \operatorname{div}_x v(t,x)\big|_{x=x(t,X)}\,J(t,X)$ und damit

$$(*) = \int_{\Omega}\Big[ f_t(t,x(t,X)) + \sum_{k=1}^{d} f_{x_k}(t,x(t,X))\,v_k(t,X) + f(t,x(t,X))\,\operatorname{div}_x v(t,x)\big|_{x=x(t,X)}\Big]\,J(t,X)\,dX$$

$$\operatorname{div}(sF) = \nabla s\cdot F + s\operatorname{div}F$$

$$\overset{\text{Trafo}}{=} \int_{\Omega(t)} f_t(t,x) + \operatorname{div}_x\big(f(t,x)\,v(t,x)\big)\,dx$$

□

Merke

$$\frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx = \int_{\Omega(t)} f_t(t,x) + \operatorname{div}_x\big(f(t,x)\underbrace{v(t,x)}_{\text{Flußgeschwindigkeit}}\big)\,dx$$

Verwendung in mathematischen Modellierung: Erhaltungssätze

$$\frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx = 0 \quad \forall_t \implies f_t(t,x) + \operatorname{div}_x\big(\underbrace{f(t,x)\,v(t,x)}_{=:q(t,x)}\big) = 0 \quad \forall_{t,x}$$

Modellierung $q$ (Phänomenologie)

i) Fick'sches Gesetz (Stoff Konzentrationen)

ii) Fourier'sches Gesetz (Temperaturen)

iii) Ohm'sches Gesetz (Ladungen)

$$\left.\begin{array}{l} \text{i), ii), iii)} \end{array}\right\} q(t,x) = -K \nabla_x f(t,x)$$

Damit

$$f_t(t,x) - \operatorname{div}(\underbrace{K}_{} \nabla_x f(t,x)) = 0 \quad (= s(t,x)$$

Quellen/Senken in $\Omega(t)$)

$K = K(t,x)$ PDGL erster Ordnung in Zeit, 2ter Ordnung im Ort

Anderes Modell:

$q(t,x) = a\, f(t,x)$ mit $a \in \mathbb{R}^d$ Transport in Richtung $a$

Gleichung

$$f_t(t,x) + \operatorname{div}_x(a\, f(t,x))$$

$$= f_t(t,x) + a \cdot \nabla_x f(t,x) = 0$$

Transport Gleichung

PDGL (PDE) erster Ordnung