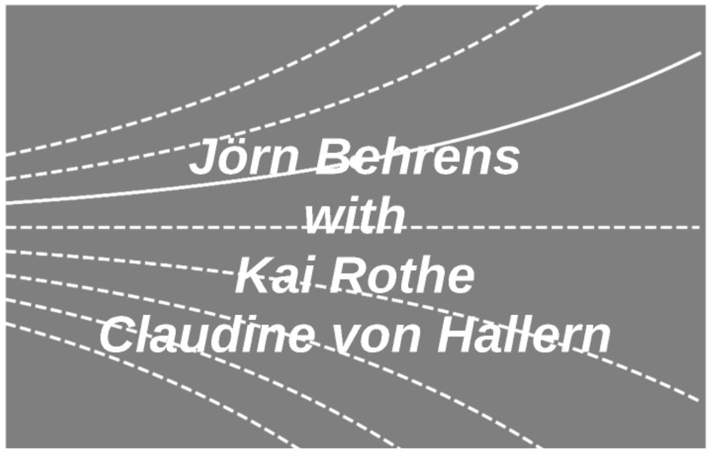
Differential Equations I



Boundary and Eigenvalue Problems

Chapter 6.13

Recall: Powerseries Approach

Summary:
Consider the IVP

$$y'' + y = \cos(2x), \quad \text{with} \quad y(0) = 0, y'(0) = 1.$$

1. Use the power series approach: $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k$.
2. Compute y' and y'' from this series.
3. Obtain a_0 and a_1 from initial values.
4. Insert series in ODE, use the power series for $\cos$.
5. Compare coefficients and obtain y as a pwer series.
6. If possible, obtain closed form for y from power series.

Remarks:

- If non-zero initial conditions $y(x_0) = y_0, y'(x_0) = y_1, x_0 \neq 0$ are given, use the approach

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

- In general a closed form cannot be expected. Then the power series of $y(x)$, or even just the first members of it, need to suffice.
- The main advantage of the power series is its "simple" derivation!

Summary:
Consider the IVP

$$y'' + y = \cos(2x), \quad \text{with} \quad y(0) = 0, y'(0) = 1.$$

1. Use the power series approach: $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k$.
2. Compute y' and y'' from this series.
3. Obtain a_0 and a_1 from initial values.
4. Insert series in ODE, use the power series for $\cos$.
5. Compare coefficients and obtain y as a pwer series.
6. If possible, obtain closed form for y from power series.

Remarks:

- If non-zero initial conditions $y(x_0) = y_0, y'(x_0) = y_1, x_0 \neq 0$ are given, use the approach

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

- In general a closed form cannot be expected. Then the power series of $y(x)$, or even just the first members of it, need to suffice.

- The main advantage of the power series is its "simple" derivation!

Boundary Value Problems

Definition: (Differential Operator of 2[nd] Order)
Let $I \subset \mathbb{R}$ be a closed intervall and $a_0(x) \neq 0$, $a_1(x)$, $a_2(x)$, $r(x)$ continuous functions. Then

$$D[y] := a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x)$$

defines a differential operator that maps twice differentiable functions $y(x)$ on I into continuous functions $D[y]$.

Definition: (Sturm Boundary Conditions)
Let the differential equation

$$D[y] = r(x)$$

be given as before. Furthermore, let

$$R_1(y) := \alpha_1 y(a) + \beta_1 y'(a), \quad R_2(y) := \alpha_2 y(b) + \beta_2 y'(b),$$

with $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$, $\alpha_k^2 + \beta_k^2 > 0$, $k = 1, 2$.
Then the ODE may conform to the Sturm boundary conditions

$$R_k(y) = \gamma_k \quad (k = 1, 2).$$

Remark: (Linear System of Equations)
Question: Solvability of the ODE with boundary conditions.

- General Solution: $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$, where $c_1, c_2 \in \mathbb{R}$, $\{y_1, y_2\}$ fundamental system of the homogeneous ODE $D[y] = 0$ and y_p particular solution of the inhomogeneous ODE $D[y] = r(x)$.
- Derivative: $y'(x) = c_1 y_1'(x) + c_2 y_2'(x) + y_p'(x)$.
- This yields for the boundary conditions:

$$\begin{aligned} \alpha_1[c_1 y_1(a) + c_2 y_2(a) + y_p(a)] + \beta_1[c_1 y_1'(a) + c_2 y_2'(a) + y_p'(a)] &= \gamma_1 \\ \alpha_2[c_1 y_1(b) + c_2 y_2(b) + y_p(b)] + \beta_2[c_1 y_1'(b) + c_2 y_2'(b) + y_p'(b)] &= \gamma_2 \end{aligned}$$

- Reformulation:

$$\begin{aligned} (\alpha_1 y_1(a) + \beta_1 y_1'(a))c_1 + (\alpha_1 y_2(a) + \beta_1 y_2'(a))c_2 &= \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a) \\ (\alpha_2 y_1(b) + \beta_2 y_1'(b))c_1 + (\alpha_2 y_2(b) + \beta_2 y_2'(b))c_2 &= \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b). \end{aligned}$$

- Use definitions for R_1, R_2 and

$$r_1 = \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a), \quad r_2 = \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b)$$

  obtain linear system of equations

$$\begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

- If the linear system is solvable, then the ODE with boundary conditions is solvable. Thus,

$$\det \begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \overset{!}{\neq} 0$$

2

6.

Definition: (Differential Operator of 2nd Order)
Let $I \subset \mathbb{R}$ be a closed intervall and $a_0(x) \neq 0$, $a_1(x)$, $a_2(x)$, $r(x)$ continuous functions. Then

$$D[y] := a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x)$$

defines a differential operator that maps twice differentiable functions $y(x)$ on I into continuous functions $D[y]$.

Remarks:

- Consider the ODE $D[y] = r(x)$.
- With initial conditions

  $$y(\xi) = \eta_a, \quad y'(\xi) = \gamma_a, \quad \xi \in I,\ \eta_a, \gamma_a \in \mathbb{R},$$

  there is a unique solution on I according to the proposition.
- Question: What if apart from position ξ conditions at other positions are required? 1

Remark:
Boundary problems (other than initial value problems) are not always solvable!

Definition: (Sturm Boundary Conditions)
Let the differential equation

$$D[y] = r(x)$$

be given as before. Furthermore, let

$$R_1(y) := \alpha_1 y(a) + \beta_1 y'(a), \quad R_2(y) := \alpha_2 y(b) + \beta_2 y'(b),$$

with $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$, $\alpha_k^2 + \beta_k^2 > 0$, $k = 1, 2$.
Then the ODE may conform to the Sturm boundary conditions

$$R_k(y) = \gamma_k \quad (k = 1, 2).$$

Remark: (Linear System of Equations)
Question: Solvability of the ODE with boundary conditions.

- General Solution: $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$, where $c_1, c_2 \in \mathbb{R}$, $\{y_1, y_2\}$ fundamental system of the homogeneous ODE $D[y] = 0$ and y_p particular solution of the inhomogeneous ODE $D[y] = r(x)$.
- Derivative: $y'(x) = c_1 y_1'(x) + c_2 y_2'(x) + y_p'(x)$.
- This yields for the boundary conditions:

$$\begin{aligned} \alpha_1[c_1 y_1(a) + c_2 y_2(a) + y_p(a)] + \beta_1[c_1 y_1'(a) + c_2 y_2'(a) + y_p'(a)] &= \gamma_1 \\ \alpha_2[c_1 y_1(b) + c_2 y_2(b) + y_p(b)] + \beta_2[c_1 y_1'(b) + c_2 y_2'(b) + y_p'(b)] &= \gamma_2 \end{aligned}$$

- Reformulation:

$$\begin{aligned} (\alpha_1 y_1(a) + \beta_1 y_1'(a))c_1 + (\alpha_1 y_2(a) + \beta_1 y_2'(a))c_2 &= \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a) \\ (\alpha_2 y_1(b) + \beta_2 y_1'(b))c_1 + (\alpha_2 y_2(b) + \beta_2 y_2'(b))c_2 &= \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b). \end{aligned}$$

- Use definitions for R_1, R_2 and

$$r_1 = \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a), \quad r_2 = \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b)$$

  obtain linear system of equations

$$\begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

- If the linear system is solvable, then the ODE with boundary conditions is solvable. Thus,

$$\det \begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \overset{!}{\neq} 0$$

2

Self Adjoint Differential Operators

Note: (Differential Operators)
Consider differential operators

$$D : C^2([a,b],\mathbb{R}) \to W$$

- $C^2([a,b],\mathbb{R})$ is the set of twice differentiable functions on interval $I = [a,b]$.
- W is a set of continuous functions, the image domain of D.

Definition: (Adjoint Differential Operator of n^{th} Order)
Let

$$D[y] := \sum_{\nu=0}^{n} a_\nu(x) y^{(n-\nu)}$$

be a linear differential operator of n^{th} order, $a_\nu(x)$ a given $(n-\nu)$-times differentiable functions on I, $a_0(x) \neq 0$, which is applied to $y(x)$, an arbitrary n-times differentiable function on I.
The adjoint differential operator corresponding to $D[y]$ is defined by

$$D^*[y] = \sum_{\nu=0}^{n} (-1)^{n-\nu} (a_\nu(x) y(x))^{(n-\nu)}.$$

Definition: (Self Adjoint Differential Operator of n^{th} Order)
A differential operator $D[y]$ is called self adjoint, if

$$D^*[y] = D[y]$$

holds for all n-times differentiable functions y on I.

Definition: (Sturm-Liouville Differential Operator)
Let $p(x)$ be continuously differentiable and positive on interval $I = [a,b]$, $q(x)$ and $z(x)$ continuous on I. We call

$$L[y] = (p(x)y')' + q(x)y$$

Sturm-Liouville differential operator and

$$L[y] = z(x)$$

Sturm-Liouville differential equation.

Example: A self adjoint differential operator for $n = 2$:
We already computed:

$$\begin{aligned} D[y] &= a_0(x)y'' + a_1(x)y' + a_2(x)y \\ D^*[y] &= (a_0(x)y)'' - (a_1(x)y)' + a_2(x)y \\ &= a_0(x)y'' + (2a_0'(x) - a_1(x))y' + (a_0''(x) - a_1'(x) + a_2(x))y. \end{aligned}$$

With $D^*[y] = D[y]$ it follows:

$$\begin{aligned} 2a_0' - a1 = a_1 &\Rightarrow a_0' = a_1 \\ a_0'' - a_1' + a_2 = a_2 &\Rightarrow a_0'' = a_1'. \end{aligned}$$

One obtains

$$D[y] = (a_0(x)y')' + a_2(x)y.$$

Note: (Differential Operators)
Consider differential operators

$$D : C^2([a,b],\mathbb{R}) \to W$$

- $C^2([a,b],\mathbb{R})$ is the set of twice differentiable functions on interval $I = [a,b]$.
- W is a set of continuous functions, the image domain of D.

Definition: (Adjoint Differential Operator of n^{th} Order)
Let

$$D[y] := \sum_{\nu=0}^{n} a_\nu(x) y^{(n-\nu)}$$

be a linear differential operator of n^{th} order, $a_\nu(x)$ a given $(n-\nu)$-times differentiable functions on I, $a_0(x) \neq 0$, which is applied to $y(x)$, an arbitrary n-times differentiable function on I.
The adjoint differential operator corresponding to $D[y]$ is defined by

$$D^*[y] = \sum_{\nu=0}^{n} (-1)^{n-\nu} (a_\nu(x) y(x))^{(n-\nu)}.$$

Example: $n = 2$

$$\begin{aligned} D[y] &= a_0(x)y'' + a_1(x)y' + a_2(x)y \\ D^*[y] &= (a_0(x)y)'' - (a_1(x)y)' + a_2(x)y \\ &= a_0(x)y'' + (2a_0'(x) - a_1(x))y' + (a_0''(x) - a_1'(x) + a_2(x))y. \end{aligned}$$

Definition: (Self Adjoint Differential Operator of n^{th} Order)
A differential operator $D[y]$ is called self adjoint, if

$$D^*[y] = D[y]$$

holds for all n-times differentiable functions y on I.

Example: A self adjoint differential operator for $n = 2$:
We already computed:

$$\begin{aligned} D[y] &= a_0(x)y'' + a_1(x)y' + a_2(x)y \\ D^*[y] &= (a_0(x)y)'' - (a_1(x)y)' + a_2(x)y \\ &= a_0(x)y'' + (2a_0'(x) - a_1(x))y' + (a_0''(x) - a_1'(x) + a_2(x))y. \end{aligned}$$

With $D^*[y] = D[y]$ it follows:

$$\begin{aligned} 2a_0' - a1 = a_1 \quad &\Rightarrow \quad a_0' = a_1 \\ a_0'' - a_1' + a_2 = a_2 \quad &\Rightarrow \quad a_0'' = a_1'. \end{aligned}$$

One obtains

$$D[y] = (a_0(x)y')' + a_2(x)y.$$

3

Definition: (Sturm-Liouville Differential Operator)
Let $p(x)$ be continuously differentiable and positive on interval $I = [a, b]$, $q(x)$ and $z(x)$ continuous on I. We call

$$L[y] = (p(x)y')' + q(x)y$$

Sturm-Liouville differential operator and

$$L[y] = z(x)$$

Sturm-Liouville differential equation.

Examples: (Bessel's Differential Equation)

1. For the ODE

$$y'' + e^x y' + xy = 0$$

one obtains

$$s(x) = \int \frac{e^x - 0}{1} dx = e^x,$$

and so the self adjoint form

$$(e^{e^x} y')' - xe^{e^x} y = 0.$$

2. For Bessel's ODE

$$x^2 y'' + xy' + (x^2 - n^2)y = 0, \quad (x > 0)$$

one obtains

$$s(x) = \int \frac{x - 2x}{x^2} dx = -\ln x,$$

i.e., the ODE is multiplied with $e^{s(x)} = \frac{1}{x}$; then the self adjoint form is

$$xy'' + y' + (x - \frac{n^2}{x})y = (xy')' + (x - \frac{n^2}{x})y = 0.$$

Generalization of Self Adjoint Differential Operators

Remark: (Integral Relation of Differential Operators)

- Use the scalar product for two on $I = [a, b]$ continuous functions $f, g : I \to \mathbb{R}$:

$$(f, g) = \int_a^b f(x)g(x)\, dx.$$

- By partial integration (twice) one obtains

$$\begin{aligned}(D[u], v) &= \int_a^b [a_0u'' + a_1u' + a_2u]v\, dx \\ &= \int_a^b u[(a_0v)'' - (a_1v)' + a_2v]\, dx + [u'a_0v]_a^b + [ua_1v]_a^b - [u(a_0v)']_a^b \\ &= (u, D^*[v]) + [u'a_0v]_a^b + [ua_1v]_a^b - [u(a_0v)']_a^b.\end{aligned}$$

- We may generalize the relation from linear algebra

$$(f[\mathbf{x}], \mathbf{y}) = (\mathbf{x}, f^*(\mathbf{y}))$$

for linear mappings $f : \mathbb{R}^n \to \mathbb{R}^m$ and their adjoint f^* with Euclidean scalar product to differential oberators, if

$$[u'a_0v]_a^b + [ua_1v]_a^b - [u(a_0v)']_a^b = 0.$$

- Thus u, v, a_0, a_1 need to conform to specific boundary conditions (e.g., $u(a) = u(b) = v(a) = v(b) = 0$). Then

$$(D[u], v) = (u, D^*[v]).$$

Observation: (Integral relation of self adjoint differential operators)
For self adjoint differential operators L the conditions simplify. We have:

$$\begin{aligned}(L[u], v) &= \int_a^b [(pu')' + qu]v\, dx \\ &= \int_a^b (-u'pv' + uqv)\, dx + [pu'v]_a^b \\ &= \int_a^b u[(pv')' + qv]\, dx + [pu'v]_a^b - [upv']_a^b \\ &= (u, L[v]) + [p(u'v - uv')]_a^b.\end{aligned}$$

The relation $(L[u], v) = (u, L[v])$ holds, if

$$[p(x)(u'(x)v(x) - u(x)v'(x))]_a^b = 0.$$

Proposition: (Self Adjoint Sturm-Liouville Eigen Value Problem)
Let $L[y] = (p(x)y')' + q(x)y$ be the Sturm-Liouville differential operator for $x \in [a, b]$ with cont. diff. function $p(x) > 0$, cont. diff. function $q(x)$ and cont. function $w(x) > 0$, $\lambda \in \mathbb{R}$ a parameter and $\alpha_k, \beta_k \in \mathbb{R}$ with $\alpha_k^2 + \beta_k^2 > 0$ $(k = 1, 2)$.
Then the Sturm-Liouville eigen value problem

$$L[y] + \lambda w(x)y = 0, \quad \alpha_1y(a) + \beta_1y'(a) = 0, \ \alpha_2y(b) + \beta_2y'(b) = 0$$

is self adjoint.

Non-trivial solutions $y_\lambda(x)$ corresponding to given parameters λ are called eigen-functions (if they exist). The corresponding parameters λ are called eigenvalues of the Sturm-Liouville eigen value problem.

Consider: (Boundary Value Problem)
Let us seek the solution on interval $I = [a, b]$ of

$$\begin{aligned}-L[y] &= \lambda w(x)y, \\ R_1(y) &= \alpha_1y(a) + \beta_1y'(a) = 0, \\ R_2(y) &= \alpha_2y(b) + \beta_2y'(b) = 0;\end{aligned}$$

with L Sturm-Liouville differential operator, $\lambda \in \mathbb{R}$ a parameter, $\alpha_k, \beta_k \in \mathbb{R}$ with $\alpha_k^2 + \beta_k^2 > 0$ $(k = 1, 2)$, $w(x)$ a positive continuous function on I.

Assume $C^2([a, b], \mathbb{R})$ as domain of L, more precisely the subset $M \subset C^2([a, b], \mathbb{R})$ of functions fulfilling the boundary conditions!. The elements in M are called test functions.

Definition: (General Self Adjoint Differential Operator)
Let L be a self adjoint differential operator of 2^{nd} order on $I = [a, b]$, and $M \subset C^2([a, b], \mathbb{R})$ the set of all functions fulfilling given boundary conditions $x = a$ and $x = b$ (test functions).
If for all $u, v \in M$ it holds that

$$(L[u], v) = (u, L[v]),$$

we call L (general) self adjoint differential operator on M. The corresponding boundary value problem is also called self adjoint.

Remark: (Integral Relation of Differential Operators)

- Use the scalar product for two on $I = [a, b]$ continuous functions $f, g : I \to \mathbb{R}$:

$$(f, g) = \int_a^b f(x)g(x)\ dx.$$

- By partial integration (twice) one obtains

$$\begin{aligned} (D[u], v) &= \int_a^b [a_0 u'' + a_1 u' + a_2 u] v\ dx \\ &= \int_a^b u[(a_0 v)'' - (a_1 v)' + a_2 v]\ dx + [u' a_0 v]_a^b + [u a_1 v]_a^b - [u(a_0 v)']_a^b \\ &= (u, D^*[v]) + [u' a_0 v]_a^b + [u a_1 v]_a^b - [u(a_0 v)']_a^b. \end{aligned}$$

- We may generalize the relation from linear algebra

$$(f(\mathbf{x}), \mathbf{y}) = (\mathbf{x}, f^*(\mathbf{y}))$$

  for linear mappings $f : \mathbb{R}^n \to \mathbb{R}^m$ and their adjoint f^* with Euclidean scalar product to differential oberators, if

$$[u' a_0 v]_a^b + [u a_1 v]_a^b - [u(a_0 v)']_a^b = 0.$$

- Thus u, v, a_0, a_1 need to conform to specific boundary conditions (e.g., $u(a) = u(b) = v(a) = v(b) = 0$). Then

$$(D[u], v) = (u, D^*[v]).$$

Observation: (Integral relation of self adjoint differential operators)
For self adjoint differential operators L the conditions simplify. We have:

$$\begin{aligned}(L[u],v) &= \int_a^b [(pu')' + qu]v\;dx \\ &= \int_a^b (-u'pv' + uqv)\;dx + [pu'v]_a^b \\ &= \int_a^b u[(pv')' + qv]\;dx + [pu'v]_a^b - [upv']_a^b \\ &= (u, L[v]) + [p(u'v - uv')]_a^b.\end{aligned}$$

The relation $(L[u],v) = (u,L[v])$ holds, if

$$[p(x)(u'(x)v(x) - u(x)v'(x))]_a^b = 0.$$

Consider: (Boundary Value Problem)
Let us seek the solution on interval $I = [a, b]$ of

$$\begin{aligned} -L[y] &= \lambda w(x) y, \\ R_1(y) &= \alpha_1 y(a) + \beta_1 y'(a) = 0, \\ R_2(y) &= \alpha_2 y(b) + \beta_2 y'(b) = 0; \end{aligned}$$

with L Sturm-Liouville differential operator, $\lambda \in \mathbb{R}$ a parameter, $\alpha_k, \beta_k \in \mathbb{R}$ with $\alpha_k^2 + \beta_k^2 > 0$ ($k = 1, 2$), $w(x)$ a positive continuous function on I.

Assume $C^2([a, b], \mathbb{R})$ as domain of L, more precisely the subset $M \subset C^2([a, b], \mathbb{R})$ of functions fulfilling the boundary conditions!. The elements in M are called test functions.

Definition: (General Self Adjoint Differential Operator)
Let L be a self adjoint differential operator of 2^{nd} order on $I = [a, b]$, and $M \subset C^2([a, b], \mathbb{R})$ the set of all functions fulfilling given boundary conditions $x = a$ and $x = b$ (test functions).
If for all $u, v \in M$ it holds that

$$(L[u], v) = (u, L[v]),$$

we call L (general) self adjoint differential operator on M. The corresponding boundary value problem is also called self adjoint.

Remarks: (Sufficient Conditions for Self Adjoint Differential Operators)

1. If all functions in M fulfill boundary conditions $R_1(y) = R_2(y) = 0$, then L is a self adjoint operator on M.

2. Let $p(a) = p(b) > 0$ and M the set of all functions that fulfill periodic boundary conditions; i.e.,

   $$y \in M \Rightarrow y(a) = y(b) \text{ and } y'(a) = y'(b).$$

   Then L is a self adjoint operator on M.

3. If $p(x) > 0$ for $x \in]a, b[$ and $p(a) = p(b) = 0$, then L is a self adjoint operator for all $u, v \in C^2([a, b], \mathbb{R})$.

Proposition: (Self Adjoint Sturm-Liouville Eigen Value Problem)
Let $L[y] = (p(x)y')' + q(x)y$ be the Sturm-Liouville differential operator for $x \in [a, b]$ with cont. diff. function $p(x) > 0$, cont. diff. function $q(x)$ and cont. function $w(x) > 0$, $\lambda \in \mathbb{R}$ a parameter and $\alpha_k, \beta_k \in \mathbb{R}$ with $\alpha_k^2 + \beta_k^2 > 0$ $(k = 1, 2)$.
Then the Sturm-Liouville eigen value problem

$$L[y] + \lambda w(x)y = 0, \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \; \alpha_2 y(b) + \beta_2 y'(b) = 0$$

is self adjoint.

Non-trivial solutions $y_\lambda(x)$ corresponding to given parameters λ are called eigenfunctions (if they exist). The corresponding parameters λ are called eigenvalues of the Sturm-Liouville eigen value problem.


Generalization of Self Adjoint Differential Operators
Self Adjoint Differential Operators
Boundary Value Problems
Differential Equations I
Boundary and Eigenvalue Problems
Recall: Powerseries Approach