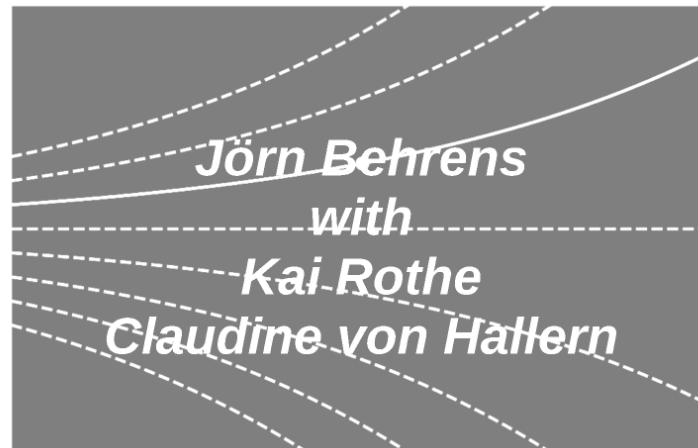


# Differential Equations I



Boundary and Eigenvalue Problems

Chapter 6.13

## Recall: Powerseries Approach

### Summary:

Consider the IVP

$$y'' + y = \cos(2x), \quad \text{with } y(0) = 0, y'(0) = 1.$$

1. Use the power series approach:  $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{k=0}^{\infty} a_k x^k$ .
2. Compute  $y'$  and  $y''$  from this series.
3. Obtain  $a_0$  and  $a_1$  from initial values.
4. Insert series in ODE, use the power series for  $\cos$ .
5. Compare coefficients and obtain  $y$  as a power series.
6. If possible, obtain closed form for  $y$  from power series.

### Remarks:

- If non-zero initial conditions  $y(x_0) = y_0, y'(x_0) = y_1, x_0 \neq 0$  are given, use the approach

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

- In general a closed form cannot be expected. Then the power series of  $y(x)$ , or even just the first members of it, need to suffice.
- The main advantage of the power series is its "simple" derivation!

**Summary:**

Consider the IVP

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- The main advantage of the power series is its “simple” derivation!

# Boundary Value Problems

## Definition: (Differential Operator of 2<sup>nd</sup> Order)

Let  $I \subset \mathbb{R}$  be a closed interval and  $a_0(x) \neq 0$ ,  $a_1(x)$ ,  $a_2(x)$ ,  $r(x)$  continuous functions. Then

$$D[y] := a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x)$$

defines a differential operator that maps twice differentiable functions  $y(x)$  on  $I$  into continuous functions  $D[y]$ .

## Definition: (Sturm Boundary Conditions)

Let the differential equation

$$D[y] = r(x)$$

be given as before. Furthermore, let

$$R_1(y) := \alpha_1 y(a) + \beta_1 y'(a), \quad R_2(y) := \alpha_2 y(b) + \beta_2 y'(b),$$

with  $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$ ,  $\alpha_k^2 + \beta_k^2 > 0$ ,  $k = 1, 2$ .

Then the ODE may conform to the **Sturm boundary conditions**

$$R_k(y) = \gamma_k \quad (k = 1, 2).$$

## Remark: (Linear System of Equations)

**Question:** Solvability of the ODE with boundary conditions.

• **General Solution:**  $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$ , where  $c_1, c_2 \in \mathbb{R}$ ,  $\{y_1, y_2\}$  fundamental system of the homogeneous ODE  $D[y] = 0$  and  $y_p$  particular solution of the inhomogeneous ODE  $D[y] = r(x)$ .

• **Derivative:**  $y'(x) = c_1 y_1'(x) + c_2 y_2'(x) + y_p'(x)$ .

• **This yields for the boundary conditions:**

$$\begin{aligned} \alpha_1 c_1 y_1(a) + \alpha_1 c_2 y_2(a) + \beta_1 [c_1 y_1'(a) + c_2 y_2'(a) + y_p'(a)] &= \gamma_1 \\ \alpha_2 c_1 y_1(b) + \alpha_2 c_2 y_2(b) + \beta_2 [c_1 y_1'(b) + c_2 y_2'(b) + y_p'(b)] &= \gamma_2 \end{aligned}$$

• **Reformulation:**

$$\begin{aligned} (\alpha_1 y_1(a) + \beta_1 y_1'(a)) c_1 + (\alpha_1 y_2(a) + \beta_1 y_2'(a)) c_2 &= \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a) \\ (\alpha_2 y_1(b) + \beta_2 y_1'(b)) c_1 + (\alpha_2 y_2(b) + \beta_2 y_2'(b)) c_2 &= \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b) \end{aligned}$$

• **Use definitions for  $R_1, R_2$  and**

$$r_1 = \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a), \quad r_2 = \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b)$$

**obtain linear system of equations**

$$\begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

• **If the linear system is solvable, then the ODE with boundary conditions is solvable. Thus,**

$$\det \begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \neq 0$$

2

**Definition:** (Differential Operator of 2<sup>nd</sup> Order)

Let  $I \subset \mathbb{R}$  be a closed interval and  $a_0(x) \neq 0$ ,  $a_1(x)$ ,  $a_2(x)$ ,  $r(x)$  continuous functions. Then

$$D[y] := a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x)$$

defines a differential operator that maps twice differentiable functions  $y(x)$  on  $I$  into continuous functions  $D[y]$ .

**Remarks:**

- Consider the ODE  $D[y] = r(x)$ .
- With initial conditions

$$y(\xi) = \eta_a, \quad y'(\xi) = \gamma_a, \quad \xi \in I, \quad \eta_a, \gamma_a \in \mathbb{R},$$

there is a unique solution on  $I$  according to the proposition.

- **Question:** What if apart from position  $\xi$  conditions at other positions are required?



**Remark:**

Boundary problems (other than initial value problems) are not always solvable!

**Definition:** (Sturm Boundary Conditions)

Let the differential equation

$$D[y] = r(x)$$

be given as before. Furthermore, let

$$R_1(y) := \alpha_1 y(a) + \beta_1 y'(a), \quad R_2(y) := \alpha_2 y(b) + \beta_2 y'(b),$$

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- This yields for the boundary conditions:

$$\begin{aligned}\alpha_1[c_1y_1(a) + c_2y_2(a) + y_p(a)] + \beta_1[c_1y_1'(a) + c_2y_2'(a) + y_p'(a)] &= \gamma_1 \\ \alpha_2[c_1y_1(b) + c_2y_2(b) + y_p(b)] + \beta_2[c_1y_1'(b) + c_2y_2'(b) + y_p'(b)] &= \gamma_2\end{aligned}$$

- Reformulation:

$$\begin{aligned}(\alpha_1y_1(a) + \beta_1y_1'(a))c_1 + (\alpha_1y_2(a) + \beta_1y_2'(a))c_2 &= \gamma_1 - \alpha_1y_p(a) - \beta_1y_p'(a) \\ (\alpha_2y_1(b) + \beta_2y_1'(b))c_1 + (\alpha_2y_2(b) + \beta_2y_2'(b))c_2 &= \gamma_2 - \alpha_2y_p(b) - \beta_2y_p'(b).\end{aligned}$$

- Use definitions for  $R_1, R_2$  and

$$r_1 = \gamma_1 - \alpha_1y_p(a) - \beta_1y_p'(a), \quad r_2 = \gamma_2 - \alpha_2y_p(b) - \beta_2y_p'(b)$$

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2

# Self Adjoint Differential Operators

**Note: (Differential Operators)**  
Consider differential operators

$$D: C^2([a, b], \mathbb{R}) \rightarrow W$$

- $C^2([a, b], \mathbb{R})$  is the set of twice differentiable functions on interval  $I = [a, b]$ .
- $W$  is a set of continuous functions, the image domain of  $D$ .

**Definition: (Sturm-Liouville Differential Operator)**

Let  $p(x)$  be continuously differentiable and positive and  $q(x)$  and  $z(x)$  continuous on  $I$ . We call

$$L[y] = (p(x)y')' + q(x)y$$

Sturm-Liouville differential operator and

$$L[y] = z(x)$$

Sturm-Liouville differential equation.

**Definition: (Adjoint Differential Operator of  $n^{\text{th}}$  Order)**

Let

$$D[y] = \sum_{k=0}^{n-1} a_k(x)y^{(k)}$$

be a linear differential operator of  $n^{\text{th}}$  order,  $a_k(x)$  a given  $(n - k)$ -times differentiable functions on  $I$ ,  $a_0(x) \neq 0$ , which is applied to  $y(x)$ , an arbitrary  $n$ -times differentiable function on  $I$ .

The adjoint differential operator corresponding to  $D[y]$  is defined by

$$D^*[y] = \sum_{k=0}^{n-1} (-1)^{k+n-1} (a_k(x)y(x))^{(k+n-k)}$$

**Definition: (Self Adjoint Differential Operator of  $n^{\text{th}}$  Order)**

A differential operator  $D[y]$  is called self adjoint, if

$$D^*[y] = D[y]$$

holds for all  $n$ -times differentiable functions  $y$  on  $I$ .

**Example: A self adjoint differential operator for  $n = 2$**

We already computed:

$$\begin{aligned} D[y] &= -u(x)y'' + u'(x)y' + u(x)y \\ D^*[y] &= -(u(x)y)'' + (u(x)y)' + u(x)y \\ &= -u(x)y'' + (2u(x)' - u(x)y)' + u(x)y + u(x)y \end{aligned}$$

With  $D^*[y] = D[y]$  it follows:

$$\begin{aligned} 2u(x)' - u(x)y' &= -u(x)y' \\ u(x)' + u(x) &= u(x) \end{aligned}$$

One obtains

$$D[y] = (u(x)y')' + u(x)y \quad \text{③}$$

Source: [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [38] [39] [40] [41] [42] [43] [44] [45] [46] [47] [48] [49] [50] [51] [52] [53] [54] [55] [56] [57] [58] [59] [60] [61] [62] [63] [64] [65] [66] [67] [68] [69] [70] [71] [72] [73] [74] [75] [76] [77] [78] [79] [80] [81] [82] [83] [84] [85] [86] [87] [88] [89] [90] [91] [92] [93] [94] [95] [96] [97] [98] [99] [100]

**Note:** (Differential Operators)  
Consider differential operators

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**Definition:** (Adjoint Differential Operator of  $n^{\text{th}}$  Order)

Let

$$D[y] := \sum_{\nu=0}^n a_{\nu}(x)y^{(n-\nu)}$$

be a linear differential operator of  $n^{\text{th}}$  order,  $a_{\nu}(x)$  a given  $(n - \nu)$ -times differentiable functions on  $I$ ,  $a_0(x) \neq 0$ , which is applied to  $y(x)$ , an arbitrary  $n$ -times differentiable function on  $I$ .

The **adjoint differential operator** corresponding to  $D[y]$  is defined by

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**Example:**  $n = 2$

$$D[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y$$

$$\begin{aligned} D^*[y] &= (a_0(x)y)'' - (a_1(x)y)' + a_2(x)y \\ &= a_0(x)y'' + (2a_0'(x) - a_1(x))y' + (a_0''(x) - a_1'(x) + a_2(x))y. \end{aligned}$$

**Definition:** (Self Adjoint Differential Operator of  $n^{\text{th}}$  Order)  
A differential operator  $D[y]$  is called **self adjoint**, if

$$D^*[y] = D[y]$$

holds for all  $n$ -times differentiable functions  $y$  on  $I$ .

**Example:** A self adjoint differential operator for  $n = 2$ :

We already computed:

$$\begin{aligned} D[y] &= a_0(x)y'' + a_1(x)y' + a_2(x)y \\ D^*[y] &= (a_0(x)y)'' - (a_1(x)y)' + a_2(x)y \\ &= a_0(x)y'' + (2a_0'(x) - a_1(x))y' + (a_0''(x) - a_1'(x) + a_2(x))y. \end{aligned}$$

With  $D^*[y] = D[y]$  it follows:

$$\begin{aligned} 2a_0' - a_1 &= a_1 \quad \Rightarrow \quad a_0' = a_1 \\ a_0'' - a_1' + a_2 &= a_2 \quad \Rightarrow \quad a_0'' = a_1'. \end{aligned}$$

One obtains

$$D[y] = (a_0(x)y')' + a_2(x)y.$$

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**Definition:** (Sturm-Liouville Differential Operator)

Let  $p(x)$  be continuously differentiable and positive on interval  $I = [a, b]$ ,  $q(x)$  and  $z(x)$  continuous on  $I$ . We call

$$L[y] = (p(x)y')' + q(x)y$$

Sturm-Liouville differential operator and

$$L[y] = z(x)$$

Sturm-Liouville differential equation.



**Examples:** (Bessel's Differential Equation)

1. For the ODE

$$y'' + e^x y' + xy = 0$$

one obtains

$$s(x) = \int \frac{e^x - 0}{1} dx = e^x,$$

and so the self adjoint form

$$(e^{e^x} y')' - x e^{e^x} y = 0.$$

2. For Bessel's ODE

$$x^2 y'' + xy' + (x^2 - n^2)y = 0, \quad (x > 0)$$

one obtains

$$s(x) = \int \frac{x - 2x}{x^2} dx = -\ln x,$$

i.e., the ODE is multiplied with  $e^{s(x)} = \frac{1}{x}$ ; then the self adjoint form is

$$xy'' + y' + \left(x - \frac{n^2}{x}\right)y = (xy')' + \left(x - \frac{n^2}{x}\right)y = 0.$$

# Generalization of Self Adjoint Differential Operators

**Remark:** (Integral Relation of Differential Operators)

- Use the scalar product for two in  $I = [a, b]$  continuous functions  $f, g: I \rightarrow \mathbb{R}$ :
 
$$(f, g) = \int_a^b f(x)g(x) dx.$$
- By partial integration (twice) one obtains
 
$$\begin{aligned} (L[u], v) &= \int_a^b (pu'' + qu)v dx \\ &= \int_a^b (u''(pv) - (u'v)') dx + \int_a^b (u'v)' dx + \int_a^b uv dx \\ &= (u, L[v]) + [u'v]_a^b - [uv]_a^b + \int_a^b uv dx. \end{aligned}$$
- We may generalize the relation from linear operators
 
$$(L[u], v) = (u, L[v])$$
 to linear mappings  $L: \mathcal{D} \rightarrow \mathcal{D}'$  and their adjoint  $L^*$  with Euclidean scalar product to differential operators, i.e.
 
$$[u'v]_a^b + [uv]_a^b - [uv]_a^b = 0$$
- This  $u, v, u', v'$  need to conform to specific boundary conditions (e.g.  $u(a) = u(b) = u'(a) = u'(b) = 0$ ). Then
 
$$(L[u], v) = (u, L[v]).$$

**Proposition:** (Self-Adjoint Sturm-Liouville Eigen-Value Problem)

Let  $L[u] = (p(x)u'(x))' + q(x)u(x)$  be the Sturm-Liouville differential operator for  $x \in [a, b]$  with cont. diff. function  $p(x) > 0$ , cont. diff. function  $q(x)$  and cont. function  $w(x) > 0$ ,  $\lambda \in \mathbb{R}$  a parameter and  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$  with  $\alpha_1^2 + \alpha_2^2 > 0$  ( $k = 1, 2$ ). Then the Sturm-Liouville eigen-value problem
 
$$L[u] + \lambda w(x)u = 0, \quad \alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \beta_1 u(b) + \beta_2 u'(b) = 0$$
 is self-adjoint. Non-trivial solutions  $y_k(x)$  corresponding to given parameters  $\lambda$  are called **eigenfunctions** (if they exist). The corresponding parameters  $\lambda$  are called **eigenvalues** of the Sturm-Liouville eigen-value problem.

**Observation:** (Integral relation of self-adjoint differential operators)  
For self-adjoint differential operators  $L$  the conditions simplify. We have:

$$\begin{aligned} (L[u], v) &= \int_a^b (pu'' + qu)v dx \\ &= \int_a^b (-u'(pv') + uv) dx + [pu'v]_a^b \\ &= \int_a^b u'([pv']) dx + [pu'v]_a^b - [pu'v]_a^b \\ &= (u, L[v]) + [p(u'v - uv')]_a^b. \end{aligned}$$

The relation  $(L[u], v) = (u, L[v])$  holds, if
 
$$[p(x)(u'(x)v(x) - u(x)v'(x))]_a^b = 0.$$

**Consider:** (Boundary Value Problem)

Let us seek the solution on interval  $I = [a, b]$  of

$$\begin{aligned} -L[u] &= \lambda w(x)u, \\ R_1(u) &= \alpha_1 u(a) + \beta_1 u'(a) = 0, \\ R_2(u) &= \alpha_2 u(b) + \beta_2 u'(b) = 0, \end{aligned}$$

with  $L$  Sturm-Liouville differential operator,  $\lambda \in \mathbb{R}$  a parameter,  $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$  with  $\alpha_1^2 + \beta_1^2 > 0$  ( $k = 1, 2$ ),  $w(x)$  a positive continuous function on  $I$ .

Assume  $C^2([a, b], \mathbb{R})$  as domain of  $L$ , more precisely the subset  $M \subset C^2([a, b], \mathbb{R})$  of functions fulfilling the boundary conditions! The elements in  $M$  are called **test functions**.

**Definition:** (General Self-Adjoint Differential Operator)

Let  $L$  be a self-adjoint differential operator of 2<sup>nd</sup> order on  $I = [a, b]$ , and  $M \subset C^2([a, b], \mathbb{R})$  the set of all functions fulfilling given boundary conditions  $x = a$  and  $x = b$  (test functions).

If for all  $u, v \in M$  it holds that

$$(L[u], v) = (u, L[v]),$$

we call  $L$  (general) self-adjoint differential operator on  $M$ . The corresponding boundary value problem is also called **self-adjoint**.

**Remark:** (Integral Relation of Differential Operators)

- Use the scalar product for two on  $I = [a, b]$  continuous functions  $f, g : I \rightarrow \mathbb{R}$ :

$$(f, g) = \int_a^b f(x)g(x) dx.$$

- By partial integration (twice) one obtains

$$\begin{aligned}(D[u], v) &= \int_a^b [a_0 u'' + a_1 u' + a_2 u] v dx \\ &= \int_a^b u [(a_0 v)'' - (a_1 v)' + a_2 v] dx + [u' a_0 v]_a^b + [u a_1 v]_a^b - [u (a_0 v)']_a^b \\ &= (u, D^*[v]) + [u' a_0 v]_a^b + [u a_1 v]_a^b - [u (a_0 v)']_a^b.\end{aligned}$$

- We may generalize the relation from linear algebra

$$(f(\mathbf{x}), \mathbf{y}) = (\mathbf{x}, f^*(\mathbf{y}))$$

for linear mappings  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and their adjoint  $f^*$  with Euclidean scalar product to differential operators, if

$$[u' a_0 v]_a^b + [u a_1 v]_a^b - [u (a_0 v)']_a^b = 0.$$

- Thus  $u, v, a_0, a_1$  need to conform to specific boundary conditions (e.g.,  $u(a) = u(b) = v(a) = v(b) = 0$ ). Then

$$(D[u], v) = (u, D^*[v]).$$

**Observation:** (Integral relation of self adjoint differential operators)  
 For self adjoint differential operators  $L$  the conditions simplify. We have:

$$\begin{aligned}
 (L[u], v) &= \int_a^b [(pu')' + qu]v \, dx \\
 &= \int_a^b (-u'pv' + uqv) \, dx + [pu'v]_a^b \\
 &= \int_a^b u[(pv')' + qv] \, dx + [pu'v]_a^b - [upv']_a^b \\
 &= (u, L[v]) + [p(u'v - uv')]_a^b.
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The relation  $(L[u], v) = (u, L[v])$  holds, if

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**Consider:** (Boundary Value Problem)

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$$\begin{aligned} -L[y] &= \lambda w(x)y, \\ R_1(y) &= \alpha_1 y(a) + \beta_1 y'(a) = 0, \\ R_2(y) &= \alpha_2 y(b) + \beta_2 y'(b) = 0; \end{aligned}$$

with  $L$  Sturm-Liouville differential operator,  $\lambda \in \mathbb{R}$  a parameter,  $\alpha_k, \beta_k \in \mathbb{R}$  with  $\alpha_k^2 + \beta_k^2 > 0$  ( $k = 1, 2$ ),  $w(x)$  a positive continuous function on  $I$ .

Assume  $C^2([a, b], \mathbb{R})$  as domain of  $L$ , more precisely the subset  $M \subset C^2([a, b], \mathbb{R})$  of functions fulfilling the boundary conditions!. The elements in  $M$  are called **test functions**.

**Definition:** (General Self Adjoint Differential Operator)

Let  $L$  be a self adjoint differential operator of 2<sup>nd</sup> order on  $I = [a, b]$ , and  $M \subset C^2([a, b], \mathbb{R})$  the set of all functions fulfilling given boundary conditions  $x = a$  and  $x = b$  (test functions).

If for all  $u, v \in M$  it holds that

$$(L[u], v) = (u, L[v]),$$

we call  $L$  (general) self adjoint differential operator on  $M$ . The corresponding boundary value problem is also called self adjoint.

**Remarks:** (Sufficient Conditions for Self Adjoint Differential Operators)

1. If all functions in  $M$  fulfill boundary conditions  $R_1(y) = R_2(y) = 0$ , then  $L$  is a self adjoint operator on  $M$ .
2. Let  $p(a) = p(b) > 0$  and  $M$  the set of all functions that fulfill periodic boundary conditions; i.e.,

$$y \in M \Rightarrow y(a) = y(b) \text{ and } y'(a) = y'(b).$$

Then  $L$  is a self adjoint operator on  $M$ .

3. If  $p(x) > 0$  for  $x \in ]a, b[$  and  $p(a) = p(b) = 0$ , then  $L$  is a self adjoint operator for all  $u, v \in C^2([a, b], \mathbb{R})$ .

**Proposition:** (Self Adjoint Sturm-Liouville Eigen Value Problem)

Let  $L[y] = (p(x)y')' + q(x)y$  be the Sturm-Liouville differential operator for  $x \in [a, b]$  with cont. diff. function  $p(x) > 0$ , cont. diff. function  $q(x)$  and cont. function  $w(x) > 0$ ,  $\lambda \in \mathbb{R}$  a parameter and  $\alpha_k, \beta_k \in \mathbb{R}$  with  $\alpha_k^2 + \beta_k^2 > 0$  ( $k = 1, 2$ ).

Then the **Sturm-Liouville eigen value problem**

$$L[y] + \lambda w(x)y = 0, \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \quad \alpha_2 y(b) + \beta_2 y'(b) = 0$$

is self adjoint.

Non-trivial solutions  $y_\lambda(x)$  corresponding to given parameters  $\lambda$  are called **eigenfunctions** (if they exist). The corresponding parameters  $\lambda$  are called **eigenvalues** of the Sturm-Liouville eigen value problem.



**Boundary Value Problems**

Consider the boundary value problem

$$y'' + p(x)y' + q(x)y = r(x), \quad y(a) = \alpha, \quad y(b) = \beta$$

where  $a < b$  and  $\alpha, \beta$  are constants. The boundary value problem is said to be **regular** if the boundary conditions are not both homogeneous and if the boundary conditions are not both of the form  $y(a) = 0$  and  $y(b) = 0$ .

The boundary value problem is said to be **self-adjoint** if the boundary conditions are of the form

$$y(a) \cos \theta - y'(a) \sin \theta = \alpha$$

$$y(b) \cos \theta - y'(b) \sin \theta = \beta$$

for some constant  $\theta$ .

The boundary value problem is said to be **regular and self-adjoint** if the boundary conditions are of the form

$$y(a) \cos \theta - y'(a) \sin \theta = \alpha$$

$$y(b) \cos \theta - y'(b) \sin \theta = \beta$$

for some constant  $\theta$  and if the boundary conditions are not both homogeneous and if the boundary conditions are not both of the form  $y(a) = 0$  and  $y(b) = 0$ .

**Self Adjoint Differential Operators**

Consider the differential operator

$$L[y] = -y'' + p(x)y' + q(x)y$$

where  $p(x)$  and  $q(x)$  are real-valued functions. The operator  $L$  is said to be **self-adjoint** if

$$(Ly, z) = (y, Lz)$$

for all functions  $y$  and  $z$  in the domain of  $L$ .

The operator  $L$  is said to be **regular and self-adjoint** if the boundary conditions are of the form

$$y(a) \cos \theta - y'(a) \sin \theta = \alpha$$

$$y(b) \cos \theta - y'(b) \sin \theta = \beta$$

for some constant  $\theta$  and if the boundary conditions are not both homogeneous and if the boundary conditions are not both of the form  $y(a) = 0$  and  $y(b) = 0$ .

**Generalization of Self Adjoint Differential Operators**

Consider the differential operator

$$L[y] = -p(x)y'' + p'(x)y' + q(x)y$$

where  $p(x)$  and  $q(x)$  are real-valued functions. The operator  $L$  is said to be **self-adjoint** if

$$(Ly, z) = (y, Lz)$$

for all functions  $y$  and  $z$  in the domain of  $L$ .

The operator  $L$  is said to be **regular and self-adjoint** if the boundary conditions are of the form

$$y(a) \cos \theta - p(a)y'(a) \sin \theta = \alpha$$

$$y(b) \cos \theta - p(b)y'(b) \sin \theta = \beta$$

for some constant  $\theta$  and if the boundary conditions are not both homogeneous and if the boundary conditions are not both of the form  $y(a) = 0$  and  $y(b) = 0$ .



**Differential Equations I**

**Recall: Powerseries Approach**

Consider the differential equation

$$y'' + p(x)y' + q(x)y = r(x)$$

where  $p(x)$  and  $q(x)$  are real-valued functions. The powerseries approach involves assuming a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and substituting it into the differential equation to determine the coefficients  $a_n$ .