Vo Dgl II 30.6.10

Fourier meth.

i) Reduktion auf 0-RB (Dirichlet, Neumann, gemischt)

ii) Ansatz (entsp. der RB): sin, cos, cos 1 gem

iii) Inhomogenität u. AB in Fourier r. entw.

(in x, $a_n = a_n(t)$)

iv) Gew. Dgl f. a_n

$\Rightarrow$ Lösung in Reihendarstellung.

ad i)

$$u_t - u_{xx} = f(x,t)$$
$$u(x,0) = g(x)$$
$$u(0,t) = h(t)$$
$$u_x(l,t) = k(t) \quad (\text{nicht } u(l,t) = k(t))$$

$$v(x,t) = u(x,t) - \left(h(t) \left(1-\frac{x}{l}\right)^2 + k(t) \left(\frac{x}{l}\right)^2 \frac{l}{2} \right)$$

$$v_x(x,t) = u_x(x,t) - \left(h(t) \left(1-\frac{x}{l}\right)\left(-\frac{2}{l}\right) + k(t) \right)$$

$$v(0,t) = u(0,t) - h(t) = 0$$
$$v_x(l,t) = u_x(l,t) - k(t) = 0$$

$$v_t - v_{xx} = u_t - u_{xx} + (\ldots)_t + (\ )_{xx} = \underbrace{f(x,t) + \ldots}_{\tilde{f}(x,t)}$$

$$v(x,0) = u(x,0) - \Big(\qquad \Big)\Big|_{t=0} =$$

$$= \underbrace{u(x,0)}_{g(x)} - \left(h(0) \left(1-\frac{x}{l}\right)^2 + k(0) \left(\frac{x}{l}\right)^2 \frac{l}{2} \right) = \tilde{g}(x)$$

ad i) analog, aber

$$u_x(0,t) = h(t)$$
$$u_x(l,t) = k(t)$$

$\vdots$

$$v(x,t) = u(x,t) - \left(h(t) \left(1-\frac{x}{l}\right)^2 \left(-\frac{l}{2}\right) + k(t) \left(\frac{x}{l}\right)^2 \frac{l}{2} \right)$$

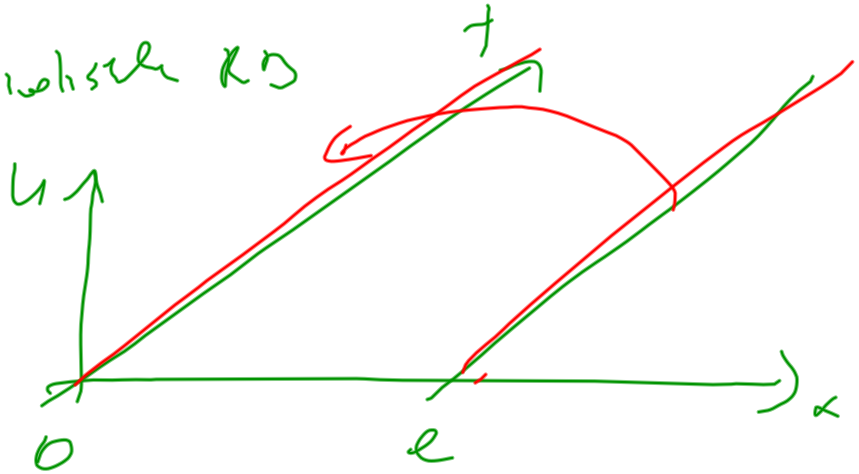
$$v_x(x,t) = u_x(x,t) - \left(h(t) \left(1-\frac{x}{l}\right) + k(t) \left(\frac{x}{l}\right) \right)$$

ad i) i.A.

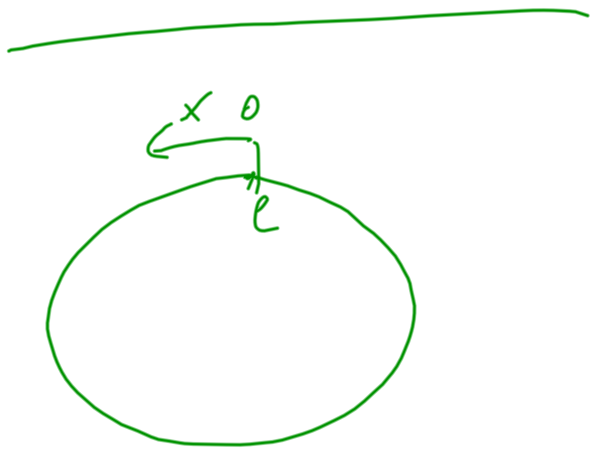
$$a\, u(0,t) + b\, u_x(0,t) = h(t)$$

$\vdots$

periodische RB



Stab



$$u(0,t) = u(l,t)$$

$$u_x(0,t) = u_x(l,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l}$$

$$u_{tt} - u_{xx} = \sum_{n=1}^{\infty} \underline{\left(a_n''(t) + \left(\frac{n\pi}{l}\right)^2 a_n(t) \right)} \sin \frac{n\pi x}{l} \overset{!}{=}$$

$$= f(x,t) = \sum_{n} \underline{h_n} \sin \frac{n\pi x}{l}$$

$$\Longrightarrow \quad a_n''(t) + \left(\frac{n\pi}{l}\right)^2 a_n(t) = \begin{cases} h_n & \\ 0 & \text{homogen} \end{cases}$$

z.B.:

$$h_n = 0 \qquad a_n(t) = A_n \sin\left(\frac{n\pi}{l} t\right) + B \cos\left(\frac{n\pi}{l} t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin \frac{n\pi}{l} t + B_n \cos \frac{n\pi}{l} t \right) \sin\left(\frac{n\pi}{l} x\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[A_n \left(\frac{n\pi}{l}\right) \cos \frac{n\pi}{l} t - B_n \left(\frac{n\pi}{l}\right) \sin\left(\frac{n\pi}{l} t\right) \right] \sin \frac{n\pi}{l} x$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{l} x\right) \overset{!}{=} g(x) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi}{l} x$$

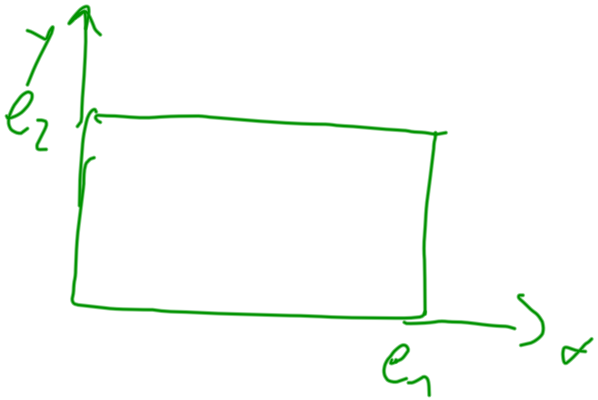
$$\Longrightarrow \boxed{B_n = d_n}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi}{l} A_n \sin \frac{n\pi}{l} x \overset{!}{=} h(x) = \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{l} x\right)$$

$$\Longrightarrow \boxed{A_n = \frac{l}{n\pi} h_n}$$

Nachteil: ?

$n > 1$



0 BB

$$u(x,y,t) = \sum_{n,m=1}^{\infty} a_{n,m}(t) \sin\left(\frac{n\pi}{l_1} x\right) \sin\left(\frac{m\pi}{l_2} y\right)$$

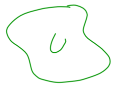
$$\Rightarrow \quad x = 0,\ x = l_1 \qquad u = 0$$

$$y = 0,\ y = l_2 \qquad u = 0$$

Existenz

$-\Delta u = f$ auf U

$u = g$ auf ∂U



Suche v mit $v|_{\partial U} = g$

$w = u - v$

$\boxed{-\Delta w = -\Delta u + \Delta v = f + \Delta v = \tilde{f}}$ auf U

$\boxed{w = g - g = 0}$ auf ∂U

allgem. elliptisch $\quad Lu = f$ auf U, $\quad u = 0$ auf ∂U

z.B. $\boxed{\begin{array}{ll} -\Delta u + u = f & \text{auf } U \\ u = 0 & \text{auf } \partial U \end{array}}$

1 stetig diffb.

$\varphi \in C_0^1(U)$

0 am Rande

$-\Delta u \varphi + u\varphi = f\varphi \quad \Big/ \int_U$

$\int_U (-\Delta u \varphi + u\varphi)\,dx = \int_U f\varphi\,dx$

$\underbrace{\int_U (\nabla u \cdot \nabla\varphi + u\varphi)\,dx - \int_{\partial U} \nabla u \cdot n \underbrace{\varphi}_{=0}\,dS}_{=0} = \int_U f\varphi\,dx$

also $\quad \int_U (\nabla u \cdot \nabla\varphi + u\varphi)\,dx = \int_U f\varphi\,dx$

Neue Formulierung:

Suche $u \in H = C_0^1(U)$ sodass

$\int_U (\nabla u \cdot \nabla\varphi + u\varphi)\,dx = \int_U f\varphi\,dx \quad \forall \varphi \in C_0^1(U)$

Gebiet $\qquad$ Inhomogenität $\qquad$ 0 RB

Falls so ein u $\exists$

u schwache Lösung

($u \in C^2(\bar{U})$ klassische Lösung)

$$\langle u, \varphi \rangle_H = \int_U (\nabla u \cdot \nabla \varphi + u\varphi)\,dx$$

i) $\langle \cdot , \cdot \rangle$ linear in beiden Einträgen
bilinear

ii) $\langle \cdot , \cdot \rangle$ symmetrisch

iii) positiv definit

$$\langle u, u \rangle = \int_U (|\nabla u|^2 + u^2)\,dx \geq 0$$

$$\langle u, u \rangle = 0 \iff u = 0$$

$\langle \cdot , \cdot \rangle$ Skalarprodukt auf H

$$\int_U f \varphi \, dx$$

$$\varphi \in C_0^1 \longrightarrow \int_U f \varphi \, dx \in \mathbb{R}$$

Funktional

<u>Satz</u> (Riesz) In einem vollst. Skalarproduktraum
(Hilbertraum)
kann jedes Funktional durch das Skalarpr. (eindeutig) dargestellt werden, d.h. $\exists! \, w \in H$

$$\int_U f \varphi \, dx = \langle w, \varphi \rangle_H \quad \forall \varphi \in H$$

Somit
$$\int_U (\nabla u\cdot\nabla\varphi + u\varphi)\,dx = \int_U f\varphi\,dx \quad \forall\varphi\in H$$

$$\langle u,\varphi\rangle = \int_U f_2\varphi\,dx \underset{\text{Riesz}}{=} \langle w,\varphi\rangle \quad \forall\varphi\in H$$

$\langle u-w,\varphi\rangle = 0 \quad \forall\varphi\in H$

$\varphi = u-w \in H$

$\langle u-w,u-w\rangle = 0 \iff u-w=0$

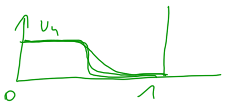
$\Longrightarrow u = w$

also $\int_U (\nabla u\cdot\nabla\varphi + u\varphi)\,dx = \int_U f\varphi\,dx \quad \forall\varphi\in H$

also is $u=w$ eindeutige schwache Lösung

i) $C_0^1(\bar U)$ ist nicht Hilbert bezgl.

$$\|u\|_H = \sqrt{\langle u,u\rangle} = \sqrt{\int_U (|\nabla u|^2 + u^2)\,dx}$$



$$u_n \xrightarrow[\|\cdot\|_H]{} u = \begin{cases} 1 & x<\frac12 \\ 0 & x\ge\frac12 \end{cases}$$

ii) f muss nicht mehr stetig sein

$\int f\varphi\,dx$

iii) also für 1 Dimension allgemeinere Lsg als aus ODE Theorie

iv) gilt auch allgemein für ellipt. Diff-operatoren

$Lu = f$ $\quad L$ elliptisch

Bspl: $\quad -u_{xx} = f = \begin{cases} 0 & x \in [0,1] \\ 1 & x \in (1,2] \end{cases}$

$u(0) = u(2) = 0$

in $[0,1]$ $\quad -u_{xx} = 0$

$u(x) = ax \qquad (b=0 \text{ wegen RB } x=0)$

$u_x = a$

in $[1,2]$ $\quad -u_{xx} = 1$

$u(x) = -\frac{(x-2)^2}{2} + b(x-2) \qquad (\text{erfüllt RB } x=2)$

$u_x = -(x-2) + b$

Versuch $\quad u \in C^1[0,2]$

$x=1 \quad u, u_x$ stetig !!!

$$\left.\begin{aligned} a \cdot 1 &= -\frac{1}{2} \pm b \\ a &= 1 + b \end{aligned}\right\} \qquad 2b = \frac{1}{2} \qquad b = \frac{1}{4} \qquad a = \frac{5}{4}$$

$\Rightarrow u(x) = \begin{cases} \frac{5}{4}x & \text{in } (0,1] \\ -\frac{(x-2)^2}{2} + \frac{1}{4}(x-2) & \text{in } (1,2] \end{cases}$

Rechenfehler!