Differential Equations I for Students of Engineering

Ordinary Differential Equations

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Chapter 1: Basics, terminology, examples

Definition (ordinary DE)

An ordinary differential equations (in brief: an ODE) is an equation with derivatives up to order $m \geq 1$ *in implicit form*

$$\boxed{F(t, u(t), u'(t), u''(t), \ldots, u^{(m)}(t)) = 0}$$

or alternatively in explicit form (i.e. solved for $u^{(m)}$*)*

$$\boxed{u^{(m)}(t) = f(t, u(t), u'(t), u''(t), \ldots, u^{(m-1)}(t))}$$

for an unknown function $u\colon I \to \mathbb{R}^n$ *of a single variable* $t \in I \subset \mathbb{R}$*.*

If the equation holds for all[‡] $t \in I$*, one calls* u *a solution of the DE on* I*.*

In contrast: *multiple* variables $\rightsquigarrow$ DE II, PDE, *partial* DE.

Caution: Also common to use $x(t)$, $y(t)$, or $y(x)$ instead of $u(t)$!

[‡] I interval or at least without single points; in boundary points use one-sided derivative.

Terminology in connection with ODEs

Brief functional notation for previous ODEs:

$$\boxed{F(\,\cdot\,,u,u',u'',\ldots,u^{(m)})\equiv 0} \quad \text{rsp.} \quad \boxed{u^{(m)}=f(\,\cdot\,,u,u',u'',\ldots,u^{(m-1)})}\,.$$

Terminology:

- m: order of the ODE (provided $u^{(m)}$ indeed occurs),
- n: number of (component) functions (of $u\colon I\to\mathbb{R}^n$),
- N: number of (component) equations (of ODE with „=“ in $\mathbb{R}^N$),
- F, f: given structure function of the ODE
  (function from domain in $\mathbb{R}\times(\mathbb{R}^n)^{1+m}$ rsp. $\mathbb{R}\times(\mathbb{R}^n)^m$ to $\mathbb{R}^N$).

From now mostly $N=n$ only (as ODE with $N\neq n$ over/underdetermined).

$N=n=1$: scalar ODE for one function,

$N=n\geq 2$: ODE system for multiple functions.

First examples of ODEs

Oversimplified examples (with $N = n$ arbitrary $\rightsquigarrow$ scalar or as a system):

- $\boxed{u' \equiv 0}$ has order $m = 1$.
  All solutions: constant functions $u(t) = C$ with $C \in \mathbb{R}^n$.
- $\boxed{u'' \equiv 0}$ has order $m = 2$.
  All solutions: affine functions $u(t) = C_1 t + C_0$ with $C_0, C_1 \in \mathbb{R}^n$.

First reasonable examples (still $N = n$ arbitrary):

- $\boxed{u' = u}$ has order $m = 1$.
  Easy-to-guess solutions: $u(t) = C\mathrm{e}^t$ with $C \in \mathbb{R}^n$.
- $\boxed{u' = \lambda u}$ with parameter $\lambda \in \mathbb{R}$ has order $m = 1$.
  Easy-to-guess solutions: $u(t) = C\mathrm{e}^{\lambda t}$ with $C \in \mathbb{R}^n$.
- $\boxed{u'' = -u}$ has order $m = 2$.
  Easy-to-guess solutions: $u(t) = C_1 \sin(t) + C_2 \cos(t)$ with $C_1, C_2 \in \mathbb{R}^n$.

(Soon: The guessed solutions are in fact the only solutions.)

Further examples of ODEs

Further examples:

- $\boxed{(u')^2 = u}$ is a scalar first-order ODE (i.e. $m = N = n = 1$).
  Easy-to-guess solutions: $u(t) = \frac{1}{4}(t - t_0)^2$ with $t_0 \in \mathbb{R}$ and $u \equiv 0$.
  (In this case *not yet all* solutions!)
- $\boxed{\begin{aligned} u_1' &= t^3 + u_1 + u_2^2\,, \\ u_2' &= t^2 u_1 u_2 \end{aligned}}$ or equivalently $\boxed{\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} t^3 + u_1 + u_2^2 \\ t^2 u_1 u_2 \end{pmatrix}}$ is a system of two first-order ODEs for an $\mathbb{R}^2$-valued function $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ (i.e. $m = 1$, $N = n = 2$).

Rule of thumb (for $N = n$, explicit form): Solution involves $\boldsymbol{m \cdot n}$ constants.

(Side remark: In general no such rule for implicit form, since e.g.:

- $3^{u'} \equiv 0$ (scalar): no solution at all (as $3^y \neq 0$ for all $y \in \mathbb{R}$),
- $u'(2 - u') \equiv 0$ (scalar): $u(t) = C$ and $u(t) = 2t + C$ „too many“ solutions.)

Initial values and boundary values

- Determination of mn constants from mn additional conditions, typical on interval I are either initial conditions (ICs)

$$u(t_0) = y_0\,,\ u'(t_0) = y_1\,,\ u''(t_0) = y_2\,,\ \ldots\,,\ u^{(m-1)}(t_0) = y_{m-1}$$

  at one given point $t_0 \in I$ with given $y_0, y_1, \ldots, y_{m-1} \in \mathbb{R}^n$ or in case $I = [t_1, t_2]$ boundary conditions (BCs)

$$r(u(t_1), u(t_2), u'(t_1), u'(t_2), \ldots, u^{(m-1)}(t_1), u^{(m-1)}(t_2)) = 0$$

  at two points t_1, t_2 with given function $r\colon (\mathbb{R}^n)^{2m} \to (\mathbb{R}^n)^m$.

- An ODE together with given initial or boundary conditions is called an initial value problem (IVP) or boundary value problem (BVP), respectively.

- In general only IVPs/BVPs (not ODEs alone) can be solved uniquely.

More terminology in connection with ODEs

More terminology:

- In an autonomous ODE, t does not occur separately, but only in form of $u(t), u'(t), \ldots, u^{(m)}(t)$, i.e. it has the slightly more specific form
$$F_0(u(t), u'(t), \ldots, u^{(m)}(t)) = 0\,.$$
- Linear ODEs come with affine dependence on $u(t), u'(t), \ldots, u^{(m)}(t)$, i.e. they have the form
$$A_m(t)u^{(m)}(t) + \ldots + A_2(t)u''(t) + A_1(t)u'(t) + A_0(t)u(t) = b(t)$$
with coefficients‡ A_k and inhomogeneity b, or in brief
$$\sum_{k=0}^{m} A_k(t)u^{(k)}(t) = b(t)\,.$$
A linear ODE with constant coefficients is one with constant A_k only. In case $b \equiv 0$ a linear ODE is homogeneous, otherwise inhomogeneous.

‡ $A_k(t) \in \mathbb{R}$ or $A_k(t) \in \mathbb{R}^{n\times n}$ (scalar or matrix coefficients), but in any case $b(t) \in \mathbb{R}^n$.

Scalar linear first-order ODEs; homogeneous case

Theorem

For I interval, $a\colon I \to \mathbb{R}$ continuous, A antiderivative of a (i.e. $A' = a$), the solutions of the scalar ODE

$$u'(t) = a(t)u(t) \qquad \textit{rsp.} \qquad u' = au$$

on I are exactly the functions u of form

$$u(t) = C\mathrm{e}^{A(t)} \qquad \textit{rsp.} \qquad u = C\mathrm{e}^{A} \qquad \textit{with constant } C \in \mathbb{R}\,.$$

Proof/verifying calculation:

- $u(t) = C\mathrm{e}^{A(t)} \rightsquigarrow u'(t) = A'(t)C\mathrm{e}^{A(t)} = a(t)C\mathrm{e}^{A(t)} = a(t)u(t) \rightsquigarrow u$ sol.
- u sol. $\rightsquigarrow (\mathrm{e}^{-A}u)' = \mathrm{e}^{-A}u' - \mathrm{e}^{-A}A'u = \mathrm{e}^{-A}(u' - au) \overset{\text{ODE}}{\equiv} 0$
  $\rightsquigarrow \mathrm{e}^{-A}u = C \rightsquigarrow u = C\mathrm{e}^{A}$. □

Corollary: solution formula for corresponding IVP with IC $u(t_0) = y_0$:

$$u(t) = y_0\mathrm{e}^{A(t)-A(t_0)}\,.$$

Application in examples; homogeneous case

Applications of theorem and IVP solution formula $u(t) = y_0 \mathrm{e}^{A(t)-A(t_0)}$:

- $\boxed{u' = \lambda u} \overset{\text{read off}}{\rightsquigarrow} a \equiv \lambda,\ A(t) = \lambda t \overset{\text{thm}}{\rightsquigarrow}$ solutions $u(t) = C\mathrm{e}^{\lambda t}$.
  (formula seen before, but now shown that these are *the only* solutions.)

- $\boxed{\text{IVP for } u'(t) = \frac{2}{t}u(t) \text{ with IC } u(1) = 5}$
  $\overset{\text{read off}}{\rightsquigarrow}$ $a(t) = \frac{2}{t},\ A(t) = 2\ln t,\ t_0 = 1,\ y_0 = 5$
  $\overset{\text{solution formula}}{\rightsquigarrow}$ solution $u(t) = 5\mathrm{e}^{A(t)-A(1)} = 5\mathrm{e}^{2\ln t} = 5t^2$.

  (valid e.g. on interval $(0,\infty)$, where $\frac{2}{t}$ defined with antiderivative $2\ln t$.)

Scalar linear first-order ODEs; inhomogeneous case

Theorem

For I interval, $a, b\colon I \to \mathbb{R}$ continuous with antiderivatives A of a, B^* of $\mathrm{e}^{-A}b$, the solutions of the scalar ODE

$$u' = au + b$$

on I are exactly the functions u of form

$$u = \mathrm{e}^{A}(B^* + C) \qquad \text{with constant } C \in \mathbb{R}\,.$$

Proof/verifying calculation similar as before!

Corollary: solution formula for corresponding IVP with IC $u(t_0) = y_0$:

$$\begin{aligned} u(t) &= \mathrm{e}^{A(t)}\left[B^*(t) - B^*(t_0) + y_0\mathrm{e}^{-A(t_0)}\right] \\ &\overset{\text{FTC}}{=} \mathrm{e}^{A(t)}\left[\int_{t_0}^{t} \mathrm{e}^{-A(s)}b(s)\,\mathrm{d}s + y_0\mathrm{e}^{-A(t_0)}\right]. \end{aligned}$$

Application in example; inhomogeneous case

Application IVP solution formula $u(t) = \mathrm{e}^{A(t)}\left[\int_{t_0}^{t} \mathrm{e}^{-A(s)} b(s)\,\mathrm{d}s + y_0 \mathrm{e}^{-A(t_0)}\right]$:

- IVP for $u'(t) = \frac{2}{t}u(t) + t\ln t$ with IC $u(1) = 5$ $\overset{\text{read off}}{\rightsquigarrow}$ $b(t) = t\ln t$.

  Then by solution formula ($A(t) = 2\ln t$, $t_0 = 1$, $y_0 = 5$ as before):

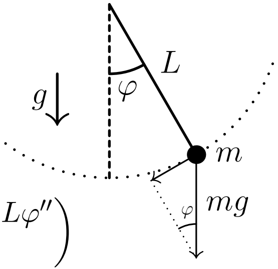
$$u(t) = \mathrm{e}^{2\ln t}\left[\int_1^t \mathrm{e}^{-2\ln s} s\ln s\,\mathrm{d}s + 5\right] = t^2\left[\int_1^t \frac{\ln s}{s}\,\mathrm{d}s + 5\right]$$
$$= t^2\left[\frac{1}{2}(\ln t)^2 - \frac{1}{2}(\ln 1)^2 + 5\right] = \frac{1}{2}t^2(\ln t)^2 + 5t^2$$

  (valid once more on interval $(0,\infty)$).

Example: ODEs of a pendulum

The ODE of a simple physical pendulum is the scalar non-linear ODE

$$\varphi''(t) = -(g/L)\sin(\varphi(t))$$



where
- $\varphi(t)$: displacement angle at time t,
- g: gravity acceleration (constant > 0),
- L: length of thread (constant > 0).

(Derivation: tang. force: $mg\sin(\varphi)$, tang. accel.: $L\varphi''$
$\rightsquigarrow$ equation of motion $mL\varphi'' = -mg\sin(\varphi)$.)

The linearized pendulum ODE (cf. harmonic oscillator) is the scalar ODE

$$\varphi''(t) = -(g/L)\varphi(t)$$

(Small-angle approximation of the preceding, as $\sin(\varphi) \approx \varphi$ for small φ).
All solutions: $\varphi(t) = C_1\sin(\omega t) + C_2\cos(\omega t)$ with $\omega := \sqrt{g/L}$, $C_i \in \mathbb{R}$.

Example: general equations of motion

Equations of motion for (point-like) particle of mass $m > 0$:

$$m\,\vec{x}''(t) = \vec{F}(t, \vec{x}(t), \vec{x}'(t))$$

is a system of 3 ODEs for the position vector $\vec{x}\colon I \to \mathbb{R}^3$ (here once with arrows on top of all 3d vectors). The force $\vec{F}$ acting on the particle is e.g.:

- gravity $\vec{F}_{\mathrm{grav}}(t, \vec{x}, \vec{v}) = -m\,g\,\vec{\mathrm{e}}_3$ with gravity acceleration g,
- Lorentz force $\vec{F}_{\mathrm{Lor}}(t, \vec{x}, \vec{v}) = q\,\vec{E}(t, \vec{x}) + q\,\vec{v} \times \vec{B}(t, \vec{x})$ with charge q and time-position-dependent electric field $\vec{E}$ and magnetic field $\vec{B}$,
- air drag (no wind; not point-like) $\vec{F}_{\mathrm{drag}}(t, \vec{x}, \vec{v}) = -C_{\mathrm{drag}}|\vec{v}|\vec{v}$ with C_{drag} dependent on cross section, drag coefficient of particle and air density

or possibly the sum of some of these terms.

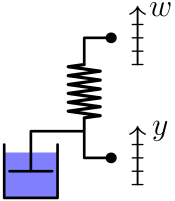
With view towards such ODEs, another word for solution is trajectory.

Example: ODE of control loop element

ODE of a simple control loop element (order 1, scalar, linear, inhomogeneous)

$$y'(t) = -\lambda y(t) + \lambda w(t)$$

for output y with given input w and constant λ.



(Derivation (with d damping and k spring constant):
- damping force $-dy'$,
- spring force $k(w-y)$ (Hooke's law with balance at $w = y$),
- force balance $-dy' + k(w-y) = 0$ gives ODE with $\lambda = \frac{k}{d}$.)

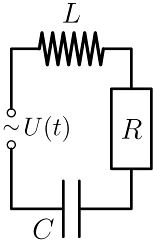
Solution with IC $y(t_0) = y_0$ (special case of previous solution formula):

$$y(t) = y_0 \mathrm{e}^{-\lambda(t-t_0)} + \lambda \int_{t_0}^{t} \mathrm{e}^{\lambda(s-t)} w(s) \, \mathrm{d}s \,.$$

Example: ODE of electrical RLC circuit

ODE of electrical RLC circuit (order 2, scalar, linear, inhomogeneous)

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = U'(t)$$



for electric current I with the following given:

- inductance L, resistance R, capacitance C (const.),
- applied voltage U.

(Derivation:
- Ohm's law: $U_{\mathrm{R}} = RI$,
- capacitor charging current: $I = CU_{\mathrm{C}}'$,
- coil/inductor voltage: $U_{\mathrm{L}} = LI'$,
- Kirchhoff rule $U_{\mathrm{L}} + U_{\mathrm{R}} + U_{\mathrm{C}} = U \overset{\text{differentiate}}{\rightsquigarrow}$ ODE.)

Soon: Solutions are damped oscillations.

Side remark: reduction-to-first-order principle

An ODE of arbitrary order m

$$F(\,\cdot\,, u, u', \ldots, u^{(m-1)}, u^{(m)}) \equiv 0 \qquad \text{for } u\colon I \to \mathbb{R}^n$$

can be rewritten purely formally as a system of order 1

$$\begin{aligned} u_0' &= u_1\,, \\ u_1' &= u_2\,, \\ \vdots \quad &\quad \vdots \\ u_{m-2}' &= u_{m-1}\,, \\ F(\,\cdot\,, u_0, u_1, \ldots, u_{m-1}, u_{m-1}') &\equiv 0 \end{aligned} \qquad \text{for } \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} : I \to (\mathbb{R}^n)^m\,.$$

Great for theory, less useful for practical computations!

Example: $LI'' + RI' + \frac{1}{C}I = U'$ turns into $\begin{cases} I_0' = I_1 \\ LI_1' + RI_1 + \frac{1}{C}I_0 = U' \end{cases}.$

Chapter 2: Methods for solving non-linear ODEs

First solutions formulas for *linear* ODEs were in the previous chapter, and their general treatment follows. For *non-linear* ODEs, in contrast, there is no general approach. In a sense one may only try around. Among possible approaches this chapter treats, for non-linear first-order ODEs, the methods

- separation of variables,
- changes of variable (specifically Bernoulli and Riccati ODEs),
- exact ODEs

plus some special cases of non-linear second-order ODEs.

2.1 Separation of variables

Separation of variables: Try to rearrange an ODE in form $g(u)u' = h$, i.e. with all u rsp. $u(t)$ on the left where also u' is, all separate t on the right. *If* this works, one can solve (at least in principle).

Illustration by examples:

- $\boxed{u' = \frac{t^2}{2u}} \overset{\text{rearrange}}{\rightsquigarrow} 2uu' = t^2$

  $\overset{\text{antideriv.}}{\rightsquigarrow} u(t)^2 = \frac{1}{3}t^3 + C \overset{\text{solve}}{\rightsquigarrow} u(t) = \pm\sqrt{\frac{1}{3}t^3 + C}.$

- $\boxed{u' = u(u-1)} \overset{\text{rearrange}}{\rightsquigarrow} \frac{u'}{u(u-1)} \equiv 1$

  $\overset{\text{antideriv.}}{\rightsquigarrow} \ln\left|1 - \frac{1}{u(t)}\right| = t + C \overset{\text{solve}}{\rightsquigarrow} u(t) = \left(1 \pm \mathrm{e}^{t+C}\right)^{-1}.$

  Here used (auxiliary calculation!): $\frac{1}{x(x-1)}$ has antiderivative $\ln\left|1 - \frac{1}{x}\right|$.

  Caution! Initial division by $u(u-1)$ allowed only if $0 \neq u \neq 1$.
  Plugging in shows that $u \equiv 0$ and $u \equiv 1$ are indeed additional solutions.
  (All solutions except $u \equiv 0$ can also be written as $u(t) = \left(1 + \widetilde{C}\mathrm{e}^t\right)^{-1}$.)

Separation of variables, general background

General principle behind:

Principle (ODEs with separated variables)

Fix intervals I, J and continuous $g\colon J \to \mathbb{R}$, $h\colon I \to \mathbb{R}$ with antiderivatives G, H. Then, for differentiable $u\colon I \to J$, the scalar ODE

$$g(u(t))u'(t) = h(t) \qquad \textit{rsp.} \qquad g(u)u' = h$$

on I is equivalent by taking antiderivatives to the equation

$$G(u(t)) = H(t) + C \qquad \textit{with constant of integration } C \in \mathbb{R}\,.$$

Proof: $\frac{\mathrm{d}}{\mathrm{d}t}G(u(t)) = G'(u(t))u'(t) = g(u(t))u'(t)$ and $H'(t) = h(t)$. □

- reduces ODE to derivative-free equation, which can be solved for $u(t)$.
  (Formally: If g is free of zeros, the solutions are $u(t) = G^{-1}(H(t) + C)$ with the differentiable inverse function G^{-1} of G.)
- computation of antiderivatives G, H and solving for $u(t)$ can be involved!

Sample application: exponential population model

Most simple population model: Ansatz with change $p'(t)$ proportional to population $p(t)$ at each time $t \in \mathbb{R}$ gives the IVP already seen

$$p' = \lambda\, p\,, \qquad\qquad p(t_0) = p_0$$

with:
- change rate $\lambda \in \mathbb{R}$ (equals birth rate minus death rate),
- initial population $p_0 \in \mathbb{R}$ at initial time $t_0 \in \mathbb{R}$, in reality $p_0 \geq 0$.

The solutions already determined

$$p(t) = p_0 \mathrm{e}^{\lambda(t-t_0)}$$

exhibit for $\lambda > 0$ exponential growth, for $\lambda < 0$ exponential decay.

But this model does not reflect bounded capacity (e.g. in habitat), as $\lim_{t\to\infty} p(t) = \infty$ for $\lambda > 0 \rightsquigarrow$ unrealistic at least over longer time!

Sample application: logistic population model

Slightly refined population model with maximal capacity $M > p_0 > 0$:

$$p' = \lambda\, p\,(M - p)\,, \qquad p(t_0) = p_0\,.$$

Solve by separation of variables (case $M = 1$, $\lambda = -1$ already seen):

$\overset{\text{rearrange}}{\rightsquigarrow}$ $\frac{p'}{p(M-p)} = \lambda$ $\overset{\text{antideriv.}}{\rightsquigarrow}$ $\frac{1}{M}\ln\frac{p(t)}{M-p(t)} = \lambda(t-t_0) + \frac{1}{M}\ln C$ with $C > 0$

$\overset{\text{solve}}{\rightsquigarrow}$ $p(t) = M\frac{C\mathrm{e}^{M\lambda(t-t_0)}}{C\mathrm{e}^{M\lambda(t-t_0)}+1}\,.$

Here used (auxiliary calculation!): $\frac{1}{x(M-x)}$ has antidervative $\frac{1}{M}\ln\frac{x}{M-x}$.

From $p_0 = p(t_0) = M\frac{C}{C+1}$ determine additionally $C = \frac{p_0}{M-p_0}$, find solution

$$p(t) = M\frac{p_\mathrm{e}(t)}{p_\mathrm{e}(t) + M - p_0} \qquad \text{with abbreviation } p_\mathrm{e}(t) = p_0\mathrm{e}^{M\lambda(t-t_0)}\,.$$

For $\lambda > 0$ known as logistic growth with in particular $\lim_{t\to\infty} p(t) = M$.

2.2 Changes of variable

Occassionally one can simply and then solve ODE by changes of variable. The two basic types are:

- New function y replaces u (change of dependent variable): $y(t) = Y(u(t))$ and slightly more generally $y(t) = Y_t(u(t))$,
- New „time“ s replaces t (change of independent variable): $t = T(s)$, $\widetilde{u}(s) = u(T(s))$

with differentiably reversable maps Y, Y_t, T (in application just specific terms).

There is no general rule for finding changes of variable, but for specific cases (some on the next slides) useful changes of variable are known.

In general, with each change of variable one attempts the 3-step principle:

- determine transformed ODE with new variable,
- solve in terms of the new variable,
- transform solution back to original variable.

Change of variable for Bernoulli ODEs

Bernoulli ODEs are scalar ODEs (in general for positive u) of type

$$\boxed{u'(t) = a(t)u(t) + b(t)u(t)^\alpha}$$

with given coefficient functions a, b and parameter[‡] $\alpha \in \mathbb{R} \setminus \{0, 1\}$.
The change of variable $y(t) := u(t)^{1-\alpha}$ transforms by

$$\begin{aligned} y'(t) = (1-\alpha)u(t)^{-\alpha}u'(t) &= (1-\alpha)\big[a(t)u(t)^{1-\alpha} + b(t)\big] \\ &= (1-\alpha)\big[a(t)y(t) + b(t)\big]\,, \end{aligned}$$

to a linear ODE for y, for which the solution formula of Chapter 1 applies.

Example: $\boxed{u'(t) = u(t) + tu(t)^2}$

$\overset{\text{read off}}{\rightsquigarrow}$ $\alpha = 2$, $a \equiv 1$, $b(t) = t$, new ODE: $y'(t) = -y(t) - t$

$\overset{\text{sol. formula}}{\rightsquigarrow}$ $y(t) = \mathrm{e}^{-t}[(1-t)\mathrm{e}^{t} + C] = 1-t + C\mathrm{e}^{-t}$

$\overset{\text{transf. back}}{\rightsquigarrow}$ $u(t) = y(t)^{-1} = \big(1-t + C\mathrm{e}^{-t}\big)^{-1}$

[‡]Cases $\alpha = 0$ and $\alpha = 1$ excluded above, since in these the ODE is directly linear.

Change of variable for Riccati ODEs

Riccati ODEs are scalar ODEs of type

$$\boxed{u'(t) = a(t)u(t) + b(t)u(t)^2 + c(t)}$$

with coefficients a, b, c. Given a special solution u_0 (which one does know or can guess) the change of variable $y(t) := \big(u(t) - u_0(t)\big)^{-1}$ transforms by

$$y'(t) = \frac{u_0'(t) - u'(t)}{(u(t) - u_0(t))^2} = \ldots = -\big[a(t) + 2u_0(t)b(t)\big]y(t) - b(t)$$

to a linear ODE for y, for which the solution formula of Chapter 1 applies.

Example: $\boxed{u'(t) = 3tu(t) - tu(t)^2 - 2t \text{ with special solution } u_0 \equiv 1}$

$\overset{\text{read off}}{\rightsquigarrow}$ $a(t) = 3t$, $b(t) = -t$, $c(t) = -2t$, new ODE: $y'(t) = -ty(t) + t$

$\overset{\text{sol. formula}}{\rightsquigarrow}$ $y(t) = \mathrm{e}^{-\frac{1}{2}t^2}\big(\mathrm{e}^{\frac{1}{2}t^2} + C\big) = 1 + C\mathrm{e}^{-\frac{1}{2}t^2}$

$\overset{\text{transf. back}}{\rightsquigarrow}$ $u(t) = y(t)^{-1} + u_0(t) = \big(1 + C\mathrm{e}^{-\frac{1}{2}t^2}\big)^{-1} + 1$

Change of variable for similarity ODEs

Similarity ODEs are scalar ODEs (on intervals where $t \neq 0$) of type

$$u'(t) = f\left(\frac{u(t)}{t}\right)$$

with structure function f. The change of variable $y(t) = \frac{u(t)}{t}$ transforms by

$$y'(t) = \frac{u'(t)}{t} - \frac{u(t)}{t^2} = \frac{1}{t}\left[f\left(\frac{u(t)}{t}\right) - \frac{u(t)}{t}\right] = \frac{1}{t}\big[f(y(t)) - y(t)\big]$$

to an ODE for y, solvable (in principle) by separation of variables.

Example: $u'(t) = 1 + \frac{u(t)}{t} + \left(\frac{u(t)}{t}\right)^2$

$\overset{\text{read off}}{\rightsquigarrow}$ $f(x) = 1 + x + x^2$, new ODE: $y'(t) = \frac{1}{t}[1 + y(t)^2]$

$\overset{\text{sep. var.}}{\rightsquigarrow}$ $y(t) = \tan(\ln|t| + C)$

$\overset{\text{transf. back}}{\rightsquigarrow}$ $u(t) = t\,y(t) = t\tan(\ln|t| + C)$

Change of variable for Euler-Cauchy ODEs

Euler-Cauchy ODEs are scalar, linear, homogeneous ODEs of special type

$$a_m t^m u^{(m)}(t) + \ldots + a_2 t^2 u''(t) + a_1 t u'(t) + a_0 u(t) \equiv 0$$

with constants $a_k \in \mathbb{R}$, but in effect with *non*-constant coefficients $a_k t^k$. One checks (on intervals where $t > 0$): The change of variable $t = \mathrm{e}^s$, $\widetilde{u}(s) = u(\mathrm{e}^s)$ transforms to an ODE for $\widetilde{u}$ with constant coefficients, and this ODE can be solved (in principle) by methods of the next chapter.

Example (which works out already now): $\boxed{t^2 u''(t) + t u'(t) + u(t) = 0}$

From $\widetilde{u}(s) = u(\mathrm{e}^s)$ deduce first $\widetilde{u}'(s) = \mathrm{e}^s u'(\mathrm{e}^s)$ and then with ODE $\widetilde{u}''(s) = (\mathrm{e}^s)^2 u''(\mathrm{e}^s) + \mathrm{e}^s u'(\mathrm{e}^s) = -u(\mathrm{e}^s) = -\widetilde{u}(s)$ (new ODE).

$$\overset{\text{Chap. 1 or. 3}}{\rightsquigarrow} \widetilde{u}(s) = C_1 \sin(s) + C_2 \cos(s)$$

$$\overset{\text{transf.back}}{\rightsquigarrow} u(t) = \widetilde{u}(\ln t) = C_1 \sin(\ln t) + C_2 \cos(\ln t)$$

2.3 Exact ODEs

3-step procedure for given scalar ODE of type

$$\boxed{f(t,u(t)) + g(t,u(t))u'(t) = 0}$$

(with f, g defined on „suitably good“ domain $D \subset \mathbb{R}^2$):

- Check if the integrability criterion

  $$\boxed{\frac{\partial f}{\partial x}(t,x) = \frac{\partial g}{\partial t}(t,x)} \qquad \text{for all } (t,x) \in D$$

  is valid. If „yes“, one has an exact ODE. Only then proceed further!

- Integrate $\int f(t,x)\,\mathrm{d}t$ and $\int g(t,x)\,\mathrm{d}x$. By choosing constants of integration reach a common result $\Psi(t,x)$.

- Determine solutions u by solving $\Psi(t,u(t)) = C$ for $u(t)$.

Example for solving an exact ODE

Example: Solve $\boxed{1 + \frac{u}{t^2} - \frac{1}{t}u' \equiv 0}$ as follows:

$\overset{\text{read off}}{\rightsquigarrow}$ $f(t,x) = 1 + \frac{x}{t^2}$, $g(t,x) = -\frac{1}{t}$

check integrability criterion: $\frac{\partial f}{\partial x}(t,x) = \frac{1}{t^2}$, $\frac{\partial g}{\partial t}(t,x) = \frac{1}{t^2}$, ✓

integrate: $\int f(t,x)\,\mathrm{d}t = t - \frac{x}{t} + \mathrm{const}(x)\,,$

$\int g(t,x)\,\mathrm{d}x = -\frac{x}{t} + \mathrm{const}(t)$

$\overset{\text{choose consts}}{\rightsquigarrow}$ potential: $\Psi(t,x) = t - \frac{x}{t}$

$\overset{\text{principle}}{\rightsquigarrow}$ equation (equivalent to ODE): $t - \frac{u(t)}{t} = C$

$\overset{\text{solve}}{\rightsquigarrow}$ solutions: $u(t) = t^2 - Ct$

Exact ODEs, general background

General principle behind:

Principle (exact ODEs)

On open $D \subset \mathbb{R}^2$ consider both partial derivatives $f = \frac{\partial \Psi}{\partial t}$ and $g = \frac{\partial \Psi}{\partial x}$ of a C^1 function $\Psi \colon D \to \mathbb{R}$ of the variables $(t, x) \in D$. Then the scalar ODE

$$f(t, u(t)) + g(t, u(t))u'(t) = 0$$

is called exact[‡] and, for differentiable $u \colon I \to \mathbb{R}$ with $(t, u(t)) \in D$ on intervals I, is equivalent by taking antiderivatives to the equation

$$\Psi(t, u(t)) = C \qquad \text{with constant of integration } C \in \mathbb{R}\,.$$

Proof: Chain rule gives $\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t, u(t)) = f(t, u(t)) + g(t, u(t))u'(t)$. □

- Same as with separation of variables: reduces ODE to derivative-free equation, which can ideally be solved for $u(t)$.

[‡] The word „exact" comes from an analogous structure in the theory of differential forms.

Potentials of 2d vector fields

Hence, for given f, g, we wonder in which cases there is a Ψ such that $f = \frac{\partial\Psi}{\partial t}$ and $g = \frac{\partial\Psi}{\partial x}$ hold. Indeed, these two conditions are summarized in

$$\binom{f}{g} = \nabla\Psi \qquad \Big(\text{vector field } \binom{f}{g} \text{ equals gradient of } \Psi\Big)\,,$$

and then fall under the following general notion $\Big($with $\binom{f}{g}$ replaced by $V\Big)$:

Definition (potentials)

A potential/antiderivative of a vector field $V\colon D \to \mathbb{R}^2$ on open $D \subset \mathbb{R}^2$ is a differentiable function $\Psi\colon D \to \mathbb{R}$ such that $\nabla\Psi = V$ holds on D.

Integrability criterion for 2d vector fields

The integrability criterion $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}$ is then nothing but precisely the following Analysis-3 criterion spelled out for $V = \binom{f}{g}$:

Theorem (integrability criterion for 2d vector fields)

Consider a simply connected domain[‡] D *in* $\mathbb{R}^2$ *and a* C^1 *vector field* $V\colon D \to \mathbb{R}^2$ *of the variables* $(t, x) \in D$*. Then there exists a potential for* V *if and only if the integrability criterion* $\boxed{\frac{\partial V_1}{\partial x} = \frac{\partial V_2}{\partial t}}$ *is satisfied on* D*.*

Side remark (cf. Analysis 3): In general one may express a potential Ψ of a vector field with integrability criterion satisfied via oriented line integrals. For explicit computation of Ψ, however, it usually suffices to integrate separately with respect to t and x (as seen in the earlier example).

[‡] A domain is a non-empty, open, and connected subset. A domain in $\mathbb{R}^2$ is simply connected if it contains (roughly speaking) "no holes".

Treatment of non-exact ODEs, integrating factors

If $f(t,u(t)) + g(t,u(t))u'(t) = 0$ is not exact, the following may help:

- Principle: Seek equivalent exact ODE

$$h(t,u(t))f(t,u(t)) + h(t,u(t))g(t,u(t))u'(t) = 0 \qquad (*)$$

  with integrating factor $h(t,u(t)) \neq 0$ to be determined.

- Make ansatz e.g. $h(t,x) = \varphi(t)$, $h(t,x) = \varphi(x)$, $h(t,x) = \varphi(t+x)$, or $h(t,x) = \varphi(tx)$. Integrability criterion $\frac{\partial(hf)}{\partial x} = \frac{\partial(hg)}{\partial t}$ for $(*)$ yields ODE for φ, from which one can (hopefully) determine φ and then h. (But: This is a trial-and-error method. No guarantee that it really helps!)

- After successful determination of h one proceeds as discussed earlier.

Example for determining an integrating factor

Example: Treat $\boxed{2t+t^2+u + (1+t^2+u)u' \equiv 0}$ as follows:

$\overset{\text{read off}}{\rightsquigarrow}$ $f(t,x) = 2t + t^2 + x$, $g(t,x) = 1 + t^2 + x$
check integrability criterion: $\frac{\partial f}{\partial x}(t,x) = 1$, $\frac{\partial g}{\partial t}(t,x) = 2t \rightsquigarrow$ not exact

ansatz $h(t,x) = \varphi(t+x)$ for integrating factor h
$\overset{\text{int.crit.}}{\rightsquigarrow}$ $\varphi'(t+x)f(t,x) + \varphi(t+x)\frac{\partial f}{\partial x}(t,x) = \varphi'(t+x)g(t,x) + \varphi(t+x)\frac{\partial g}{\partial t}(t,x)$
$\overset{\text{compute}}{\rightsquigarrow}$ $\varphi'(t+x)(2t-1) = \varphi(t+x)(2t-1)$, also $\varphi'(s) = \varphi(s)$
$\overset{\text{solve}}{\rightsquigarrow}$ $\varphi(s) = \mathrm{e}^s$, thus integrating factor $h(t,x) = \mathrm{e}^{t+x}$

equivalent, exact ODE: $\mathrm{e}^{t+u}(2t + t^2 + u) + \mathrm{e}^{t+u}(1 + t^2 + u)u' \equiv 0$
$\overset{\text{as before}}{\rightsquigarrow}$ potential $\Psi(t,x) = \mathrm{e}^{t+x}(t^2+x)$,
equivalent equation $\mathrm{e}^{t+u(t)}(t^2+u(t)) = C$ not explicitly solvable

2.4 Specific second-order ODEs

Specific types of non-linear second-order ODE can be solved (in principle) by reduction to first order:

- type $\boxed{F(t,u'(t),u''(t))=0}$ („no $u(t)$“):
  Solve as first-order ODE for u', then determine u.

- type $\boxed{u''(t)=g(u(t))}$ (explicit, autonomous, „no $u'(t)$“):
  Deduce $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}u'(t)^2 = u''(t)u'(t) \overset{\text{ODE}}{=} g(u(t))u'(t) = \frac{\mathrm{d}}{\mathrm{d}t}G(u(t))$ with antiderivative G of g, solve first-order ODE $\frac{1}{2}u'(t)^2 = G(u(t))+C$.

- type $\boxed{u''(t)=f_0(u(t),u'(t))}$ (explicit, autonomous):
  Change of variable $y(x)=u'(u^{-1}(x))=\frac{1}{(u^{-1})'(x)}$ leads to first-order ODE $y'(x)=\frac{f_0(x,y(x))}{y(x)}$ for $y(x)$. Solve for y, then determine u^{-1}, u.
  (Works essentially if y has no zeros, produces invertible u!)

Chapter 3: Linear ODEs and linear systems of ODEs

This chapter delves into the treatment of linear ODEs and linear systems of ODEs and discusses their general mathematical theory along with further methods for explicitly computing solutions (beyond the earlier solutions formulas for the scalar first-order case).

We here work (for reasons to be revealed soon) with solutions $u\colon I \to \mathbb{K}^n$ where I is still an interval[‡] in $\mathbb{R}$, but now $\mathbb{K}$ stands as a wildcard for either $\mathbb{R}$ or $\mathbb{C}$ (real or complex numbers).

[‡]Empty intervals and intervals which consist of a single point are always excluded now.

3.1 General solution theory for linear ODEs

We consider, for $m, n \in \mathbb{N}$, a general linear system of ODEs

$$\sum_{k=0}^{m} A_k(t)u^{(k)}(t) = b(t)$$

for $u \in \mathrm{C}^m(I, \mathbb{K}^n)$ with matrix coefficients $A_k \in \mathrm{C}^0(I, \mathbb{K}^{n\times n})$ and inhomogeneity $b \in \mathrm{C}^0(I, \mathbb{K}^n)$. With the differential operator

$$\mathcal{L}\colon\ \mathrm{C}^m(I, \mathbb{K}^n) \to \mathrm{C}^0(I, \mathbb{K}^n)\,,\ \mathcal{L}[u](t) := \sum_{k=0}^{m} A_k(t)u^{(k)}(t)$$

the above inhomogeneous system and its homogeneous counterpart read

$$\mathcal{L}[u] = b \quad \text{on } I \qquad\qquad \text{and} \qquad\qquad \mathcal{L}[u] \equiv 0 \quad \text{on } I\,.$$

Concrete example (with $m = 2$, $n = 2$, $\mathbb{K} = \mathbb{R}$):

$$\underbrace{\begin{pmatrix} u_1''(t) \\ u_2''(t) \end{pmatrix}}_{A_2(t)=\mathbf{I}_2 \text{ (identity matrix)}} + \underbrace{\begin{pmatrix} \mathrm{e}^t & -5 \\ t^3 & 2t \end{pmatrix}}_{A_1(t)} \underbrace{\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix}}_{u'(t)} + \underbrace{\begin{pmatrix} 1 & t^7 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}}_{A_0(t)} \underbrace{\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}}_{u(t)} = \underbrace{\begin{pmatrix} 2t-1 \\ -3 \end{pmatrix}}_{b(t)}$$

Solution structure for linear ODEs

Theorem: *For linear systems (as before), it holds:*

(1) *The differential operator $\mathcal{L}$ is linear* (*i.e.* $\mathcal{L}[ru+sv] = r\mathcal{L}[u] + s\mathcal{L}[v]$ *for all* $u, v \in \mathrm{C}^m(I, \mathbb{K}^n)$ *and* $r, s \in \mathbb{K}$).

(2) *The solutions of the homogeneous system* $\mathcal{L}[u] \equiv 0$ *on* I *form a vector (sub-)space (of* $\mathrm{C}^m(I, \mathbb{K}^n)$*), the solutions vector space of the system.*

(3) *Given one solution* u_0 *of* $\mathcal{L}[u] = b$ *on* I *one obtains all solutions of* $\mathcal{L}[u] = b$ *on* I *in form* $\boxed{u = u_0 + u_\mathrm{h}}$ *with solutions* u_h *of* $\mathcal{L}[u] \equiv 0$ *on* I (*general inhom. sol. = special/particular inhom. sol. + general hom. sol.*).

Proof: (1):

$$\begin{aligned}\mathcal{L}[ru+sv](t) &= \textstyle\sum_{k=0}^m A_k(t)(ru+sv)^{(k)}(t) \\ &= r\textstyle\sum_{k=0}^m A_k(t)u^{(k)}(t) + s\textstyle\sum_{k=0}^m A_k(t)v^{(k)}(t) \\ &= r\mathcal{L}[u](t) + s\mathcal{L}[v](t)\end{aligned}$$

(2): $\mathcal{L}[u] \equiv 0,\ \mathcal{L}[v] \equiv 0 \implies \mathcal{L}[ru+sv] \overset{(1)}{=} r\mathcal{L}[u] + s\mathcal{L}[v] \equiv r\cdot 0 + s\cdot 0 = 0$

(3): $\mathcal{L}[u_\mathrm{h}] \equiv 0 \overset{\mathcal{L}[u_0]=b}{\Longrightarrow} \mathcal{L}[\underbrace{u_0+u_\mathrm{h}}_{=:u}] = b$ and $\mathcal{L}[u] = b \overset{\mathcal{L}[u_0]=b}{\Longrightarrow} \mathcal{L}[\underbrace{u-u_0}_{=:u_\mathrm{h}}] \equiv 0$ □

IVPs and degrees of freedom for linear ODEs

Theorem: *For linear systems (as before) with $A_m = \mathbf{I}_n$ (explicit form):*

(1) *Existence and uniqueness theorem for linear IVPs: For arbitrary $t_0 \in I$ and $y_0, y_1, \ldots, y_{m-1} \in \mathbb{K}^n$, there exist a unique solution to*

$$\mathcal{L}[u] = b \text{ on } I\,, \qquad u(t_0) = y_0,\, u'(t_0) = y_1,\, \ldots,\, u^{(m-1)}(t_0) = y_{m-1}\,.$$

(2) *The solutions vector space of $\mathcal{L}[u] \equiv 0$ on I has dimension mn.*

Decisive: By Part (2) one may express every solution u of $\mathcal{L}[u] \equiv 0$ on I in terms of an arbitrary basis $u_1, u_2, \ldots, u_{mn}$ of the solutions vector space as

$$\boxed{u = C_1u_1 + C_2u_2 + \ldots + C_{mn}u_{mn}} \qquad \text{with } C_1, C_2, \ldots, C_{mn} \in \mathbb{K}\,.$$

This confirms the rule of thumb „solution with mn constants" of Chapter 1.

Proof: (1) is proved in the later Chapter 6 (and then directly in wider generality).
(2): For each $t_0 \in I$, the linear map $u \mapsto (u(t_0), \ldots, u^{(m-1)}(t_0))$ from the solutions vector space of $\mathcal{L}[u] \equiv 0$ to $(\mathbb{K}^n)^m \cong \mathbb{K}^{mn}$ is one-to-one and onto by (1), hence preserves bases. So, the solutions vector space has the dimension mn of $\mathbb{K}^{mn}$. □

3.2 Scalar linear ODEs with constant coefficients

For a scalar homogeneous linear ODE of arbitrary order $m \in \mathbb{N}$ with constant coefficients $a_k \in \mathbb{K}$ and leading coefficient $a_m \neq 0$

$$\mathcal{L}[u] := \sum_{k=0}^{m} a_k u^{(k)} \equiv 0 \qquad (*)$$

for $u \in \mathrm{C}^m(I, \mathbb{K})$, try the exponential ansatz $u(t) = \mathrm{e}^{\lambda t}$ with $\lambda \in \mathbb{K}$: In view of $u^{(k)}(t) = \lambda^k \mathrm{e}^{\lambda t}$ and $\mathcal{L}[u](t) = \big(\sum_{k=0}^{m} a_k \lambda^k\big)\mathrm{e}^{\lambda t}$ obtain a solution u of $(*)$ if and only if $\lambda \in \mathbb{K}$ is zero of the characteristic polynomial

$$p(\lambda) := \sum_{k=0}^{m} a_k \lambda^k \,.$$

In the sequel, this very basic idea will be extended and refined:

Examples for exponential solutions; homogeneous case

- example $\boxed{u'' = u}$ or equivalently $\boxed{u'' - u \equiv 0}$:

  $\overset{\text{read off}}{\rightsquigarrow}$ characteristic polynomial: $p(\lambda) = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$,

  zeros: $\lambda_1 = 1$ and $\lambda_2 = -1$ with multiplicities $d_1 = d_2 = 1$

  $\overset{\text{principle}}{\rightsquigarrow}$ general solution: $u(t) = C_1 \mathrm{e}^t + C_2 \mathrm{e}^{-t}$ with $C_1, C_2 \in \mathbb{K}$

  $\big(= R_1 \cosh(t) + R_2 \sinh(t)$ with $R_1 = C_1 + C_2$, $R_2 = C_1 - C_2\big)$

- example $\boxed{u''' + 3u'' - 4u \equiv 0}$:

  $\overset{\text{read off}}{\rightsquigarrow}$ characteristic polynomial: $p(\lambda) = \lambda^3 + 3\lambda^2 - 4$

  $\overset{\text{solve } p(\lambda) \overset{!}{=} 0}{\rightsquigarrow}$ zeros: $\lambda_1 = 1$ with $d_1 = 1$ and $\lambda_2 = -2$ with $d_2 = 2$,

  $p(\lambda) = (\lambda-1)(\lambda+2)^2$

  $\overset{\text{principle}}{\rightsquigarrow}$ general solution: $u(t) = C_1 \mathrm{e}^t + C_2 \mathrm{e}^{-2t} + C_3 t \mathrm{e}^{-2t}$

Exponential solutions in case of non-real zeros

Caution! Over $\mathbb{K} = \mathbb{C}$ a polynomial p of degree m always has m zeros (counted with multiplicity!!), and the multiplicities d_i satisfy $\sum_{i=1}^{\ell} d_i = m$. Over $\mathbb{K} = \mathbb{R}$ this is not true in general. Hence, even for real coefficients $a_k \in \mathbb{R}$ one may need to calculate over $\mathbb{C}$:

- example $\boxed{u'' = -u}$ or equivalently $\boxed{u'' + u \equiv 0}$ (cf. Chapter 1):

  $\overset{\text{read off}}{\rightsquigarrow}$ characteristic polynomial: $p(\lambda) = \lambda^2 + 1 = (\lambda - \mathbf{i})(\lambda + \mathbf{i})$,
  zeros: $\lambda_1 = \mathbf{i}$ and $\lambda_2 = -\mathbf{i}$ with $d_1 = d_2 = 1$

  $\overset{\text{principle}}{\rightsquigarrow}$ general complex sol.: $u(t) = C_1 \mathrm{e}^{\mathbf{i}t} + C_2 \mathrm{e}^{-\mathbf{i}t}$ with $C_1, C_2 \in \mathbb{C}$

  $\overset{\mathrm{Re}(\cdot)}{\rightsquigarrow}$ general real sol.: $u_{\mathbb{R}}(t) = R_1 \cos(t) + R_2 \sin(t)$ with $R_1, R_2 \in \mathbb{R}$
  (where $R_1 = \mathrm{Re}(C_1 + C_2)$, $R_2 = \mathrm{Im}(C_2 - C_1)$)

Here, the last step was computing the real part $u_{\mathbb{R}} = \mathrm{Re}(u)$ with the help of $\mathrm{e}^{\mathbf{i}t} = \cos(t) + \mathbf{i}\sin(t)$. In general cases with real coefficients $a_k \in \mathbb{R}$, one analogously obtains all real solutions as the real parts (or alternatively as the imaginary parts) of the complex solutions. Later more on this!

Exponential solutions of scalar homogeneous linear ODEs

The background behind the computations is:

Principle (exponential solutions; homogeneous case)

(1) *If* $\lambda \in \mathbb{K}$ *is a multiplicity-*d *zero of the charact. polynomial* p*, then* $\mathrm{e}^{\lambda t}, t\mathrm{e}^{\lambda t}, t^2\mathrm{e}^{\lambda t}, \ldots, t^{d-1}\mathrm{e}^{\lambda t}$ *are* d *linearly independent solutions of* $(*)$*.*

(2) *If* $\lambda_1, \ldots, \lambda_\ell \in \mathbb{K}$ *are distinct zeros of* p *with correspond. multiplicities* $d_1, \ldots, d_\ell \in \mathbb{N}$*, then also* $u(t) = \sum_{i=1}^{\ell} \sum_{j=0}^{d_i-1} C_{i,j} t^j \mathrm{e}^{\lambda_i t}$ *with* $C_{i,j} \in \mathbb{K}$ *is a solution of* $(*)$*. In case* $\sum_{i=1}^{\ell} d_i = m$ *this is the general solution.*

Proof: (1): By induction: For $d=1$, see beginning of 3.2. For $d \ge 2$, factorize $p(z) = \widetilde{p}(z)(z-\lambda)$ such that λ is a multiplicity-$(d-1)$ zero of $\widetilde{p}$. By induction hypothesis (IH), $u_j(t) := t^j \mathrm{e}^{\lambda t}$ with $j \le d-2$ solve $\widetilde{\mathcal{L}}[u] \equiv 0$, where $\widetilde{\mathcal{L}}$ is the differential operator corresponding to $\widetilde{p}$. For all $j \le d-1$, then deduce:

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t}-\lambda\big)u_j(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big(t^j \mathrm{e}^{\lambda t}\big) - \lambda t^j \mathrm{e}^{\lambda t} = (jt^{j-1}+\lambda t^j-\lambda t^j)\mathrm{e}^{\lambda t} = j u_{j-1}(t)\,,$$

$$\mathcal{L}[u_j] = \widetilde{\mathcal{L}}\big[\big(\tfrac{\mathrm{d}}{\mathrm{d}t}-\lambda\big)u_j\big] = j\widetilde{\mathcal{L}}[u_{j-1}] \overset{\text{IH}}{\equiv} 0\,.$$

(2): Now follows by theory in 3.1 ($\mathcal{L}$ linear, solutions vector space has dim. m). □

Exponential solutions of scalar, *in*homogeneous linear ODEs

For a scalar *in*homogeneous, linear ODE with constant coefficients $a_k \in \mathbb{K}$ (where still $m \in \mathbb{N}$ and $a_m \neq 0$)

$$\mathcal{L}[u] := \sum_{k=0}^{m} a_k u^{(k)} = b \qquad (**)$$

with characteristic polynomial $p(\lambda) := \sum_{k=0}^{m} a_k \lambda^k$ the main issue left is determining a special solution u_0. Here is a rule for approaching this:

Theorem (exponential solutions in case of exponential inhomogeneity)

In case $b(t) = \sum_{h=0}^{q} b_h t^h \mathrm{e}^{\zeta t}$ *with* $q \in \mathbb{N}_0$, $b_h, \zeta \in \mathbb{K}$ *the ansatz*

$$u_0(t) = \begin{cases} \sum_{h=0}^{q} B_h t^h \mathrm{e}^{\zeta t}, & \text{if } \zeta \text{ is not a zero of } p \\ \sum_{h=0}^{q} B_h t^{d+h} \mathrm{e}^{\zeta t}, & \text{if } \zeta \text{ is a multiplicity-}d \text{ zero of } p \end{cases}$$

with suitably determined $B_h \in \mathbb{K}$ *yields a special solution* u_0 *of* $(**)$.

This can be verified via induction on q, but here we omit this proof.

Example: oscillator equation with inhomogeneity

$\boxed{u''(t) + \omega^2 u(t) = \mathrm{e}^{\mathbf{i}\xi t}}$ with $\omega, \xi \in \mathbb{R}$ has $p(\lambda) = \lambda^2 + \omega^2 = (\lambda - \mathbf{i}\omega)(\lambda + \mathbf{i}\omega)$.

- In case $\xi \neq \pm\omega$ observe that $\mathbf{i}\xi$ is not a zero of p.
  $\overset{\text{ansatz}}{\rightsquigarrow}$ special solution $u_0(t) = B_0 \mathrm{e}^{\mathbf{i}\xi t}$ with (use ODE!) $B_0 = \frac{1}{\omega^2 - \xi^2}$
- In case $\xi = \pm\omega$ observe that $\mathbf{i}\xi$ is a multiplicity-1 zero of p.
  $\overset{\text{ansatz}}{\rightsquigarrow}$ special solution $u_0(t) = B_0 t \mathrm{e}^{\mathbf{i}\xi t}$ with (use ODE!) $B_0 = -\frac{\mathbf{i}}{2\xi}$

The general solution then is $u = u_0 + u_\mathrm{h}$ with $u_\mathrm{h}(t) = C_1 \mathrm{e}^{\mathbf{i}\omega t} + C_2 \mathrm{e}^{-\mathbf{i}\omega t}$.

Interpretation: oscillator of eigenfrequency $\frac{|\omega|}{2\pi}$ driven with frequency $\frac{|\xi|}{2\pi}$.
For $\xi \neq \pm\omega$ get bounded u, but resonance $\xi = \pm\omega$ produces $\lim_{t\to\infty} |u(t)| = \infty$.

For $\boxed{u''(t) + \omega^2 u(t) = \cos(\xi t)}$ taking the real part yields more realistic *real* solutions
$$u_\mathbb{R}(t) = \begin{Bmatrix} \frac{1}{\omega^2 - \xi^2} \cos(\xi t), & \text{if } \xi \neq \pm\omega \\ \frac{1}{2\xi} t \sin(\xi t), & \text{if } \xi = \pm\omega \end{Bmatrix} + R_1 \cos(\omega t) + R_2 \sin(\omega t).$$

Addendum to Section 3.1: fundamental systems

For general linear systems (with operator $\mathcal{L}[u] := \sum_{k=0}^{m} A_k u^{(k)}$ and $A_k \in \mathrm{C}^0(I, \mathbb{K}^{n\times n})$ as in Section 3.1), we additionally introduce:

Definition

A basis of the solutions vector space of the homogeneous system $\mathcal{L}[u] \equiv 0$ on I is called a fundamental system (FS) for $\mathcal{L}[u] \equiv 0$ on I.

For case $A_m = \mathbf{I}_n$ seen before: representation of general solution u with FS out of mn basis solutions u_i of $\mathcal{L}[u] \equiv 0$ and with mn constants $C_i \in \mathbb{K}$:

- for $\mathcal{L}[u] \equiv 0$ (homogeneous): $u = C_1 u_1 + C_2 u_2 + \ldots + C_{mn} u_{mn}$,
- for $\mathcal{L}[u] = b$ (inhomogeneous): $u = u_0 + C_1 u_1 + C_2 u_2 + \ldots + C_{mn} u_{mn}$ with one special/particular solution u_0 of $\mathcal{L}[u] = b$.

3.3 First-order linear systems of ODEs

We restrict considerations on linear systems of ODEs to the first-order case (cf. end of Chapter 1 for a corresponding reduction), i.e. to homogeneous and inhomogeneous linear systems of $n \in \mathbb{N}$ differential equations

$$u' = Au \qquad \text{and} \qquad u' = Au + b\,,$$

respectively, for $u \in \mathrm{C}^1(I, \mathbb{K}^n)$ with coefficient matrix $A \in \mathrm{C}^0(I, \mathbb{K}^{n \times n})$ and inhomogeneity $b \in \mathrm{C}^0(I, \mathbb{K}^n)$.

Fundamental matrices

Definition

Whenever $u_1, u_2, \ldots, u_n \in \mathrm{C}^1(I, \mathbb{K}^n)$ *are solutions of the system* $u' = Au$ *with* $A \in \mathrm{C}^0(I, \mathbb{K}^{n\times n})$*, one calls the matrix function*[‡]

$$W(t) := \Big(u_1(t) \Big| u_2(t) \Big| \ldots \Big| u_n(t) \Big) \in \mathbb{K}^{n\times n}$$

a solutions matrix for $u' = Au$ *on* I*. If* $u_1, u_2, \ldots, u_n$ *is even an FS, one calls* W *a fundamental (solutions) matrix (FM) for* $u' = Au$ *on* I*.*

Every solutions matrix W solves the matrix ODE $W' = AW$.
(Verification: $W_i' = u_i' = Au_i = AW_i = (AW)_i$ with index i for ith column)

Important for theory: For an FM, the matrix $W(t)$ is invertible at all $t \in I$.
(Verification via basis preservation from proof in Section 3.1 and linear algebra:
$u_1, \ldots, u_n$ basis solutions $\overset{3.1}{\Longleftrightarrow}$ $u_1(t), \ldots, u_n(t)$ basis in $\mathbb{K}^n$ $\overset{\text{lin alg}}{\Longleftrightarrow}$ $W(t)$ invertible)

‡For orders $m \geq 2$, one may build $W(t) \in \mathbb{K}^{mn\times mn}$ by inserting into its ith column the vectors $u_i(t)$, $u_i'(t)$, …, $u^{(m-1)}(t)$ (below each other). However, this is not needed here.

Fundamental matrices and abstract solution formulas

An FM W for the homogeneous system $u' = Au$ allows for writing down:

a representation of the general solution with constant vector $C \in \mathbb{K}^n$ ...

- for $u' = Au$ (homogeneous): $u(t) = W(t)C$,
- for $u' = Au + b$ (inhomogeneous): $u(t) = W(t)[B^*(t)+C]$ with antiderivative B^* of $W^{-1}b$.

(Verification that $u_0 = WB^*$ is a special solution of $u' = Au + b$:
$u_0' = (WB^*)' = W'B^* + W(B^*)' = AWB^* + WW^{-1}b = Au_0 + b$.)

a solution formula for the IVP with IC $u(t_0) = y_0$ (where $t_0 \in I$, $y_0 \in \mathbb{K}^n$) ...

- for $u' = Au$ (homogeneous): $u(t) = W(t)W(t_0)^{-1}y_0$,
- for $u' = Au + b$ (inhom.): $u(t) = W(t)\left[\int_{t_0}^{t} W(s)^{-1}b(s)\,\mathrm{d}s + W(t_0)^{-1}y_0\right]$

(The solution formulas of Chapter 1 correspond to the special case $n = 1$.)

Main issue left: compute FS and FM, respectively. More on this follows:

Eigenvalue-eigenvector solutions

Consider a homogeneous linear system of $n \in \mathbb{N}$ ODEs with constant coefficients

$$u' = Au \tag{hS}$$

for $u \in \mathrm{C}^1(I, \mathbb{K}^n)$ with constant coefficient matrix $A \in \mathbb{K}^{n\times n}$.

The exponential ansatz $u(t) = \mathrm{e}^{\lambda t} v$ with $\lambda \in \mathbb{K}$ and vector $v \in \mathbb{K}^n$ gives

$$u'(t) - Au(t) = \lambda \mathrm{e}^{\lambda t} v - \mathrm{e}^{\lambda t} Av = \mathrm{e}^{\lambda t}(\lambda v - Av)$$

and thus yields a solution of (hS) if and only if $Av = \lambda v$ holds, i.e. if and only if v is an eigenvalue of the matrix A for the eigenvalue λ (or $v = 0$).

One may call such solutions eigenvalue-eigenvector solutions.
We now generalize these as follows:

Exponential solutions of homogeneous linear systems

Convention: For $s \in \mathbb{N}$, call $v \in \mathbb{K}^n$ a step-s generalized eigenvector (GEV) of $A \in \mathbb{K}^{n\times n}$ for the eigenvalue $\lambda \in \mathbb{K}$ if $(A-\lambda\mathbf{I}_n)^s v = 0 \neq (A-\lambda\mathbf{I}_n)^{s-1}v$ holds.
(Then: If v is step-s GEV, then $(A-\lambda\mathbf{I}_n)^j v$ is step-$(s-j)$ GEV for same eigenvalue. Step-1 generalized eigenvectors are nothing but eigenvectors.)

Theorem: *For $A \in \mathbb{K}^{n\times n}$, there holds*:

(1) *Whenever $v \in \mathbb{K}^n$ is a step-s GEV of A for the eigenvalue $\lambda \in \mathbb{K}$, then $u(t) = \mathrm{e}^{\lambda t}\sum_{j=0}^{s-1}\frac{1}{j!}t^j v_j$ with $v_j := (A-\lambda\mathbf{I}_n)^j v$ is a solution of $u' = Au$.*

(2) *Given a basis of $\mathbb{K}^n$ out of generalized eigenvectors of A, the corresponding solutions of the preceding type form an FS for $u' = Au$.*

Proof: (1): Use $\lambda v_j - Av_j = -v_{j+1}$ and $v_s = 0$ plus an index shift to compute:

$$\begin{aligned} u'(t) - Au(t) &= \mathrm{e}^{\lambda t}\sum_{j=0}^{s-1}\left(\frac{j}{j!}t^{j-1}v_j + \frac{1}{j!}t^j\lambda v_j - \frac{1}{j!}t^j Av_j\right) \\ &= \mathrm{e}^{\lambda t}\left(\sum_{j=1}^{s-1}\frac{1}{(j-1)!}t^{j-1}v_j - \sum_{j=0}^{s-2}\frac{1}{j!}t^j v_{j+1}\right) = 0\,. \end{aligned}$$

(2): As observed earlier: n linearly independent solutions are basis solutions. □

Example: exponential solutions of a homogeneous system

In (favorable) case of $\boxed{u' = Au \text{ with } A := \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$ proceed this way:

$\overset{\text{read off}}{\rightsquigarrow}$ characteristic polynomial: $p(\lambda) = \det \begin{pmatrix} \lambda-2 & -1 & 3 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{pmatrix} = (\lambda-2)^3$

$\overset{\text{read off}}{\rightsquigarrow}$ 2 is the sole eigenvalue of A and has algebraic multiplicity 3.

Then use $A-2\mathbf{I}_3 = \begin{pmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in computing the (generalized) eigenvectors:

- $\left(\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow$ linearly independent eigenvectors e.g. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$
- $\left(\begin{array}{ccc|c} 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow$ step-2 GEV e.g. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ with $(A-2\mathbf{I}_3)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $\left(\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow$ no GEV (also clear from dimension argument)

$\overset{\text{thm}}{\rightsquigarrow}$ general solution: $C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + C_3 e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$,

an FS: $e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e^{2t}\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$, an FM: $e^{2t}\begin{pmatrix} 1 & 0 & t \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (both not unique!)

Real fundamental system in case of non-real eigenvalues

For real $A \in \mathbb{R}^{n\times n}$ and a (G)EV $v \in \mathbb{C}^n$ of A for eigenvalue $\lambda \in \mathbb{C}$, also $\overline{v}$ is a (G)EV of A for the eigenvalue $\overline{\lambda}$ (as e.g. in EV case: $A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\overline{v}$). Thus, non-real eigenvalues, (G)EVs, and solutions of $u' = Au$ occur in pairs conjugate to each other. This is the starting point for:

Principle (general real solution and real fundamental system for $u' = Au$)

For real coefficients $A \in \mathbb{R}^{n\times n}$ and a non-real[‡] solution v of $u' = Au$ with complex-conjugate solution $\overline{v}$, one may ...

- *replace a term $C_1v + C_2\overline{v}$ (with $C_i \in \mathbb{C}$) in the general complex solution, for the general real solution, by $R_1\mathrm{Re}(v) + R_2\mathrm{Im}(v)$ (with $R_i \in \mathbb{R}$),*
- *correspondingly replace basis solutions v and $\overline{v}$ of a complex FS, for a real FS, by $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$.*

Proof: In essence observe $u' = Au \implies \mathrm{Re}(u)' = \mathrm{Re}(u') = \mathrm{Re}(Au) = A\mathrm{Re}(u)$ and $\mathrm{Re}(C_1v + C_2\overline{v}) = R_1\mathrm{Re}(v) + R_2\mathrm{Im}(v)$ for suitable choice of constants. □

[‡] The „non-real“ assumption is to be understood the way that it excludes $v = Cv_0$ with $C \in \mathbb{C}$ and $\mathbb{R}^n$-valued v_0. This ensures linear independency of v and $\overline{v}$ over $\mathbb{C}$.

Example: computing real FS despite non-real eigenvalues (1)

In the exemplary case $u' = Au$ with $A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{pmatrix}$ proceed this way:

$\overset{\text{compute}}{\rightsquigarrow}$ characteristic polynomial: $\lambda^4+4\lambda^2 = \lambda^2(\lambda-2\mathbf{i})(\lambda+2\mathbf{i})$

$\overset{\text{read off}}{\rightsquigarrow}$ A has multiplicity-2 eigenvalue 0 and multiplicity-1-eigenvalues $\pm 2\mathbf{i}$.

For eigenvalue 0 get EV $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and step-2 GEV $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ with $A\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

For eigenvalues $\pm 2\mathbf{i}$ use $A \mp 2\mathbf{i} \cdot \mathbf{I}_4$: $\left(\begin{array}{cccc|c} \mp 2\mathbf{i} & 0 & 1 & 0 & 0 \\ 0 & \mp 2\mathbf{i} & 0 & 1 & 0 \\ -2 & 2 & \mp 2\mathbf{i} & 0 & 0 \\ 2 & -2 & 0 & \mp 2\mathbf{i} & 0 \end{array}\right) \rightsquigarrow$ EV $\begin{pmatrix} 1 \\ -1 \\ \pm 2\mathbf{i} \\ \mp 2\mathbf{i} \end{pmatrix}$.

$\overset{\text{thm}}{\rightsquigarrow}$ a complex FS: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathrm{e}^{2\mathbf{i}t}\begin{pmatrix} 1 \\ -1 \\ 2\mathbf{i} \\ -2\mathbf{i} \end{pmatrix}, \mathrm{e}^{-2\mathbf{i}t}\begin{pmatrix} 1 \\ -1 \\ -2\mathbf{i} \\ 2\mathbf{i} \end{pmatrix}$

$\overset{\text{principle}}{\rightsquigarrow}$ a real FS: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ t \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \cos(2t) \\ -\cos(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{pmatrix}, \begin{pmatrix} \sin(2t) \\ -\sin(2t) \\ 2\cos(2t) \\ -2\cos(2t) \end{pmatrix}$

Example: computing real FS despite non-real eigenvalues (2)

In the exemplary case $\boxed{u' = Au \text{ with } A := \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}}$ proceed this way:

$\overset{\text{compute}}{\rightsquigarrow}$ characteristic polynomial: $(\lambda^2 - 2\lambda + 2)^2 = (\lambda - 1 - \mathbf{i})^2 (\lambda - 1 + \mathbf{i})^2$

$\overset{\text{read off}}{\rightsquigarrow}$ A has multiplicity-2 eigenvalues $1+\mathbf{i}$ and $1-\mathbf{i}$.

For eigenvalue $1+\mathbf{i}$, solve linear systems with coefficients $A-(1+\mathbf{i})\mathbf{I}_4$ to find EV $\begin{pmatrix} 1 \\ -\mathbf{i} \\ 0 \\ 0 \end{pmatrix}$ and step-2 GEV $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\mathbf{i} \end{pmatrix}$ with $(A-(1+\mathbf{i})\mathbf{I}_4)\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\mathbf{i} \end{pmatrix} = \begin{pmatrix} 1 \\ -\mathbf{i} \\ 0 \\ 0 \end{pmatrix}$.

For eigenvalue $1-\mathbf{i}$, obtain conjugate results with $-\mathbf{i}$ in place of $\mathbf{i}$.

$\underset{e^{(1\pm\mathbf{i})t} = e^t e^{\pm\mathbf{i}t}}{\overset{\text{thm}}{\rightsquigarrow}}$ a complex FM: $e^t \begin{pmatrix} e^{\mathbf{i}t} & e^{-\mathbf{i}t} & te^{\mathbf{i}t} & te^{-\mathbf{i}t} \\ -\mathbf{i}e^{\mathbf{i}t} & \mathbf{i}e^{-\mathbf{i}t} & -\mathbf{i}te^{\mathbf{i}t} & \mathbf{i}te^{-\mathbf{i}t} \\ 0 & 0 & e^{\mathbf{i}t} & e^{-\mathbf{i}t} \\ 0 & 0 & -\mathbf{i}e^{\mathbf{i}t} & \mathbf{i}e^{-\mathbf{i}t} \end{pmatrix}$

$\overset{\text{principle}}{\rightsquigarrow}$ a real FM: $e^t \begin{pmatrix} \cos(t) & \sin(t) & t\cos(t) & t\sin(t) \\ \sin(t) & -\cos(t) & t\sin(t) & -t\cos(t) \\ 0 & 0 & \cos(t) & \sin(t) \\ 0 & 0 & \sin(t) & -\cos(t) \end{pmatrix}$

Addendum: matrix exponential

Addendum: Alternatively, one can use the matrix exponential series

$$e^M := \sum_{k=0}^{\infty} \frac{1}{k!} M^k \in \mathbb{K}^{n\times n} \qquad \text{for } M \in \mathbb{K}^{n\times n}$$

(M^k denotes k-fold matrix product; convergence of series is entry-wise)
to obtain an FM W for $u' = Au$ with $A \in \mathbb{K}^{n\times n}$ „simply“ as $W(t) = e^{tA}$.
(Background: $\frac{d}{dt} e^{tA} = \frac{d}{dt} \sum_{k=1}^{\infty} \frac{1}{k!}(tA)^k = A \sum_{k=1}^{\infty} \frac{1}{(k-1)!}(tA)^{k-1} = Ae^{tA}$.)

However, computation of e^{tA} in general requires determining a normal form of A, which still relies on computing GEVs. Only once this is achieved one may compute e^{tA} with the help of the following rules (for $\lambda_i \in \mathbb{K}$ and $J, M, N, T \in \mathbb{K}^{n\times n}$):

- rule $e^{\operatorname{diag}(\lambda_1,\ldots,\lambda_n)} = \operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$ for diagonal matrices,
- exponential law $e^{M+N} = e^M e^N = e^N e^M$ <u>only in case $MN = NM$</u>,
- transformation rule $e^{TJT^{-1}} = Te^J T^{-1}$ for invertible T.

The variation of constants

A solutions formula for $u' = Au+b$ based on an FM for $u' = Au$ has already been discussed. A more concrete rereading of this formula is:

Principle (solving *in*homogeneous systems by variation of constants)

Whenever $u_1, u_2, \dots, u_n$ is an FS for $u' = Au$ on I with $A \in \mathrm{C}^0(I, \mathbb{K}^{n\times n})$, one obtains all solutions of $u' = Au + b$ on I with $b \in \mathrm{C}^0(I, \mathbb{K}^n)$ in form $\boxed{u(t) = K_1(t)u_1(t)+ \ldots +K_n(t)u_n(t)}$ with functions $K_i \in \mathrm{C}^1(I, \mathbb{K})$ such that $K_i'(t)$ solve the linear system $\boxed{K_1'(t)u_1(t)+ \ldots +K_n'(t)u_n(t) = b(t)}$.

- K_i take the place of the constants C_i in the general solution of $u' = Au$.
- application in computations: first solve linear system (corresponds to inverting FM), then integrate K_i' to find K_i. In principle analogous is:

Proof: In view of $u'-Au = \sum_{i=1}^n \left[K_i'u_i+K_i\left(u_i'-Au_i\right)\right] \overset{u_i \text{ solve}}{=} \sum_{i=1}^n K_i'u_i$ the ansatz leads to the system for $K_i'(t)$ with parameter $t \in I$. Rewriting this system as $W(t)K'(t) = b(t)$ with FM $W := (u_1|\ldots|u_n)$ and $K' := (K_1', \dots, K_n')$, the solutions $K'(t) = W(t)^{-1}b(t)$ are contin. in t, and K_i exist as antiderivatives. □

Example for applying the variation of constants

For $\boxed{u'(t) = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} u(t) + \begin{pmatrix} t \\ -1 \\ -2 \end{pmatrix}}$ an FM $\mathrm{e}^{2t}\begin{pmatrix} 1 & 0 & t \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has been determined.

$\overset{\text{read off}}{\rightsquigarrow}$ linear system of eqns: $\begin{cases} \mathrm{e}^{2t}K_1'(t) + t\mathrm{e}^{2t}K_3'(t) = t\,, \\ 3\mathrm{e}^{2t}K_2'(t) + \mathrm{e}^{2t}K_3'(t) = -1\,, \\ \mathrm{e}^{2t}K_2'(t) = -2 \end{cases}$

$\overset{\text{solve}}{\rightsquigarrow}$ $K_1'(t) = -4t\mathrm{e}^{-2t}$, $K_2'(t) = -2\mathrm{e}^{-2t}$, $K_3'(t) = 5\mathrm{e}^{-2t}$

$\overset{\text{antider.}}{\rightsquigarrow}$ $K_1(t) = (2t+1)\mathrm{e}^{-2t} + C_1$, $K_2(t) = \mathrm{e}^{-2t} + C_2$, $K_3(t) = -\frac{5}{2}\mathrm{e}^{-2t} + C_3$

From ansatz obtain general solution of *in*homogeneous system as follows:

$$\begin{aligned} u(t) &= K_1(t)\mathrm{e}^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + K_2(t)\mathrm{e}^{2t}\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + K_3(t)\mathrm{e}^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \\ &= \left(2t+1+C_1\mathrm{e}^{2t}\right)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(1+C_2\mathrm{e}^{2t}\right)\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + \left(-\tfrac{5}{2}+C_3\mathrm{e}^{2t}\right)\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -t/2+1 \\ 1/2 \\ 1 \end{pmatrix} + C_1\mathrm{e}^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\mathrm{e}^{2t}\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + C_3\mathrm{e}^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Scalar linear equation versus first-order linear system

Concluding remark: A scalar linear equation of order m

$$\sum_{k=0}^{m} a_k u^{(k)} \equiv 0 \qquad \text{(slG)}$$

with leading coefficient $a_m = 1$ is equivalent (compare end of Chapter 1; components of $v\colon I \to \mathbb{K}^m$ correspond to $u, u', u'', \ldots, u^{(m-1)}\colon I \to \mathbb{K}$) with the system of m first-order linear equations

$$v' = Av \qquad \text{with } A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{m-2} & -a_{m-1} \end{pmatrix}.$$

One checks (e.g. by Laplace expansion along last row) that (slG) and A have the same characteristic polynomial and then grasps the background reasons for the similarity of the methods in Sections 3.2 and 3.3.

3.4 d'Alembert reduction

A method for solving linear ODE with non-constant coefficients is based on:

Principle (reducing order from 2 to 1 based on a known solution)

For $a_0, a_1, b\colon I \to \mathbb{K}$, consider the scalar linear second-order ODE

$$u'' + a_1 u' + a_0 u = b \qquad \text{on } I\,. \tag{$*$}$$

Whenever a known solution u_0 of the homogeneous version of $(*)$ has no zeros, then $\boxed{u = wu_0}$ is a solution of $(*)$ if and only if w solves

$$w'' + \left(\frac{2u_0'}{u_0} + a_1\right) w' = \frac{b}{u_0} \qquad \text{on } I\,. \tag{$**$}$$

Proof: $u = wu_0$ in $(*)$ yields: $w''u_0 + 2w'u_0' + wu_0'' + a_1 w' u_0 + a_1 w u_0' + a_0 w u_0 = b$ $\overset{u_0 \text{ solves}}{\Longleftrightarrow}$ $w''u_0 + 2w'u_0' + a_1 w' u_0 = b$ $\overset{\text{division by } u_0 \neq 0}{\Longleftrightarrow}$ $(**)$. □

Decisive: For ODE $(**)$ of order 1 in w' have a solution formula. From w' determine w by integration, and then get all solutions $u = wu_0$ of $(*)$.

Side remark: In a similar way one can use a known solution to reduce scalar equations from order m to $m-1$ and first-order systems from n equations to $n-1$ equations. The above, however, should be the case which is useful most frequently.

Example for applying d'Alembert's reduction

In case of $\boxed{u'' + \frac{1}{t^2}u' - \frac{1}{t^3}u \equiv 0}$ with solution $u_0(t) = t$, proceed this way:

$\overset{\text{read off}}{\rightsquigarrow}$ $a_1(t) = \frac{1}{t^2}$, $a_0(t) = -\frac{1}{t^3}$, $b \equiv 0$

$\overset{\text{principle}}{\rightsquigarrow}$ first-order ODE $w'' + \left(\frac{2}{t}+\frac{1}{t^2}\right)w' \equiv 0$ for w'

$\overset{\text{sol. formula}}{\rightsquigarrow}$ $w'(t) = C\mathrm{e}^{-2\ln(|t|)+1/t} = \frac{C}{t^2}\mathrm{e}^{1/t}$

$\overset{\text{antideriv.}}{\rightsquigarrow}$ $w(t) = C_1\mathrm{e}^{1/t} + C_2$ (with choice $C_1 = -C$)

$\overset{u=wu_0}{\rightsquigarrow}$ general solutions of original ODE: $u(t) = C_1 t\mathrm{e}^{1/t} + C_2 t$

(Valid this way on each interval I such that $0 \notin I$.)

3.5 On linear boundary value problems

Reminder: A boundary value problem (BVP) combines an ODE (instead of the ICs considered before) with boundary conditions (BCs) which involve evaluations $u(t_1)$ and $u(t_2)$ of solutions u at two points t_1 and t_2. While the typical case is $I = [t_1, t_2]$, the theory works even for arbitrary $t_1, t_2 \in I$.

BVPs differ from IVPs inasmuch as existence and uniqueness of solutions may fail in specific cases, but still one has good criteria for their availability:

Boundary value problems for first-order linear systems

Consider, for $u\colon I \to \mathbb{K}^n$, a first-order linear boundary value problem

$$u' = Au+b \text{ on } I \qquad \text{with BC} \quad \Gamma_1 u(t_1)+\Gamma_2 u(t_2) = y\,, \qquad \text{(BVP)}$$

where $t_1, t_2 \in I$, $\Gamma_1, \Gamma_2 \in \mathbb{K}^{n\times n}$ and $y \in \mathbb{K}^n$ are given.

Based on an FM the following criterion decides on solvability of the BVP:

Theorem (solvability criterion for linear BVPs; existence and uniqueness)

Fix $t_1, t_2 \in I$, $\Gamma_1, \Gamma_2 \in \mathbb{K}^{n\times n}$, and an FM W for $u' = Au$ on I. Then, (BVP) is uniquely solvable for all $b \in \mathrm{C}^0(I, \mathbb{K}^n)$ and all $y \in \mathbb{K}^n$ if and only if $\Gamma_1 W(t_1)+\Gamma_2 W(t_2) \in \mathbb{K}^{n\times n}$ is an invertible matrix.

Proof: Write general solution of $u' = Au+b$ as $u(t) = u_0(t)+W(t)C$ with $C \in \mathbb{K}^n$. Plug this into BC to find $\big(\Gamma_1 W(t_1)+\Gamma_2 W(t_2)\big)C = y - \Gamma_1 u_0(t_1) - \Gamma_2 u_0(t_2)$, which is always uniquely solvable (only) for invertible $\Gamma_1 W(t_1)+\Gamma_2 W(t_2)$. □

Remark: IVP corresponds to special case $\Gamma_1 = \mathbf{I}_n$ (or invertible at least), $\Gamma_2 = 0$.

Boundary value problems for second-order linear equations

The general scalar second-order linear boundary value problem reads

$$u''+a_1u'+a_0u = b \text{ on } I \qquad \text{with BC} \quad \Gamma_1\begin{pmatrix} u(t_1) \\ u'(t_1) \end{pmatrix}+\Gamma_2\begin{pmatrix} u(t_2) \\ u'(t_2) \end{pmatrix} = y\,,$$

where now $\Gamma_1, \Gamma_2 \in \mathbb{K}^{2\times 2}$, $y \in \mathbb{K}^2$. Specifically, for $\Gamma_1 = \begin{pmatrix} \gamma_{11} & 0 \\ \gamma_{21} & 0 \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} \gamma_{12} & 0 \\ \gamma_{22} & 0 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ this reduces to the BVP with zero-order BCs:

$$u''+a_1u'+a_0u = b \text{ on } I \qquad \text{with BCs} \quad \begin{aligned} \gamma_{11}u(t_1)+\gamma_{12}u(t_2) &= y_1 \\ \gamma_{21}u(t_1)+\gamma_{22}u(t_2) &= y_2 \end{aligned} \qquad \text{(BVP2)}$$

The solvability criterion then carries forward from the equivalent first-order system $v' = Av+\begin{pmatrix} 0 \\ b \end{pmatrix}$ for $v = \begin{pmatrix} u \\ u' \end{pmatrix}$ (cf. earlier) to the above equation:

Corollary (solvability criterion for scalar second-order linear BVPs)

Fix $t_1, t_2 \in I$, $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22} \in \mathbb{K}$, and an FS u_1, u_2 for $u''+a_1u'+a_0u \equiv 0$. Then, (BVP2) *is uniquely solvable for all $b \in \mathrm{C}^0(I, \mathbb{K})$ and all $y_1, y_2 \in \mathbb{K}$ if and only if $\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}\begin{pmatrix} u_1(t_1) & u_2(t_1) \\ u_1(t_2) & u_2(t_2) \end{pmatrix} \in \mathbb{K}^{2\times 2}$ is an invertible matrix.*

Example for BVP solvability, boundary eigenvalue problem

For $\boxed{u''+\omega^2 u = b \text{ with BCs } u(t_1)=y_1,\ u(t_2)=y_2}$ ($\omega \in \mathbb{R}\setminus\{0\}$ parameter), start from FS $\cos(\omega t), \sin(\omega t)$ for $u''+\omega^2 u \equiv 0$ and $\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solvability criterion: above BVP always uniquely solvable

$$\iff \begin{pmatrix} \cos(\omega t_1) & \sin(\omega t_1) \\ \cos(\omega t_2) & \sin(\omega t_2) \end{pmatrix} \text{ invertible } \overset{\det(\cdot)}{\iff} \sin(\omega(t_2-t_1)) \neq 0$$
$$\iff \omega \neq \frac{k\pi}{t_2-t_1} \text{ for all } k \in \mathbb{Z}.$$

For the exceptional values $\omega = \frac{k\pi}{t_2-t_1}$, $k \in \mathbb{Z}\setminus\{0\}$: Homogeneous BVP with $b \equiv 0$, $y_1 = 0$, $y_2 = 0$ has infinitely many solutions $u(t) = C\sin(\omega(t-t_1))$ with $C \in \mathbb{R}$. Inhomogeneous BVP has either infinitely many solutions (e.g. $b \equiv 0$, $y_2 = y_1$) or no solution at all (e.g. $b \equiv 0$, $y_2 \neq y_1$).

With $\mathcal{L}[u] := -u''$ write homogeneous ODE as $\mathcal{L}[u] = \omega^2 u$. One calls the infinitely many (!) exceptional values $\omega^2 = \left(\frac{k\pi}{t_2-t_1}\right)^2$, $k \in \mathbb{N}$, the eigenvalues and the corresponding solutions $\not\equiv 0$ the eigenfunctions of the operator $\mathcal{L}$.

3.6 The Laplace transform

An alternative approach to linear ODE is based on the following concept:

Definition (Laplace transform)

The Laplace transform $\mathscr{L}f$ *of a function* $f\colon [0,\infty) \to \mathbb{K}$*, which is Riemann integrable over all* $[t_1, t_2]$ *with* $0 < t_1 < t_2 < \infty$ *(e.g. is continuous or is piecewise continuous), is defined as*

$$\mathscr{L}f(s) := \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \in \mathbb{C} \qquad \text{for suitable } s \in \mathbb{C}\,.$$

For $\mathscr{L}f = F$ *one also writes* $f(t) \circ\!\!-\!\!\!-\!\!\bullet\, F(s)$ *or* $F(s) \bullet\!\!-\!\!\!-\!\!\circ\, f(t)$ *(Doetsch symbol).*

- Basic existence assertion: In case of at most exponential growth $|f(t)| \le C\mathrm{e}^{\gamma_0 t}$ for all $t \ge 0$ with fixed $C \in [0,\infty)$, $\gamma_0 \in \mathbb{R}$, the transform $F(s)$ is defined at least on the half-plane of $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \gamma_0$.
  (Justification: From $|\mathrm{e}^{-st}f(t)| = \mathrm{e}^{-\mathrm{Re}(s)t}|f(t)| \le C\mathrm{e}^{(\gamma_0-\mathrm{Re}(s))t}$ deduce absolute convergence $\int_0^\infty |\mathrm{e}^{-st}f(t)|\,\mathrm{d}t \le C\int_0^\infty \mathrm{e}^{(\gamma_0-\mathrm{Re}(s))t}\,\mathrm{d}t < \infty$.)

On inversion of the Laplace transform

Remark: The Laplace transform is closely related to the Fourier transform, defined for $g\colon \mathbb{R} \to \mathbb{K}$ as $\mathscr{F}g(\xi) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathbf{i}\xi t} g(t)\,\mathrm{d}t \in \mathbb{C}$ for $\xi \in \mathbb{R}$. More precisely, there holds $\mathscr{L}f(\gamma+\mathbf{i}\xi) = \mathscr{F}(f_\gamma)(\xi)$ with abbreviation $f_\gamma(t) := \begin{cases} \mathrm{e}^{-\gamma t} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Essential advantage of Laplace transform over Fourier transform:
Still defined without problems for functions/solutions of exponential growth.

Decisive: Continuous f of at most exponential growth are fully determined by their Laplace transform $\mathscr{L}f$ (and even by $\mathscr{F}(f_\gamma)$ for a single $\gamma > \gamma_0$). In this sense, the Laplace transform is one-to-one and does not lose information. For such f, g, it holds $\mathscr{L}f = \mathscr{L}g \implies f = g$. In fact, there are even inversion formulas, which express f in terms of $\mathscr{L}f$ (or $\mathscr{F}(f_\gamma)$).

For reasons of time, no details and proofs on this!

Examples of Laplace transforms

Table of correspondencies (important Laplace-transform pairs):

$$t^k \circ\!\!-\!\!\bullet \frac{k!}{s^{k+1}} \quad \text{for } k \in \mathbb{N}_0 \quad (F \text{ defined where } \mathrm{Re}(s) > 0)$$

$$\mathrm{e}^{\lambda t} \circ\!\!-\!\!\bullet \frac{1}{s-\lambda} \quad \text{for } \lambda \in \mathbb{K} \quad (F \text{ defined where } \mathrm{Re}(s) > \mathrm{Re}(\lambda))$$

$$\left.\begin{aligned} \mathrm{e}^{\alpha t}\cos(\omega t) &\circ\!\!-\!\!\bullet \frac{s-\alpha}{(s-\alpha)^2+\omega^2} \\ \mathrm{e}^{\alpha t}\sin(\omega t) &\circ\!\!-\!\!\bullet \frac{\omega}{(s-\alpha)^2+\omega^2} \end{aligned}\right\} \text{for } \alpha,\omega \in \mathbb{R} \quad (F \text{ defined where } \mathrm{Re}(s) > \alpha)$$

Verifications: 1.) For $f(t) = t^k$, argue by induction on k:

Base ($k=0$): $F(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t \overset{\text{FTC}}{=} \left[-\frac{1}{s}\mathrm{e}^{-st}\right]_{t=0}^\infty = \lim_{t\to\infty}\left(-\frac{1}{s}\mathrm{e}^{-st}\right) + \frac{1}{s} = \frac{1}{s}$

Step ($k \geq 1$): $F(s) = \int_0^\infty \mathrm{e}^{-st} t^k \mathrm{d}t \overset{\text{ibp}}{=} \underbrace{\left[-\frac{1}{s}\mathrm{e}^{-st}t^k\right]_{t=0}^\infty}_{=0} + \frac{k}{s}\underbrace{\int_0^\infty \mathrm{e}^{-st}t^{k-1}\mathrm{d}t}_{=\frac{(k-1)!}{s^k}} = \frac{k!}{s^{k+1}}$

2.) For $f(t) = \mathrm{e}^{\lambda t}$ compute: $F(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{\lambda t}\,\mathrm{d}t \overset{\text{FTC}}{=} \left[\frac{1}{\lambda - s}\mathrm{e}^{(\lambda-s)t}\right]_{t=0}^\infty = \frac{1}{s-\lambda}$

3.) Use $\mathrm{e}^{\alpha t}\cos(\omega t) = \frac{\mathrm{e}^{(\alpha+\mathbf{i}\omega)t}+\mathrm{e}^{(\alpha-\mathbf{i}\omega)t}}{2}$ and $\mathrm{e}^{\alpha t}\sin(\omega t) = \frac{\mathrm{e}^{(\alpha+\mathbf{i}\omega)t}-\mathrm{e}^{(\alpha-\mathbf{i}\omega)t}}{2\mathbf{i}}$ and linearity on next slide to deduce remaining claims from the one for $f(t) = \mathrm{e}^{\lambda t}$. □

Calculation rules for Laplace transforms

Calculation rules (for integrable $f, g\colon [0,\infty) \to \mathbb{K}$ of at most exponential growth; transforms defined where $\mathrm{Re}(s) > \gamma_0$ for growth exponent γ_0; always $F = \mathscr{L}f$):

- linearity: $\mathscr{L}(rf{+}sg) = r\mathscr{L}f + s\mathscr{L}g$ for $r, s \in \mathbb{K}$.
- derivation rules: $\boxed{\mathscr{L}\big(f'\big)(s) = s\mathscr{L}f(s) - f(0)}$ for contin. differentiable f,
  $\mathscr{L}\big(f^{(k)}\big)(s) = s^k\mathscr{L}f(s) - \sum_{i=0}^{k-1} s^{k-i-1} f^{(i)}(0)$ for f in C^k, $k \in \mathbb{N}_0$
  (in other words: $f^{(k)}(t) \circ\!\!-\!\!\!-\!\!\bullet\, s^kF(s) - \sum_{i=0}^{k-1} s^{k-i-1} f^{(i)}(0)$).
- multiplication rule: $t^k f(t) \circ\!\!-\!\!\!-\!\!\bullet\, (-1)^k F^{(k)}(s)$ for $k \in \mathbb{N}_0$
  (in other words: $\mathscr{L}(t^k f)(s) = (-1)^k(\mathscr{L}f)^{(k)}(s)$ where $t^k f$ stands for $t \mapsto t^k f(t)$).

Proofs: 1.) Checking linearity is straightforward.

2.) $\mathscr{L}\big(f'\big)(s) = \int_0^\infty \mathrm{e}^{-st} f'(t)\mathrm{d}t \overset{\text{ibp}}{=} \big[\mathrm{e}^{-st} f(t)\big]_{t=0}^{\infty} + s\int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t = s\mathscr{L}f(s) - f(0)$.
The rule for $\mathscr{L}\big(f^{(k)}\big)$ follows iteratively.

3.) $(\mathscr{L}f)^{(k)}(s) = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}(-t)^k f(t)\mathrm{d}t = (-1)^k\mathscr{L}(t^k f)(s)$. □

Calculation rules for Laplace transforms (continued)

Integration and division rules for Laplace transforms can be obtained by „reading backwards" the derivations and multiplication rules.

Further calculations rules are (same general framework):

- scaling rule: $f(\alpha t) \circ\!\!-\!\!\bullet \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$ for $\alpha > 0$.
- exponential rule: $\mathrm{e}^{\lambda t} f(t) \circ\!\!-\!\!\bullet F(s-\lambda)$ for $\lambda \in \mathbb{K}$.
- translation rule: $f(t_0+t) \circ\!\!-\!\!\bullet \mathrm{e}^{st_0} F(s)$ for $t_0 \in \mathbb{R}$ **provided** that $f \equiv 0$ holds on $[0, t_0)$, $t_0 > 0$ or that one sets $f :\equiv 0$ on $[t_0, 0)$, $t_0 < 0$.

Sketch of proof: First use definition. Then change variables $\tau = \alpha t$, read off the claim by rearranging terms, or change variables $\tau = t_0+t$, respectively. □

Example 1: solving a linear IVP via Laplace transform

A central application of the Laplace transform is solving scalar linear IVPs with constant coefficients.

Exemplary IVP: $u'' - 4u' + 3u \equiv 0$ on $[0, \infty)$ with $u(0) = 1$, $u'(0) = 5$

For computing the solution u apply $\mathscr{L}$ to the ODE, and proceed as follows:

$$\overset{\text{linearity}}{\rightsquigarrow} \mathscr{L}(u'') - 4\mathscr{L}(u') + 3\mathscr{L}u \equiv 0$$

$$\overset{\text{derivation rule, ICs}}{\rightsquigarrow} \big(s^2\mathscr{L}u(s) - 5 - s\big) - 4\big(s\mathscr{L}u(s) - 1\big) + 3\mathscr{L}u(s) = 0$$

$$\overset{\text{solve for } \mathscr{L}u(s)}{\rightsquigarrow} \mathscr{L}u(s) = \frac{s+1}{s^2-4s+3} = \frac{2(s-1)-(s-3)}{(s-1)(s-3)} = \frac{2}{s-3} - \frac{1}{s-1}$$

(in general needs partial fraction decomposition, possibly lengthy!)

$\overset{\text{back trafo/table}}{\rightsquigarrow}$ solution of IVP: $u(t) = 2\mathrm{e}^{3t} - \mathrm{e}^{t}$

Example 2: solving another linear IVP via Laplace transform

Exemplary IVP: $\boxed{u'' + u = \sin(2t) \text{ on } [0, \infty) \text{ with } u(0) = 2,\ u'(0) = 1}$

$$\overset{\text{linearity, table}}{\rightsquigarrow} \quad \mathscr{L}(u'')(s) + \mathscr{L}u(s) = \tfrac{2}{s^2+4}$$

$$\overset{\text{derivation rule, ICs}}{\rightsquigarrow} \quad \big(s^2\mathscr{L}u(s) - 1 - 2s\big) + \mathscr{L}u(s) = \tfrac{2}{s^2+4}$$

$$\overset{\text{solve for } \mathscr{L}u(s)}{\rightsquigarrow} \quad \mathscr{L}u(s) = \tfrac{2}{(s^2+4)(s^2+1)} + \tfrac{2s+1}{s^2+1}$$

$$= \tfrac{\frac{2}{3}(s^2+4) - \frac{2}{3}(s^2+1)}{(s^2+4)(s^2+1)} + \tfrac{2s+1}{s^2+1} = \tfrac{2s}{s^2+1} + \tfrac{\frac{5}{3}}{s^2+1} - \tfrac{\frac{2}{3}}{s^2+4}$$

$\overset{\text{back trafo/table}}{\rightsquigarrow}$ solution of IVP: $u(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$

(Side remark: Both exemplary IVPs had ICs at 0. In order to apply this method with ICs at another point $t_0 \neq 0$, first implement a change of variables $t = t_0 + \tau$, $\widetilde{u}(\tau) = u(t_0 + \tau)$ to reach an IVP for $\widetilde{u}$ with ICs at 0.)

Transfer functions

For the scalar linear IVP with constant coefficients a_k and zero ICs

$$\sum_{k=0}^{m} a_k u^{(k)} = b \text{ on } [0,\infty) \quad \text{with } u(0) = \ldots = u^{(m-1)}(0) = 0\,,$$

Laplace transform and term rearrangement yield $\mathscr{L}u(s) = \frac{1}{\sum_{k=0}^{m} a_k s^k} \mathscr{L}b(s)$.
On the level of Laplace transforms one thus moves on from inhomogeneity b to solution u by multiplication with the transfer function $\frac{1}{\sum_{k=0}^{m} a_k s^k}$.

Analogous transfer functions on the level of Laplace transforms exist for ODEs $\sum_{k=0}^{m} a_k y^{(k)} = \sum_{k=0}^{\ell} c_k w^{(k)}$ (with constant a_k, c_k and zero ICs for y and w) and govern the transfer from input signal w to output signal y. This has many applications in theory/engineering of systems and control and in signal processing.

Example: For $y' + \lambda y = \lambda w$ (ODE of control loop element from Chapter 1) with $y(0) = 0$, find $\mathscr{L}y(s) = \frac{\lambda}{s+\lambda} \mathscr{L}w(s)$. Thus, the transfer function is $\frac{\lambda}{s+\lambda}$.

Chapter 4: Visualizing ODE solutions

In case of low order and few equations one can visualize ODE solutions by drawings in the plane $\mathbb{R}^2$. More precisely, this works out, in slightly different manners,

- for scalar first-order ODEs,
- for autonomous systems of two first-order ODEs,
- and for autonomous scalar second-order ODEs.

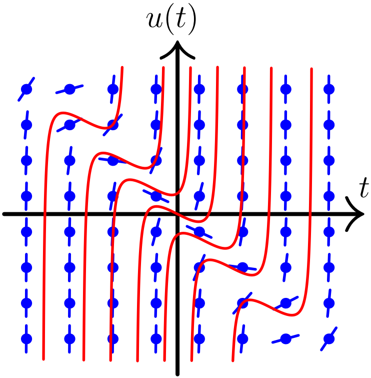
These cases are now discussed in more detail.

Scalar first-order ODEs and slope fields

A scalar first-order ODE

$$u' = f(t,u)$$

can be visualized via a slope field, which prescribes the slope $f(t,x)$ at each point $(t,x) \in \mathrm{D}_f \subset \mathbb{R}^2$. For solutions u, the derivative $u'(t)$ coincides with the prescribed slope $f(t,u(t))$ for all $t \in I$.



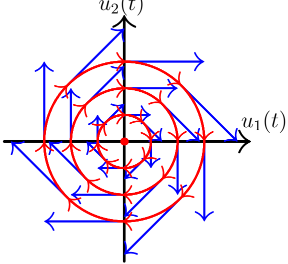
The slope field $f(t,x) = 2(t+x)^2 - \frac{1}{2}$ and some solutions of $u' = 2(t+u)^2 - \frac{1}{2}$.

Planar systems and trajectories

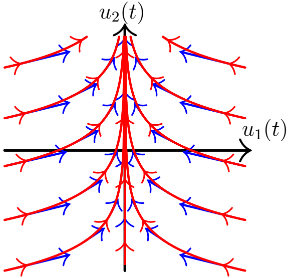
In case of an ODE system for $\mathbb{R}^2$-valued u (called a planar system)

$$u' = F(u)$$

one visualizes the vector field F by attaching to every point $x \in \mathrm{D}_F \subset \mathbb{R}^2$ the vector $F(x)$ in form of an arrow. For a solution u, the derivative $u'(t)$ equals $F(u(t))$ for all $t \in I$. Thus, the image of u stretches out along the prescribed vectors and is called a trajectory. (Its orientation is typically indicated by arrows.)



The vector field $F(x) = (x_2, -x_1)$ on $\mathbb{R}^2$ and some trajectories of the corresponding system $u_1' = u_2$, $u_2' = -u_1$.



The vector field $F(x) = (-\frac{x_1}{2}, 1)$ on $\mathbb{R}^2$ and some trajectories of the corresponding system $u_1' = -\frac{u_1}{2}$, $u_2' \equiv 1$.

Scalar second-order ODEs and phase space portraits

An autonomous scalar second-order ODE

$$u'' = f_0(u, u')$$

can be rewritten as a planar system $u' = v$, $v' = f_0(u, v)$. Its solutions (u, v) follow the vector field $F(x_1, x_2) = (x_2, f_0(x_1, x_2))$ and can be visualized as before. For instance, in case of the oscillation equation $u'' = -u$ one obtains once more the first drawing of the last slide — just with the axes marked as $u(t)$ and $v(t) = u'(t)$ rather than $u_1(t)$ and $u_2(t)$.

One calls such drawings — in case of both planar systems and scalar second-order ODEs — phase space portraits or phase space diagrams (where the phase space of an explicit order-m ODE system for $\mathbb{R}^n$-valued u is the joint target space $(\mathbb{R}^n)^m$ of the functions $(u, u', u'', \ldots, u^{(m-1)})$).

Observation: As long as solutions of IVPs are unique, trajectories in a phase space portrait must not touch or intersect each other.

Chapter 5: Long-time stability of solutions

In connection with ODEs, stability refers to continuous dependence of solutions on parameters, initial values, and/or boundary values. Up to a finite time horizon, this is usually satisfied (but the details are technical!).

This chapter directly deals with an infinite time horizon and the more intricate long-time stability in dependence on initial values, i.e. with the question whether arbitrarily small modifications of the initial values reflect merely in arbitrarily small perturbations of the solutions, even for $t \to \infty$.

In view of the reduction-to-first-order principle, the following discussion of such issues indeed focuses on the first-order case.

5.1 Ljapunov stability and equilibria

The central notion of the theory is:

Definition (stability notions)

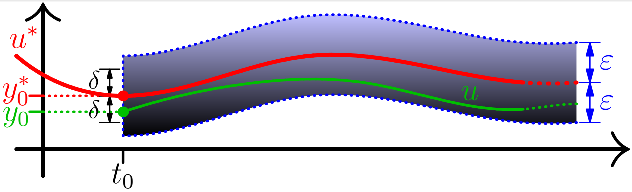
A solution $u^* \colon I \to \mathbb{K}^n$ *of an ODE system* $u' = f(t,u)$ *on an interval* I *of type* $[\alpha, \infty)$ *or* (α, ∞) *is called ...*

(1) *(Ljapunov) stable if, for each* $t_0 \in I$ *and each* $\varepsilon > 0$*, there exists some* $\delta > 0$ *such that, for initial values* $y_0^* := u^*(t_0)$ *and* $y_0 \in \mathbb{K}^n$*, it holds:*

$$|y_0 - y_0^*| < \delta \implies |u(t) - u^*(t)| < \varepsilon \text{ for all } t \geq t_0\,,$$

where u *uniquely solves* $u' = f(t,u)$ *on* $[t_0, \infty)$ *with* $u(t_0) = y_0$ *(existence and uniqueness of* u *in case* $|y_0 - y_0^*| < \delta$ *part of the requirement).*

Illustration in case $\mathbb{K}^n = \mathbb{R}$:



Uniform and asymptotic stability, instability

From the stability notion (1) of the previous slide one further derives:

Definition (stability notions; continued)

A solution u^* of $u' = f(t,u)$ on $I = [\alpha,\infty)$ or $I = (\alpha,\infty)$ is called ...

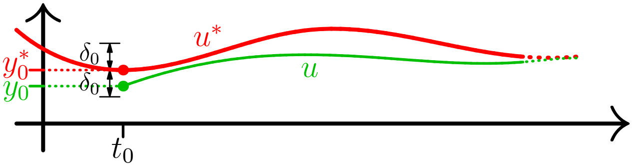
(2) *uniformly stable if it is stable and if, whenever an arbitrary $\varepsilon > 0$ is fixed, the implication in (1) holds for all $t_0 \in I$ with a single $\delta > 0$.*

(3) *asymptotically stable if it is stable and if, for each $t_0 \in I$, there exists some $\delta_0 > 0$ such that, for IVs y_0^*, y_0 and solution u as in (1), it holds:*

$$|y_0 - y_0^*| < \delta_0 \implies \lim_{t\to\infty} |u(t) - u^*(t)| = 0\,.$$

(4) *unstable if it is not stable.*

Asymptotic stability in case $\mathbb{K}^n = \mathbb{R}$:



Equilibria of autonomous systems

Specifically one is interested in stability of equilibria, as they occur in physical systems modeled by ODEs:

Definition (equilibria of autonomous ODE systems)

Whenever $x^* \in \mathrm{D}_F$ is a zero of a continuous vector field $F\colon \mathrm{D}_F \to \mathbb{K}^n$ on $\mathrm{D}_F \subset \mathbb{K}^n$, then $u^*\colon \mathbb{R} \to \mathbb{K}^n$ with $u^* :\equiv x^*$ is a constant solution of the autonomous ODE system $u' = F(u)$. In this situation one calls both x^* and u^* an equilibrium or a stationary point of $u' = F(u)$.

Observe in this regard:

- In phase space portraits, equilibria occur as single „non-moving" points.
- For equilibria of autonomous systems (specifically $u' = Au+b$ with A, b constant) there is no difference between stability and uniform stability. (Justification: One can pass from one t_0 to another by a suitable time shift.)

5.2 Equilibria of linear systems and linear stability

Basic principles (for equilibria of linear systems)

For $A \in \mathbb{K}^{n\times n}$ and $b \in \mathbb{K}^n$, there hold:

(1) *The equilibria of $u' = Au+b$ are precisely the solutions of the linear system of equations $Ax = -b$.*

(2) *The equilibria of $u' = Au$ form the vector subspace $\ker(A)$ in $\mathbb{K}^n$. In particular, the null vector is always an equilibrium of the homogeneous system $u' = Au$ and is called the null equilibrium.*

(3) *All equilibria of the inhomogeneous system $u' = Au+b$ share the stability properties of null for the homogeneous system $u' = Au$.*

Proofs: (1) and (2): clear by definition with $F(x) = Ax+b$ and $F(x) = Ax$. (3): „Translate“ stability of equilibrium u^* of $u' = Au+b$ in stability of 0 for $u' = Au$ essentially by correspondence

$$\begin{array}{c} u \text{ solution of } u' = Au+b \\ \text{with } |u(t)-u^*(t)| < \varepsilon \text{ for all } t \end{array} \iff \begin{array}{c} u-u^* \text{ solution of } u' = Au \\ \text{with } |(u-u^*)(t)-0| < \varepsilon \text{ for all } t \end{array}.$$

□

Linear stability

Theorem (on stability of equilibria of linear systems; w.l.o.g. homogeneous)

Denote by $\lambda_1, \ldots, \lambda_\ell \in \mathbb{C}$ all eigenvalues of $A \in \mathbb{K}^{n\times n}$. Then one has the following necessary and sufficient criteria for (in)stability:

(1) *If and only if $\mathrm{Re}(\lambda_i) < 0$ holds for all λ_i, the equilibria of $u' = Au$ are asymptotically stable.*

(2) *If and only if $\mathrm{Re}(\lambda_i) < 0$ or $\big\{\mathrm{Re}(\lambda_i) = 0,\ \text{g-mult}(\lambda_i) = \text{a-mult}(\lambda_i)\big\}$ holds for all λ_i, the equilibria of $u' = Au$ are stable.*

(3) *If and only if $\mathrm{Re}(\lambda_i) > 0$ or $\big\{\mathrm{Re}(\lambda_i) = 0,\ \text{g-mult}(\lambda_i) < \text{a-mult}(\lambda_i)\big\}$ holds for one λ_i at least, the equilibria of $u' = Au$ are unstable.*

Here, for the eigenvalues λ of A, we used the notations ...

- $\text{a-mult}(\lambda)$ for multiplicity of λ as zero of the characteristic polynomial of A,
- $\text{g-mult}(\lambda)$ for dimension of the λ-eigenspace of A.

Roughly, $\text{g-mult}(\lambda) = \text{a-mult}(\lambda)$ requires „sufficiently many“ λ-eigenvectors of A.

Example for linear stability analysis

Analyze ODE system $\boxed{u' = Au \text{ with } A := \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$ as follows:

Determine equilibria:

$Ax = 0 \rightsquigarrow$ linear system: $\begin{aligned} -2x_1 + x_3 &= 0 \\ x_4 &= 0 \\ -x_1 - 2x_3 &= 0 \end{aligned} \rightsquigarrow \ker(A) = \left\{ \begin{pmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{pmatrix} \middle| \, x_2 \in \mathbb{R} \right\}.$

Determine eigenvalues:

ch. polynomial: $\det \begin{pmatrix} \lambda+2 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 1 & 0 & \lambda+2 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} = \left[(\lambda+2)^2+1\right]\lambda^2 = (\lambda+2-\mathbf{i})(\lambda+2+\mathbf{i})\lambda^2$

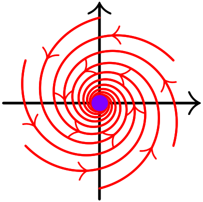
$\rightsquigarrow$ eigenvalues: $-2\pm\mathbf{i}$ with $\text{a-mult}(-2\pm\mathbf{i}) = \text{g-mult}(-2\pm\mathbf{i}) = 1$,
0 with $\text{a-mult}(0) = 2$, $\text{g-mult}(0) = 1$ (e.vectors seen above!).

Conclusion: $\text{Re}(0) = 0$, $\text{g-mult}(0) < \text{a-mult}(0) \rightsquigarrow$ all equilibria unstable!

Equilibria of planar model systems: phase space portraits

asympt. stable

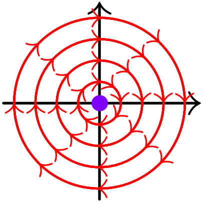
vortex point



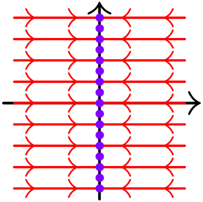
$u' = \begin{pmatrix} -1 & -4 \\ 4 & -1 \end{pmatrix} u$, e.values $-1 \pm 4\mathbf{i}$

stable, not asympt. stable

(on top: circulation point)



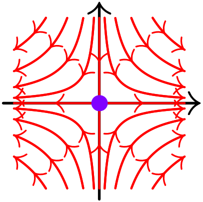
$u' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u$, e.values $\pm\mathbf{i}$



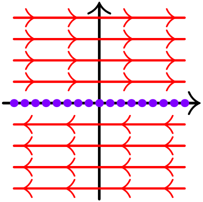
$u' = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} u$, e.values $0, -1$

unstable

(on top: saddle point)



$u' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u$, e.values $1, -1$



$u' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u$, e.value 0

Proof of the theorem on linear stability

On the proof: W.l.o.g. consider equilibrium 0, work with $\mathbb{K} = \mathbb{C}$ and $t_0 = 0$.

In view of 3.3 consider basis solutions ($\lambda \in \mathbb{C}$ eigenvalue, $v_0 \in \mathbb{C}^n$ step-s GEV)

$$u(t) = \mathrm{e}^{\lambda t} \sum_{j=0}^{s-1} \frac{1}{j!} t^j v_j$$

with $|u(0)| = |v_0| > 0$ arbitrarily small. Then $|u(t)| = \mathrm{e}^{\mathrm{Re}(\lambda)t} \big| \sum_{j=0}^{s-1} \frac{1}{j!} t^j v_j \big|$ yields:

$$\lim_{t\to\infty} |u(t)| = \begin{cases} 0 & \text{if } \mathrm{Re}(\lambda) < 0 & \rightsquigarrow \text{ asymtotically stable} \\ |v_0| & \text{if } \mathrm{Re}(\lambda) = 0,\ s = 1 & \rightsquigarrow \text{ stable, not asymtotically stable} \\ \infty & \text{otherwise} & \rightsquigarrow \text{ unstable} \end{cases}$$

From this deduce all criteria (where (2) and (3) and equivalent by negation).

Technical elaboration e.g. on (2) „$\Longrightarrow$“: Given FM W, from boundedness of basis solutions on $[0, \infty)$ get $M > 0$ such that $|W(t)x| \leq M|x|$ and $|W(0)^{-1}x| \leq M|x|$ for $x \in \mathbb{C}^n$, $t \geq 0$. For $\varepsilon > 0$, take $\delta := \frac{\varepsilon}{M^2}$. For solution $u(t) = W(t)W(0)^{-1}u(0)$ with $|u(0)| < \delta$, infer $|u(t)| \leq M^2|u(0)| < M^2\delta = \varepsilon \rightsquigarrow$ equilibrium 0 stable. □

5.3 On stability analysis for non-linear systems

Sometimes one can decide on stability of equilibria of non-linear systems by the following criteria, which resemble the linear case:

Theorem (linearization criteria for stability in non-linear systems)

For a zero x^* of $F\colon \mathrm{D}_F \to \mathbb{R}^n$ in the interior of $\mathrm{D}_F \subset \mathbb{R}^n$, assume that F is continuously differentiable in x^*. If $\lambda_1, \ldots, \lambda_\ell \in \mathbb{C}$ denote all eigenvalues of the Jacobi matrix $\mathrm{J}F(x^*) \in \mathbb{R}^{n\times n}$, the following criteria are valid:

(1) *If $\mathrm{Re}(\lambda_i) < 0$ holds for all λ_i, the equilibrium x^* of $u' = F(u)$ is asymptotically stable.*

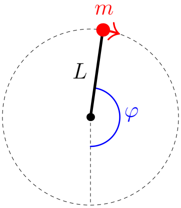
(2) *If $\mathrm{Re}(\lambda_i) > 0$ holds for one λ_i at least, the equilibrium x^* of $u' = F(u)$ is unstable.*

- These criteria are sufficient, but not necessary. If the largest real part is exactly 0, they do not help (since then higher-order effects, which are not reflected in the first derivative, may enter and may play a role).

Example: ODE of a simple physical pendulum

The ODE and equivalent ODE system of a simple physical pendulum are

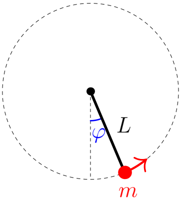
$$\varphi'' = -(g/L)\sin(\varphi) \qquad \text{and} \qquad \begin{pmatrix}\varphi' \\ v'\end{pmatrix} = \begin{pmatrix}(1/L)v \\ -g\sin(\varphi)\end{pmatrix}$$



(with displacement angle φ, velocity v, positive constants g, L; ODE considered already in Chapter 1).

The governing vector field

$$F(\varphi, v) := \begin{pmatrix}(1/L)v \\ -g\sin(\varphi)\end{pmatrix}$$



of the system has the equilibria $(k\pi, 0)$ with $k \in \mathbb{Z}$.

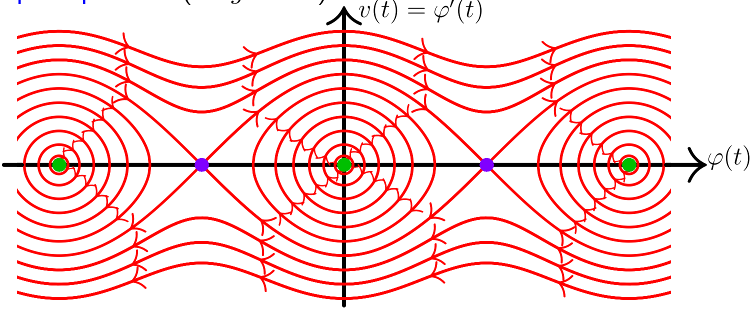
Here, even k correspond to the lower equilibrium position, odd k to the upper equilibrium position of the physical system.

Example: stability analysis for pendulum ODE

Now analyze equilibria $(k\pi, 0)$ at hand of $\mathrm{J}F(\varphi, v) = \begin{pmatrix} 0 & 1/L \\ -g\cos(\varphi) & 0 \end{pmatrix}$:

- k even („lower“ equilibria): $\mathrm{J}F(k\pi, 0) = \begin{pmatrix} 0 & 1/L \\ -g & 0 \end{pmatrix}$, e.values $\pm\mathbf{i}\sqrt{g/L}$ $\overset{\text{thm}}{\rightsquigarrow}$ stability unclear. But from picture: stable, not asymptotically stable.
- k odd („upper“ equilibria): $\mathrm{J}F(k\pi, 0) = \begin{pmatrix} 0 & 1/L \\ g & 0 \end{pmatrix}$, e.values $\pm\sqrt{g/L}$ $\overset{\text{thm, part (2)}}{\rightsquigarrow}$ unstable.

Phase space portrait (for $g{=}L{=}1$):



Proof of the criterion for asymptotic stability

Proof for part (1) of the theorem in case of symmetric matrix $A := \mathrm{J}F(x^*)$:

In this case, A has solely negative real eigenvalues and is invertible.

For simplicity, take $x^* = 0$ (otherwise analogous with subtraction of x^*).

Choose $M > 0$ with $|A^{-1}x| \le M|x|$, consider sufficiently small $\varepsilon > 0$ such that $|x| \le \varepsilon \Longrightarrow |F(x) - Ax| \le \frac{1}{2M}|x|$ holds (exploits $F(0) = 0$ and $\mathrm{J}F(0) = A$).

Now define $L\colon \mathbb{R}^n \to \mathbb{R}$ by $L(x) := -x \cdot A^{-1}x = -\sum_{i,j=1}^n x_i (A^{-1})_{ij} x_j$ (with symbol „$\cdot$“ for inner product), calculate $\nabla L(x) = -2A^{-1}x$ (uses symmetry of A).

Whenever solution u of $u' = F(u)$ satisfies $|u(t)| \le \varepsilon$, further deduce

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} L(u(t)) &= u'(t) \cdot \nabla L(u(t)) \overset{\text{ODE}}{=} F(u(t)) \cdot \nabla L(u(t)) = -2F(u(t)) \cdot A^{-1}u(t) \\ &\le -2Au(t) \cdot A^{-1}u(t) + 2|F(u(t)) - Au(t)|\,|A^{-1}u(t)| \\ &\le -2u(t) \cdot u(t) + \tfrac{1}{M}|u(t)|\, M|u(t)| = -|u(t)|^2 \le 0\,. \end{aligned}$$

Conclusion: $L(u(t))$ decreasing in t ($\rightsquigarrow$ Ljapunov function, energy interpretation).

Proof of the criterion for asymptotic stability (continued)

Here, 0 is a strict minimum point of L (thanks to choice of L and solely negative eigenvalues of A). Choose $0 < \delta < \varepsilon$ such that $|x| < \delta\,, |y| = \varepsilon \Longrightarrow L(x) < L(y)$. Then solution u with $|u(0)| < \delta$ cannot satisfy $|u(t)| = \varepsilon$ for $t > 0$ (otherwise get contradiction $L(u(0)) < L(u(t))$). Thus $|u(t)| < \varepsilon$ for $t \geq 0$. Equilibrium 0 is stable.

Next show $\lim_{t\to\infty} u(t) = 0$ (for some ε, δ as before, u solution with $|u(0)| < \delta$): For simplicity assume the limit exists. In case $x_0 := \lim_{t\to\infty} u(t) \neq 0$ deduce from $\limsup_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} L(u(t)) \leq -|x_0|^2 < 0$ (by calculation of previous slide) that $L(x_0) = \lim_{t\to\infty} L(u(t)) = -\infty$ must hold. Contradiction! The sole remaining possibility is $\lim_{t\to\infty} u(t) = 0$. Equilibrium 0 is even asymptotically stable. □

Remarks on the proof:

- In the preceding, a few technical details (in particular reasoning for existence and uniqueness of solutions u on all of $[0, \infty)$) have been suppressed.
- The generalization for non-symmetric matrices $\mathrm{J}F(x^*)$ and proofs for part (2) of the theorem are rather more difficult and are omitted here.

Chapter 6: Existence and uniqueness of solutions

Existence and uniqueness of the solution in initial value problems have been used decisively in Chapters 3 and 5. In the sequel this is underpinned with the precise mathematical statements.

6.1 Local existence and uniqueness

Denote by $\overline{\mathrm{B}}^n_\varepsilon(y_0) := \{x \in \mathbb{R}^n \,|\, |x-y_0| \le \varepsilon\}$ the closed ball in $\mathbb{R}^n$ with center $y_0 \in \mathbb{R}^n$ and radius $\varepsilon > 0$. The main theorem of this chapter is:

Picard-Lindelöf Theorem (local existence and uniqueness for IVPs)

If a continuous $f\colon \mathrm{D}_f \to \mathbb{R}^n$ *with* $[t_0-\varepsilon, t_0+\varepsilon] \times \overline{\mathrm{B}}^n_\varepsilon(y_0) \subset \mathrm{D}_f \subset \mathbb{R} \times \mathbb{R}^n$ *for* $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^n$, $\varepsilon > 0$, *satisfies the partial Lipschitz condition (pLC)*

$$|f(t,\widetilde{x}) - f(t,x)| \le L|\widetilde{x} - x| \qquad \text{for all } t \in [t_0-\varepsilon, t_0+\varepsilon]\,,\ x,\widetilde{x} \in \overline{\mathrm{B}}^n_\varepsilon(y_0)$$

with a constant $L \in [0,\infty)$, *then the IVP*

$$u' = f(t,u) \text{ on } [t_0-\delta, t_0+\delta]\,, \qquad u(t_0) = y_0$$

is uniquely solvable for each sufficiently small $\delta > 0$.

- also valid with $\mathbb{C}^n$ in place of $\mathbb{R}^n$, as one may here identify $\mathbb{C}^n = \mathbb{R}^{2n}$.

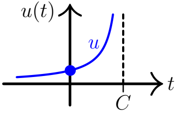
Complements to the Picard-Lindelöf theorem

- decisive sufficient criterion for pLC: If $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$ are continuous on open D_f, a pLC valid on all cylinders $[t_0-\varepsilon, t_0+\varepsilon] \times \overline{\mathrm{B}}_\varepsilon^n(y_0) \subset \mathrm{D}_f$ (with L dependent on t_0, y_0, ε), and the theorem applies.

  (Justification: For $L_i := \max_{[t_0-\varepsilon,t_0+\varepsilon]\times\overline{\mathrm{B}}_\varepsilon^n(x_0)} \left|\frac{\partial f}{\partial x_i}\right|$ and $L := \sum_{i=1}^n L_i$, find $|f(t,\widetilde{x})-f(t,x)| \overset{\text{FTC}}{=} \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} f(t,x+s(\widetilde{x}-x))\,\mathrm{d}s\right| \le \sum_{i=1}^n L_i|\widetilde{x}_i-x_i| \le L|\widetilde{x}-x|$.)

- In general existence applies only locally, i.e. for small δ:

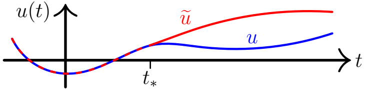
  For instance, $u(t) = \frac{-1}{t-C}$ with $C > 0$ solves the scalar ODE $u' = u^2$ on $(-\infty, C)$ with IC $u(0) = \frac{1}{C}$, but cannot be extended at $t = C$. Hence, in this case existence applies only for $\delta < C$, but not for $\delta \ge C$.



Complements to the Picard-Lindelöf theorem (continued)

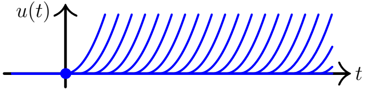
- Uniqueness is automatically global, i.e. valid on arbitrary intervals I (provided a pLC holds near each $t_0 \in I$ and $y_0 \in \mathbb{R}^n$).

  (Justification: For different solutions $u, \widetilde{u}$ of $u' = f(t,u)$ with $\widetilde{u}(t_0) = u(t_0)$ find (by continuity) largest/smallest t_* in interior of I such that $\widetilde{u}(t_*) = u(t_*)$. By Picard-Lindelöf get $\widetilde{u} = u$ on $[t_* - \delta, t_* + \delta]$, which contradicts choice of t_*.)



- In general no uniqueness without pLC!

  For instance, the scalar IVP for $u' = 2\sqrt{|u|}$ with IC $u(0) = 0$ is solved for every $C \in [0, \infty]$ by $u_C(t) = \begin{cases} 0 & \text{for } t < C \\ (t-C)^2 & \text{for } t \geq C \end{cases}$. Hence, this IVP has infinitely many solutions.



Proof of the Picard-Lindelöf theorem

The proof of the Picard-Lindelöf theorem exploits this analysis result:

Banach fixed point theorem

If A is closed subset of a complete normed space and $T\colon A \to A$ is a strict contraction, i.e. $\|T(\widetilde{u}) - T(u)\| \le \kappa \|\widetilde{u} - u\|$ for $u, \widetilde{u} \in A$ with a constant $\kappa \in [0,1)$, then there is one and just one $u \in A$ such that $T(u) = u$.

One calls $u \in A$ with $T(u) = u$ a fixed point of T and $T(u) = u$ the corresponding fixed point equation.

Proof of the Picard-Lindelöf theorem: Set $I := [t_0 - \delta, t_0 + \delta]$ (for sufficiently small $\delta \le \varepsilon$; soon more on this) and record:

- $\mathrm{C}^0(I, \mathbb{R}^n) = \{u\colon I \to \mathbb{R}^n \,|\, u \text{ continuous}\}$ is a complete normed space with norm $\|u\|_\infty := \max_{s \in I} |u(s)|$.
- $A := \mathrm{C}^0\left(I, \overline{\mathrm{B}}^n_\varepsilon(y_0)\right)$ is a closed subset of $\mathrm{C}^0(I, \mathbb{R}^n)$.

Proof of the Picard-Lindelöf theorem (continued)

Reformulation of solution property as fixed point equation (for $u \in A$):

$$u' = f(t,u) \text{ on } I\,,\; u(t_0) = y_0 \overset{\text{FTC}}{\iff} u(t) = y_0 + \int_{t_0}^{t} f(s,u(s))\,\mathrm{d}s \text{ for } t \in I$$

$$\iff T(u) = u$$

with $T\colon A \to \mathrm{C}^0(I,\mathbb{R}^n)$ defined by $T(u)(t) := y_0 + \int_{t_0}^{t} f(s,u(s))\,\mathrm{d}s$.

Now check assumptions of fixed point theorem for this T:

(1) Show $T(u) \in A$ for $u \in A$ (in order to ensure $T\colon A \to A$): For $M := \max_{s\in[t_0-\varepsilon,t_0+\varepsilon],x\in\overline{\mathrm{B}}^n_\varepsilon(y_0)} |f(s,x)|$ and $t \in I$, find

$$|T(u)(t) - y_0| = \left| \int_{t_0}^{t} f(s,u(s))\,\mathrm{d}s \right| \le |t{-}t_0|\, M \le \delta M \le \varepsilon$$

<u>provided that $\delta \le \frac{\varepsilon}{M}$</u>. Infer $T(u)(t) \in \overline{\mathrm{B}}^n_\varepsilon(y_0)$ for $t \in I$ and $T(u) \in A$.

Proof of the Picard-Lindelöf theorem (continued)

(2) For checking the strict contraction property

$$\|T(\widetilde{u}) - T(u)\|_\infty \leq \tfrac{1}{2}\,\|\widetilde{u} - u\|_\infty \qquad \text{for all } u, \widetilde{u} \in A$$

of T, first estimate, for $t \in I$,

$$\begin{aligned} |T(\widetilde{u})(t) - T(u)(t)| &= \left| \int_{t_0}^{t} \left[f(s, \widetilde{u}(s)) - f(s, u(s)) \right] \mathrm{d}s \right| \\ &\leq |t{-}t_0|\, \|f(s, \widetilde{u}(s)) - f(s, u(s))\|_\infty \\ &\overset{\text{pLC}}{\leq} \delta L\, \|\widetilde{u}{-}u\|_\infty \leq \tfrac{1}{2}\, \|\widetilde{u}{-}u\|_\infty \end{aligned}$$

provided that $\delta \leq \frac{1}{2L}$. Then take $\max_{t \in I}(\,\cdot\,)$ to arrive at the claim.

Conclusion: For $\delta \leq \min\left\{\varepsilon, \frac{\varepsilon}{M}, \frac{1}{2L}\right\}$, all assumptions of the fixed point theorem satisfied! Deduce unique solvability of $T(u) = u$ and of the IVP.

Proof of the Picard-Lindelöf theorem (final technical detail)

Finalize proof by technical reasoning, which extends uniqueness from $A = \mathrm{C}^0\left(I, \overline{\mathrm{B}}^n_\varepsilon(y_0)\right)$ to all solutions on $I = [t_0{-}\delta, t_0{+}\delta]$:

Since ε and δ may be slightly decreased, the preceding yields a solution u such that $|u(t){-}y_0| < \varepsilon$ for all $t \in (t_0{-}\delta, t_0{+}\delta)$.

If there exists a solution $\widetilde{u}$ on I such that $\widetilde{u} \notin \mathrm{C}^0\left(I, \overline{\mathrm{B}}^n_\varepsilon(y_0)\right)$, then ...

- choose (by continuity) $t_* \in (t_0{-}\delta, t_0{+}\delta)$ with $|\widetilde{u}(t_*){-}y_0| = \varepsilon$ such that $\delta_* := |t_*{-}t_0| < \delta$ is smallest possible,
- infer $u, \widetilde{u} \in A_* := \mathrm{C}^0\left([t_0{-}\delta_*, t_0{+}\delta_*], \overline{\mathrm{B}}^n_\varepsilon(y_0)\right)$ with $\widetilde{u}(t_*) \neq u(t_*)$ and thus arrive at a contradiction to already-proven uniqueness in A_*.

So, uniqueness holds even among all solutions. The proof is complete. □

6.2 Global existence

In good cases even global existence on arbitrarily given intervals I is valid:

Theorem (global existence under global pLC)

If a continuous $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the global-in-x pLC

$$|f(t,\widetilde{x}) - f(t,x)| \leq \ell(t)\,|\widetilde{x} - x| \qquad \text{for all } t \in I\,,\ x, \widetilde{x} \in \mathbb{R}^n$$

with continuous $\ell\colon I \to [0,\infty)$, then, for all $t_0 \in I$ and $y_0 \in \mathbb{R}^n$, the IVP

$$u' = f(t,u) \text{ on } I\,, \qquad u(t_0) = y_0$$

is always uniquely solvable.

The important case are linear systems $u' = Au+b$ with $A \in \mathrm{C}^0(I, \mathbb{R}^{n\times n})$, $b \in \mathrm{C}^0(I, \mathbb{R}^n)$. Higher-order cases and cases with $\mathbb{C}^n$ in place of $\mathbb{R}^n$ can be reduced as usual. The central existence claim of Section 3.1 is covered.

On the proof of the global existence theorem

Proof: W.l.o.g. consider only case $I = \mathbb{R}$ with $|f(t,\widetilde{x}) - f(t,x)| \le L|\widetilde{x} - x|$ and $|f(t,x)| \le L(1+|x|)$ for all $t \in \mathbb{R}$, $x, \widetilde{x} \in \mathbb{R}^n$ and some $L \in [1,\infty)$.

Use Picard-Lindelöf to subsequently extend solution u of IVP with $\delta > 0$:

(1) $u\colon [t_0-\delta, t_0+\delta] \to \overline{\mathrm{B}}^n_{\varepsilon_1}(y_0)$ with $\varepsilon_1 := 1+|y_0|$,

(2) $u\colon [t_0, t_0+2\delta] \to \overline{\mathrm{B}}^n_{\varepsilon_2}(u(t_0+\delta))$ with $\varepsilon_2 := 1+|u(t_0+\delta)|$,

(3) $u\colon [t_0+\delta, t_0+3\delta] \to \overline{\mathrm{B}}^n_{\varepsilon_3}(u(t_0+2\delta))$ with $\varepsilon_3 := 1+|u(t_0+2\delta)|$,

and so on.

Here, in ith step of construction exploit, for $x \in \overline{\mathrm{B}}^n_{\varepsilon_i}(u(t_0+(i-1)\delta))$, the bound $|f(t,x)| \le L\big(1+|u(t_0+(i-1)\delta)|+\varepsilon_i\big) = 2L\varepsilon_i$, in order to achieve extension step with $\min\big\{\varepsilon_i, \frac{\varepsilon_i}{2L\varepsilon_i}, \frac{1}{2L}\big\} = \frac{1}{2L} =: \delta$ by proof of Section 6.1.

In conclusion determine u on $[t_0-\delta, \infty)$. In same way treat $(-\infty, t_0+\delta]$. □

Outlook: Variational principles for ODEs

In several cases, one can motivate and derive ODEs from minimization problems for an unknown function and then speaks of variational principles. In the sequel a brief and basic introduction to this theory is given.

Variational integrals and minimizers

Consider $t_1 < t_2$ in $\mathbb{R}$, $n \in \mathbb{N}$, and a given continuous structure function $\mathcal{L}\colon [t_1,t_2]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, called Lagrange function or simply integrand.

The interest is then in minimization of the variational integral

$$\boxed{\mathcal{I}[u] := \int_{t_1}^{t_2} \mathcal{L}(t,u(t),u'(t))\,\mathrm{d}t}$$

among all functions $u\colon [t_1,t_2] \to \mathbb{R}^n$ with BCs $u(t_1)=y_1$, $u(t_2)=y_2$.

Definition (minimizers of variational integrals)

A function $u \in \mathrm{C}^1([t_1,t_2],\mathbb{R}^n)$ is called a minimizer of $\mathcal{I}$, if there holds

$$\mathcal{I}[u] \le \mathcal{I}[\widetilde{u}] \qquad \begin{array}{c}\textit{for all } \widetilde{u} \in \mathrm{C}^1([t_1,t_2],\mathbb{R}^n)\\ \textit{with } \widetilde{u}(t_1)=u(t_1)\,,\ \widetilde{u}(t_2)=u(t_2)\,.\end{array}$$

The Euler-Lagrange equation

In order to state the central connection to ODEs, the integrand of

$$\mathcal{I}[u] = \int_{t_1}^{t_2} \mathcal{L}(t, u(t), u'(t))\,\mathrm{d}t$$

is regarded as a function $\mathcal{L}(t, x, v)$ of $t \in [t_1, t_2]$, $x \in \mathbb{R}^n$, and $v \in \mathbb{R}^n$:

Theorem (Euler-Lagrange equation)

Whenever $\mathcal{L}$, $\nabla_x \mathcal{L}$ are continuous and $\nabla_v \mathcal{L}$ is even C^1 on $[t_1, t_2] \times \mathbb{R}^n \times \mathbb{R}^n$, every minimizer $u \in \mathrm{C}^2([t_1, t_2], \mathbb{R}^n)$ of $\mathcal{I}$ satisfies the second-order ODE

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\big[\nabla_v \mathcal{L}(t, u(t), u'(t))\big] = \nabla_x \mathcal{L}(t, u(t), u'(t))} \qquad \text{for } t \in [t_1, t_2]\,.$$

- This is an analog of the analysis criteria $f'(x) = 0$ and $\nabla f(x) = 0$ for minimum points — but now for minimization among functions and thus with an ODE instead of simply an equation or a system of equations.
- For a variational integral $\mathcal{I}$ of arbitrary (rather than first) order $m \in \mathbb{N}$, there is an analogous ODE of order $2m$.

Derivation/proof of the Euler-Lagrange equation

Proof of theorem: For $\varphi \in \mathrm{C}^1([t_1,t_2],\mathbb{R}^n)$ with $\varphi(t_1)=\varphi(t_2)=0$, it is

$$\mathcal{I}[u] \le \mathcal{I}[u+s\varphi] \qquad \text{for all } s \in \mathbb{R}\,.$$

The necessary criterion of minimum points (applied to $s \mapsto \mathcal{I}[u+s\varphi]$) gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathcal{I}[u+s\varphi] \overset{\substack{\text{thm differ.}\\ \text{parameter}}}{=} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathcal{L}(t,u(t){+}s\varphi(t),u'(t){+}s\varphi'(t))\,\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\Big[\nabla_x\mathcal{L}(t,u(t),u'(t))\cdot\varphi(t) + \nabla_v\mathcal{L}(t,u(t),u'(t))\cdot\varphi'(t)\Big]\,\mathrm{d}t$$

$$\overset{\text{ibp}}{=} \int_{t_1}^{t_2}\Big[\nabla_x\mathcal{L}(t,u(t),u'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\big[\nabla_v\mathcal{L}(t,u(t),u'(t))\big]\Big]\cdot\varphi(t)\,\mathrm{d}t\,.$$

Since this holds for <u>all</u> $\varphi \in \mathrm{C}^1([t_1,t_2],\mathbb{R}^n)$ with $\varphi(t_1)=\varphi(t_2)=0$, the fundamental lemma of the calculus of variations (no details on this) implies

$$0 = \nabla_x\mathcal{L}(t,u(t),u'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\big[\nabla_v\mathcal{L}(t,u(t),u'(t))\big] \qquad \text{for } t \in [t_1,t_2]\,.$$

The claim then follows by rearranging terms. □

Application 1: variational principles for equations of motion

The motion of a (point-like) particle of mass $m > 0$ under influence of a potential $V\colon \mathbb{R}^3 \to \mathbb{R}$ is governed by the variational integral

$$S[\vec{x}] := \int_{t_1}^{t_2} \left[\tfrac{1}{2}m|\vec{x}\,'(t)|^2 - V(\vec{x}(t))\right] \mathrm{d}t \qquad \text{for } \vec{x}\colon [t_1,t_2] \to \mathbb{R}^3\,,$$

known as the action functional. The corresponding Lagrange function is

$$\mathcal{L}(t,\vec{x},\vec{v}) = \tfrac{1}{2}m|\vec{v}|^2 - V(\vec{x}) = \mathrm{E}_{\mathrm{kin}}(t,\vec{x},\vec{v}) - \mathrm{E}_{\mathrm{pot}}(t,\vec{x},\vec{v})\,.$$

One computes $\nabla_{\vec{x}}\mathcal{L}(t,\vec{x},\vec{v}) = -\nabla V(\vec{x})$ and $\nabla_{\vec{v}}\mathcal{L}(t,\vec{x},\vec{v}) = m\vec{v}$ and gets as Euler-Lagrange equation the general equation of motion

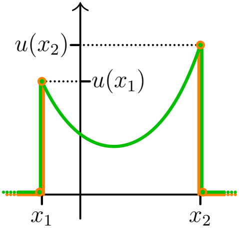
$$m\vec{x}\,'' = -\nabla V\big(\vec{x}\big)\,.$$

Read off: Acceleration occurs in direction of steepest descent of V.

Application 2: the hanging-chain variational principle

Describe freely hanging portion of rope/chain of constant mass density $\mu > 0$ as graph of $u\colon [x_1, x_2] \to \mathbb{R}$ with $x_1 < x_2$ in $\mathbb{R}$. The potential energy of the portion

$$E[u] := \mu g \int_{x_1}^{x_2} u(x)\sqrt{1+(u'(x))^2}\,\mathrm{d}x$$



(with gravity acceleration $g > 0$) is a variational integral in dimension $n = 1$. Minimizers of the type illustrated occur

(1) if one additionally fixes the length of the rope,

or

(2) if a longer rope is supported and redirected as in the picture, but the length of the hanging portion stays smaller than the sum $u(x_1)+u(x_2)$ of BVs (so the hanging portion cannot „drop down“ for physical reasons).

The standard situation (1) requires further theory for treating the additional constraint and is beyond our scope here. However, situation (2) is similar in principle and is covered by our computations to follow:

Application 2 (continued): the hanging-chain ODE

As the factor $\mu g > 0$ does not affect minimization, we are left with

$$\int_{x_1}^{x_2} u(x)\sqrt{1+(u'(x))^2}\,\mathrm{d}x$$

with corresponding Lagrange function $\mathcal{L}(x,y,s) = y\sqrt{1+s^2}$. We first compute‡ $\frac{\partial\mathcal{L}}{\partial y}(x,y,s) = \sqrt{1+s^2}$ and $\frac{\partial\mathcal{L}}{\partial s}(x,y,s) = \frac{sy}{\sqrt{1+s^2}}$ and then find the Euler-Lagrange equation

$$\left(\frac{u'u}{\sqrt{1+(u')^2}}\right)' = \sqrt{1+(u')^2}$$

or equivalently (after applying derivation rules and some rearranging!) the scalar hanging-chain ODE

$$u''u = 1+(u')^2\,.$$

‡Here, in the scalar case, the gradients ∇_y, ∇_s reduce to simple derivatives $\frac{\partial}{\partial y}, \frac{\partial}{\partial s}$.

Application 2 (continued): solving the hanging-chain ODE

For solving the ODE

$$u''u = 1+(u')^2$$

one has options. For instance, one may proceed as follows:

- first approach: follow last point on slide 36 to transform to scalar ODE $s' = \frac{1+s^2}{sy}$ for function s of variable y; via separation of variables deduce $s(y) = \pm\sqrt{C^2y^2-1}$ with $0 \neq C \in \mathbb{R}$; transform back via $(u^{-1})' = \frac{1}{s}$ to arrive at $u(x) = \pm\frac{1}{C}\cosh(C(x-x_0))$ with $x_0 \in \mathbb{R}$.
- second approach: compute $\left(\frac{u}{\sqrt{1+(u')^2}}\right)' = u'\frac{1+(u')^2-u''u}{(1+(u')^2)^{3/2}} \equiv 0$; deduce $\frac{u}{\sqrt{1+(u')^2}} = \frac{1}{C}$ with $0 \neq C \in \mathbb{R}$; via separation of variables infer once again $u(x) = \pm\frac{1}{C}\cosh(C(x-x_0))$ with $x_0 \in \mathbb{R}$.

In any case, the solutions $u(x) = \pm\frac{1}{C}\cosh(C(x-x_0))$ are shifted and scaled version of $\cosh$. Their graphs are known as catenaries.

Finally, the constants x_0 and C are determined from the BCs.

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