Differential Equations I

Week 11 / J. Behrens

(1)

Proposition: (Orthogonality in Sturm-Liouville Eigenvalue Problems)
For the coefficient functions of the homogeneous Sturm-Liouville differential equation

$$L[y] + \lambda w y = (p(x)y')' + q(x)y + \lambda w y = 0$$

with $\lambda \in \mathbb{R}$ a parameter, assume:

- For $x \in [a, b]$ let $p(x)$ be continuously differentiable,
- let $q(x), w(x)$ be continuous.
- For $x \in]a, b[$ let $p(x) > 0$ and $w(x) > 0$.

Then two non-trivial solutions $y_1(x), y_2(x) \in C^2([a, b], \mathbb{R})$ corresponding to two different parameter values $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are orthogonal i.e.,

$$\langle y_1, y_2 \rangle = \int_a^b y_1(x) y_2(x) w(x) \, dx = 0,$$

if

1. y_1 and y_2 satisfy the homogeneous boundary conditions $R_1(y) = 0 = R_2(y)$ i.e., λ_1, λ_2 are eigenvalues corresponding to eigenfunctions y_1, y_2 of the Sturm-Liouville eigenvalue problem, or
2. the coefficient function $p(x)$ fulfills the condition $p(a) = p(b) = 0$.

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Example:

- Consider: $\boxed{-y'' = \lambda y \quad , \quad y(0) = y(l) = 0}$, $x \in [0, l] \subset \mathbb{R}$

$\rightarrow \quad L[y] = y'' \qquad\qquad p \equiv 1 \, , \; w \equiv 1 \, , \; q \equiv 0$

- General Solution:

$$\left.\begin{array}{ll} \lambda \neq 0: & y(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x} \\ \lambda = 0: & y(x) = c_1 + c_2 x \end{array}\right\} \; c_1, c_2 \in \mathbb{R}$$

- Boundary Conditions:

$\lambda \leq 0 \implies$ There ex. only the trivial solution!

$\lambda > 0 \implies y_1(x) = \cos(\sqrt{\lambda}x)\,,\; y_2(x) = \sin(\sqrt{\lambda}x)$ form a fundamental system

$$\implies y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Since $y(0) = y(l) = 0 \implies 0 = y(0) = c_1 + c_2 \cdot 0 = c_1$

$$0 = y(l) = c_2 \sin(\sqrt{\lambda}l)$$

$$\implies c_1 = 0 \text{ and } c_2 = 0 \text{ or } \sin(\sqrt{\lambda}l) = 0$$

$$\iff \sqrt{\lambda}l = \pi k \,,\; k \in \mathbb{N}$$

This case correspond to the trivial solution and is therefore excluded.

- Eigenvalues: We have $\lambda_k = \frac{k^2\pi^2}{l^2}$ are Eigenvalues

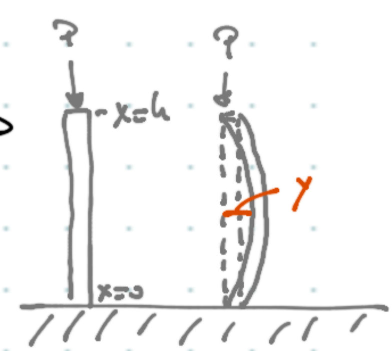
- Eigenfunctions: $y_k(x) = \sin(k\pi \frac{x}{l})$

- Orthogonality: For $\lambda_k \neq \lambda_j$ $(j \neq k)$

$$\langle y_k, y_j \rangle = \int_0^l \sin(k\pi \tfrac{x}{l}) \sin(j\pi \tfrac{x}{l})\, dx = 0$$

- Application: $-y'' = \lambda y$ with $y(0) = y(h) = 0\,,\; \lambda = \frac{P}{B}$

This equation describes the displacement of a beam of height h in relation to the weight (force) P and the bending strength B



- Solutions: $y_k(x) = c \sin\left(\sqrt{\frac{P}{B}}\, x\right)$ if $\frac{P}{B} = \frac{k^2\pi^2}{h^2}$

that means the force P is proportional to the bending strength.

- Alternatives:

i) if $P < P_1 = B\frac{\pi^2}{h^2} \Rightarrow \lambda = \frac{P}{B} < \frac{\pi^2}{h^2}$

$\Rightarrow$ Ex. only the trivial solution, so no bending

ii) if $\lambda_1 = \frac{\pi^2}{h^2} \Rightarrow P_1 = B\frac{\pi^2}{h^2} \Rightarrow y_1(x) = c \sin\left(\frac{\pi}{h} x\right)$

so the bending is of the shape of sin

Remark: P_1 is called Euler's buckling load.

2 **Proposition**: (Expansion)
Let $(y_n(x))$ be a sequence of normalized eigenfunctions, corresponding to eigenvalues λ_n of the eigenvalue problem

$$-L[y] = \omega w y, R_1(y) = 0 = R_2(y)$$

with coefficient function $p(x) > 0$ and weight function $w(x) > 0$ on $[a, b]$. Thus, it holds:

$$\langle y_k, y_j \rangle = \delta_{kj}.$$

Then each continuously differentiable function f, satisfying the boundary conditions of the eigenvalue problem, can be represented as function series

$$f(x) = \sum_{n=1}^{\infty} \langle f, y_n \rangle y_n(x).$$

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The series converges in $[a, b]$ uniformly and absolutely.

<u>Example</u>:

- Consider $-y'' = \lambda y$, $y(0) = y(\pi) = 0$

- Eigenvalues: $\lambda_k = k^2$, $k \in \mathbb{N}$
- Eigenfunctions: $y_k(x) = c \sin kx$, $c \neq 0$
- Normalized: $\langle y_k, y_k \rangle = \int_0^\pi c \sin kx \; c \sin kx \, dx = 1$

with $\int_0^\pi \sin^2 kx \, dx = \frac{\pi}{2} \Rightarrow c \sqrt{\frac{2}{\pi}}$

$\Rightarrow y_k(x) \sqrt{\frac{2}{\pi}} \sin kx$

- Series expansion (apply the proposition):

$$f(x) = \sum_{k=0}^{\infty} b_k \sqrt{\frac{2}{\pi}} \sin kx \qquad (f(0) = f(\pi) = 0)$$

with $b_k = \langle f, y_k \rangle = \int_0^\pi f(x) \sqrt{\frac{2}{\pi}} \sin kx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin kx \, dx$$

that means $f(x) = \sum_{k=0}^{\infty} \tilde{b}_k \sin kx$ with $\tilde{b}_k = \frac{\pi}{2} \int_0^\pi f(x) \sin kx \, dx$

This is the Fourier series of the function f given on $[0, \pi]$ and extended unevenly and periodically with period 2π.