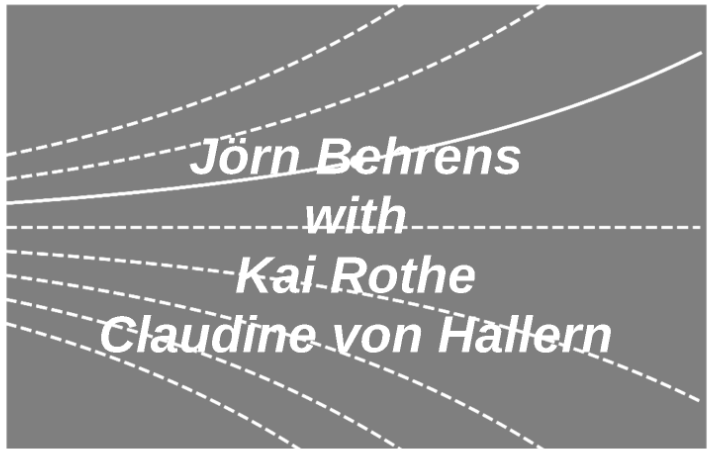
Differential Equations I



Autonomous Systems and Stability

Chapter 6.14

Recapitulation

Definition: (Dynamical System)
Consider the mapping

$$\mathbf{F} : \mathbb{R}^{n+1} \to \mathbb{R}^n \quad \text{und} \quad \mathbf{x} : \mathbb{R} \to \mathbb{R}^n,$$

$\mathbf{x}$ differenciable. The system of differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t),$$

with $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^\top$ and $\mathbf{F}(\mathbf{x}, t) = (F_1(\mathbf{x}, t), \ldots, F_n(\mathbf{x}, t))^\top$, is called dynamical System.
The space of solution curves $\mathbf{x}(t)$ is called phase space and the solution curves phase curves.

Remark: (System of first Order)
In analogy to the linear case, an ODE of n^{th} order can be reformulated as a system of n equations of first order:

- Let: $y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)}, t)$.
- Introduce: $x_1(t) = y(t)$, $x_2(t) = y'(t)$, …, $x_n(t) = y^{(n-1)}(t)$.
- The dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ with

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(x_1, x_2, \ldots, x_n, t) \end{pmatrix} =: \mathbf{F}(x_1, \ldots, x_n, t)$$

  is equivalent to the ODE of n^{th} order above.

Definition: (Autonomous System)
If the mapping $\mathbf{F}$ of the dynamical system does not depend on t i.e.,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

with $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, then the system is called autonomous system. 1

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Stability of Linear Autonomous Systems

Definition: (Equilibrium State)
If $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ with $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is an autonomous system, then values $\mathbf{x}_0 \in \mathbb{R}^n$ with the property

$$\mathbf{F}(\mathbf{x}_0) = \mathbf{0}$$

are called equilibrium or equilibrium state of the system.
These points are also called *critical, stationary* or *singular* points.

Fundamental Question: (Stability)
Does a phase curve, starting close to an equilibrium $\mathbf{x}_0$, remain in its vicinity?

Remark: (Linear Autonomous System)

- Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$.
- Let $A \in \mathbb{R}^{n \times n}$ be a matrix and set $\mathbf{F} = A$.
- Then $\mathbf{x}_0 = \mathbf{0}$ is an equilibrium.
- If A has the n pair-wise distinct eigenvalues λ_k, then the general solution of the autonomous system is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{e}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{e}_n,$$

with $\mathbf{e}_k$ the eigenvectors corresponding to λ_k.

Definition: (Properties of Equilibria)
Let $\mathbf{x}_0$ be equilibrium of the autonomous system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$. Then $\mathbf{x}_0$ is called

1. attractive, if solutions $\mathbf{x}(t)$, starting close to $\mathbf{x}_0$ converge towards the equilibrium:

$$\exists \delta > 0 : \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}_0, \quad \forall \mathbf{x}(t) \text{ with } |\mathbf{x}(0) - \mathbf{x}_0| < \delta,$$

2. stable, if solutions $\mathbf{x}(t)$, starting close to $\mathbf{x}_0$ remain in its vicinity:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ with } |\mathbf{x}(0) - \mathbf{x}_0| < \delta \ \Rightarrow\ |\mathbf{x}(t) - \mathbf{x}_0| < \epsilon\ \forall t > 0,$$

3. asymptotically stable, if attractive and stable,
4. unstable, if there are solutions that – although started close to $\mathbf{x}_0$ – deviate away from equilibrium:

$$\exists \epsilon > 0 \text{ and } t_1 > 0 : \forall \delta > 0 \text{ with } |\mathbf{x}(0) - \mathbf{x}_0| < \delta \ \Rightarrow\ |\mathbf{x}(t) - \mathbf{x}_0| > \epsilon,\ t \geq t_1.$$

Proposition: (Stability of Linear Autonomous Systems)
The equilibrium $\mathbf{x}_0$ of a linear autonomous system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = A\mathbf{x}$ is

1. asymptotically stable, if all eigenvalues of A have negative real parts,
2. stable, if no eigenvalue of A has positive real part and for eigenvalues with real part zero we have: geometric = algebraic multiplicity.
3. unstable, if an eigenvalue of A has positive real part or there is an eigenvalue with real part zero and geometric < algebraic multiplicity.

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These points are also called *critical*, *stationary* or *singular* points.

Remark: (Equilibrium)
Obviously $\mathbf{x}(t) = \mathbf{x}_0$ is a constant, time-independent solution of the autonomous system, for which the system rests in equilibrium.

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Stability of non-linear autonomous Systems

Motivation:

- Most interesting problems are non-linear.
- Since we study the behavior in a neighborhood of an equlibrium, we may *linearize*.
- Linear Approximation (Taylor-Expansion):

$$\mathbf{L}\mathbf{x} = \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0),$$

  with $\mathbf{F}'$ matrix derivative of $\mathbf{F}$.
- If $\mathbf{F}$ sufficiently smooth, we see

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

- If $\mathbf{x}_0$ is an equilibrium, we see

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

Proposition: (Stability of Non-Linear Autonomous Systems)
The equilibrium $\mathbf{x}_0$ of a non-linear autonomous System $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is

1. asymptotically stable, if all eigenvalues of the derivative matrix $\mathbf{F}'(\mathbf{x}_0)$ have negative real parts,
2. unstable, if at least one eigenvalue of $\mathbf{F}'(\mathbf{x}_0)$ has positive real part. 3

Motivation:

- Most interesting problems are non-linear.
- Since we study the behavior in a neighborhood of an equlibrium, we may *linearize*.
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$$\mathbf{L}\mathbf{x} = \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0),$$

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- If $\mathbf{x}_0$ is an equilibrium, we see

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3

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Recapitulation

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