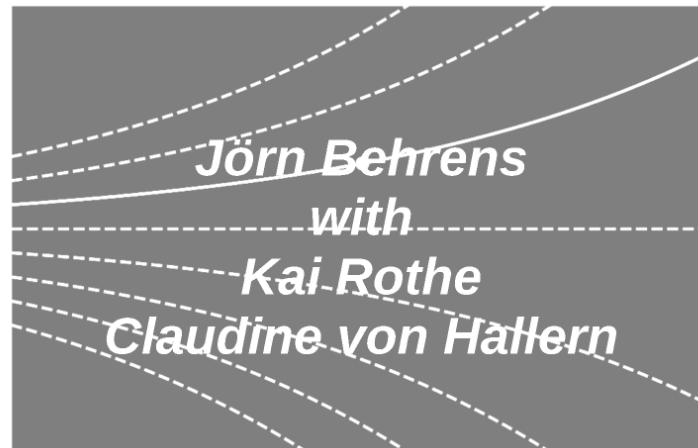


Differential Equations I



Autonomous Systems and Stability

Chapter 6.14

Recapitulation

Definition: (Dynamical System)
Consider the mapping

$$F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \quad \text{und} \quad x: \mathbb{R} \rightarrow \mathbb{R}^n,$$

x differentiable. The system of differential equations

$$\dot{x} = F(x, t),$$

with $x(t) = (x_1(t), \dots, x_n(t))^T$ and $F(x, t) = (F_1(x, t), \dots, F_n(x, t))^T$, is called **dynamical System**.

The space of solution curves $x(t)$ is called **phase space** and the solution curves **phase curves**.

Remark: (System of first Order)

In analogy to the linear case, an ODE of n^{th} order can be reformulated as a system of n equations of first order:

- Let: $y^{(n)} = f(y, y', y'', \dots, y^{(n-1)}, t)$.
- Introduce: $x_1(t) = y(t)$, $x_2(t) = y'(t)$, \dots , $x_n(t) = y^{(n-1)}(t)$.
- The dynamical system $\dot{x} = F(x, t)$ with

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(x_1, x_2, \dots, x_n, t) \end{pmatrix} =: F(x_1, \dots, x_n, t)$$

is equivalent to the ODE of n^{th} order above.

Definition: (Autonomous System)

If the mapping F of the dynamical system does not depend on t i.e.,

$$\dot{x} = F(x)$$

with $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, then the system is called **autonomous system**. **1**

Definition: (Dynamical System)

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$$\mathbf{F} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \quad \text{und} \quad \mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^n,$$

\mathbf{x} differentiable. The system of differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t),$$

with $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^{\top}$ and $\mathbf{F}(\mathbf{x}, t) = (F_1(\mathbf{x}, t), \dots, F_n(\mathbf{x}, t))^{\top}$, is called **dynamical System**.

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Definition: (Autonomous System)

If the mapping \mathbf{F} of the dynamical system does not depend on t i.e.,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

with $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, then the system is called **autonomous system**.

1

Stability of Linear Autonomous Systems

Definition: (Equilibrium State)
 If $x_0 \in \mathbb{R}^n$ with $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an autonomous system, then values $x_0 \in \mathbb{R}^n$ with the property $F(x_0) = 0$ are called **equilibrium or equilibrium state** of the system.
 These points are also called **critical, stationary or regular points**.

Fundamental Question (Stability)
 Does a phase curve, starting close to an equilibrium x_0 , remain in its vicinity?

Remark: (Linear Autonomous System)

- Consider the autonomous system $\dot{x} = F(x)$.
- Let $A \in \mathbb{R}^{n \times n}$ be a matrix and set $F = Ax$.
- Then $x_0 = 0$ is an equilibrium.
- If A has the n pairwise distinct eigenvalues λ_i , then the general solution of the autonomous system is

$$x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n,$$

with v_i the eigenvectors corresponding to λ_i .

2

Proposition: (Stability of Linear Autonomous System)

The equilibrium x_0 of a linear autonomous system $\dot{x} = F(x) = Ax$ is

- asymptotically stable, if all eigenvalues of A have negative real parts.
- stable, if no eigenvalue of A has positive real part and for eigenvalue with real part zero an even geometric = algebraic multiplicity.
- unstable, if an eigenvalue of A has positive real part or there is an eigenvalue with real part zero and geometric < algebraic multiplicity.

Definition: (Properties of Equilibria)

Let x_0 be equilibrium of the autonomous system $\dot{x} = F(x)$. Then x_0 is called

- attractive**, if solutions $x(t)$, starting close to x_0 , converge towards the equilibrium: $\exists \delta > 0, \exists \epsilon > 0, \forall t \geq 0, \forall x(0) = x_0 + \Delta x(0)$ with $|\Delta x(0)| < \delta$.
- stable**, if solutions $x(t)$, starting close to x_0 , remain in its vicinity: $\forall \epsilon > 0, \exists \delta > 0$ with $|\Delta x(0)| < \delta \Rightarrow |\Delta x(t)| < \epsilon, \forall t \geq 0$.
- asymptotically stable**, if attractive and stable.
- repulsive**, if there are solutions that - although started close to x_0 - deviate away from equilibrium: $\exists \epsilon > 0$ and $t_0 > 0, \forall \delta > 0$ with $|\Delta x(0)| < \delta \Rightarrow |\Delta x(t)| > \epsilon, \exists t \geq t_0$.

Definition: (Equilibrium State)

If $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ with $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an autonomous system, then values $\mathbf{x}_0 \in \mathbb{R}^n$ with the property

$$\mathbf{F}(\mathbf{x}_0) = \mathbf{0}$$

are called **equilibrium** or **equilibrium state** of the system.

These points are also called *critical*, *stationary* or *singular* points.

Remark: (Equilibrium)

Obviously $\mathbf{x}(t) = \mathbf{x}_0$ is a constant, time-independent solution of the autonomous system, for which the system **rests in equilibrium**.

Fundamental Question: (Stability)

Does a phase curve, starting close to an equilibrium \mathbf{x}_0 , remain in its vicinity?

Remark: (Linear Autonomous System)

- Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$.
- Let $A \in \mathbb{R}^{n \times n}$ be a matrix and set $\mathbf{F} = A$.
- Then $\mathbf{x}_0 = \mathbf{0}$ is an equilibrium.
- If A has the n pair-wise distinct eigenvalues λ_k , then the general solution of the autonomous system is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{e}_1 + \cdots + c_n e^{\lambda_n t} \mathbf{e}_n,$$

with \mathbf{e}_k the eigenvectors corresponding to λ_k .



Definition: (Properties of Equilibria)

Let \mathbf{x}_0 be equilibrium of the autonomous system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$. Then \mathbf{x}_0 is called

1. **attractive**, if solutions $\mathbf{x}(t)$, starting close to \mathbf{x}_0 converge towards the equilibrium:

$$\exists \delta > 0 : \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_0, \quad \forall \mathbf{x}(0) \text{ with } |\mathbf{x}(0) - \mathbf{x}_0| < \delta,$$

2. **stable**, if solutions $\mathbf{x}(t)$, starting close to \mathbf{x}_0 remain in its vicinity:

$$\forall \epsilon > 0 \exists \delta > 0 \text{ with } |\mathbf{x}(0) - \mathbf{x}_0| < \delta \Rightarrow |\mathbf{x}(t) - \mathbf{x}_0| < \epsilon \quad \forall t > 0,$$

3. **asymptotically stable**, if attractive and stable,

4. **unstable**, if there are solutions that – although started close to \mathbf{x}_0 – deviate away from equilibrium:

$$\exists \epsilon > 0 \text{ and } t_1 > 0 : \forall \delta > 0 \text{ with } |\mathbf{x}(0) - \mathbf{x}_0| < \delta \Rightarrow |\mathbf{x}(t) - \mathbf{x}_0| > \epsilon, \quad t \geq t_1.$$

Proposition: (Stability of Linear Autonomous Systems)

The equilibrium \mathbf{x}_0 of a linear autonomous system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = A\mathbf{x}$ is

1. asymptotically stable, if all eigenvalues of A have negative real parts,
2. stable, if no eigenvalue of A has positive real part and for eigenvalues with real part zero we have: geometric = algebraic multiplicity.
3. unstable, if an eigenvalue of A has positive real part or there is an eigenvalue with real part zero and geometric < algebraic multiplicity.

Stability of non-linear autonomous Systems

Motivation

- Most interesting problems are non-linear.
- Since we study the behavior in a neighborhood of an equilibrium, we may linearize.
- Linear Approximation (Taylor-Expansion):

$$\dot{x} = F(x_0) + F'(x_0)(x - x_0),$$

with F' matrix derivative of F .

- If F sufficiently smooth, we see

$$F(x) = F(x_0) + F'(x_0)(x - x_0).$$

- If x_0 is an equilibrium, we see

$$F(x) = F'(x_0)(x - x_0).$$

Proposition: (Stability of Non-Linear Autonomous Systems)

The equilibrium x_0 of a non-linear autonomous System $\dot{x} = F(x)$ is

1. asymptotically stable, if all eigenvalues of the derivative matrix $F'(x_0)$ have negative real parts.
2. unstable, if at least one eigenvalue of $F'(x_0)$ has positive real part. **3**

Motivation:

- Most interesting problems are non-linear.
- Since we study the behavior in a neighborhood of an equilibrium, we may *linearize*.
- Linear Approximation (Taylor-Expansion):

$$\mathbf{Lx} = \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0),$$

with \mathbf{F}' matrix derivative of \mathbf{F} .

- If \mathbf{F} sufficiently smooth, we see

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

- If \mathbf{x}_0 is an equilibrium, we see

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

Proposition: (Stability of Non-Linear Autonomous Systems)

The equilibrium \mathbf{x}_0 of a non-linear autonomous System $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is

1. *asymptotically stable*, if all eigenvalues of the derivative matrix $\mathbf{F}'(\mathbf{x}_0)$ have negative real parts,
2. *unstable*, if at least one eigenvalue of $\mathbf{F}'(\mathbf{x}_0)$ has positive real part.



Stability of Linear Autonomous Systems

Definition 1.1 (Linear Autonomous System). A linear autonomous system is a system of the form $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$ is a constant matrix and $x \in \mathbb{R}^n$ is the state vector.

Theorem 1.2 (Stability of Linear Autonomous Systems). The linear autonomous system $\dot{x} = Ax$ is asymptotically stable if and only if all eigenvalues of A have a negative real part.

Example 1.3 (Stability of a Linear Autonomous System). Consider the linear autonomous system $\dot{x} = Ax$ with $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$, both of which have negative real parts. Therefore, the system is asymptotically stable.

Recapitulation

Definition 1.1 (Linear Autonomous System). A linear autonomous system is a system of the form $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$ is a constant matrix and $x \in \mathbb{R}^n$ is the state vector.

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Stability of non-linear autonomous Systems

Definition 1.1 (Non-linear Autonomous System). A non-linear autonomous system is a system of the form $\dot{x} = f(x)$, where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a non-linear vector field and $x \in \mathbb{R}^n$ is the state vector.

Theorem 1.2 (Stability of Non-linear Autonomous Systems). The non-linear autonomous system $\dot{x} = f(x)$ is asymptotically stable if and only if the Jacobian matrix $J_f(x_0)$ at the equilibrium point x_0 has all eigenvalues with negative real parts.

Example 1.3 (Stability of a Non-linear Autonomous System). Consider the non-linear autonomous system $\dot{x} = f(x)$ with $f(x) = \begin{pmatrix} -x \\ -x^2 \end{pmatrix}$. The equilibrium point is $x_0 = 0$. The Jacobian matrix at x_0 is $J_f(x_0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. The eigenvalues of $J_f(x_0)$ are $\lambda_1 = -1$ and $\lambda_2 = 0$. Since $\lambda_2 = 0$ has a non-negative real part, the system is not asymptotically stable.

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