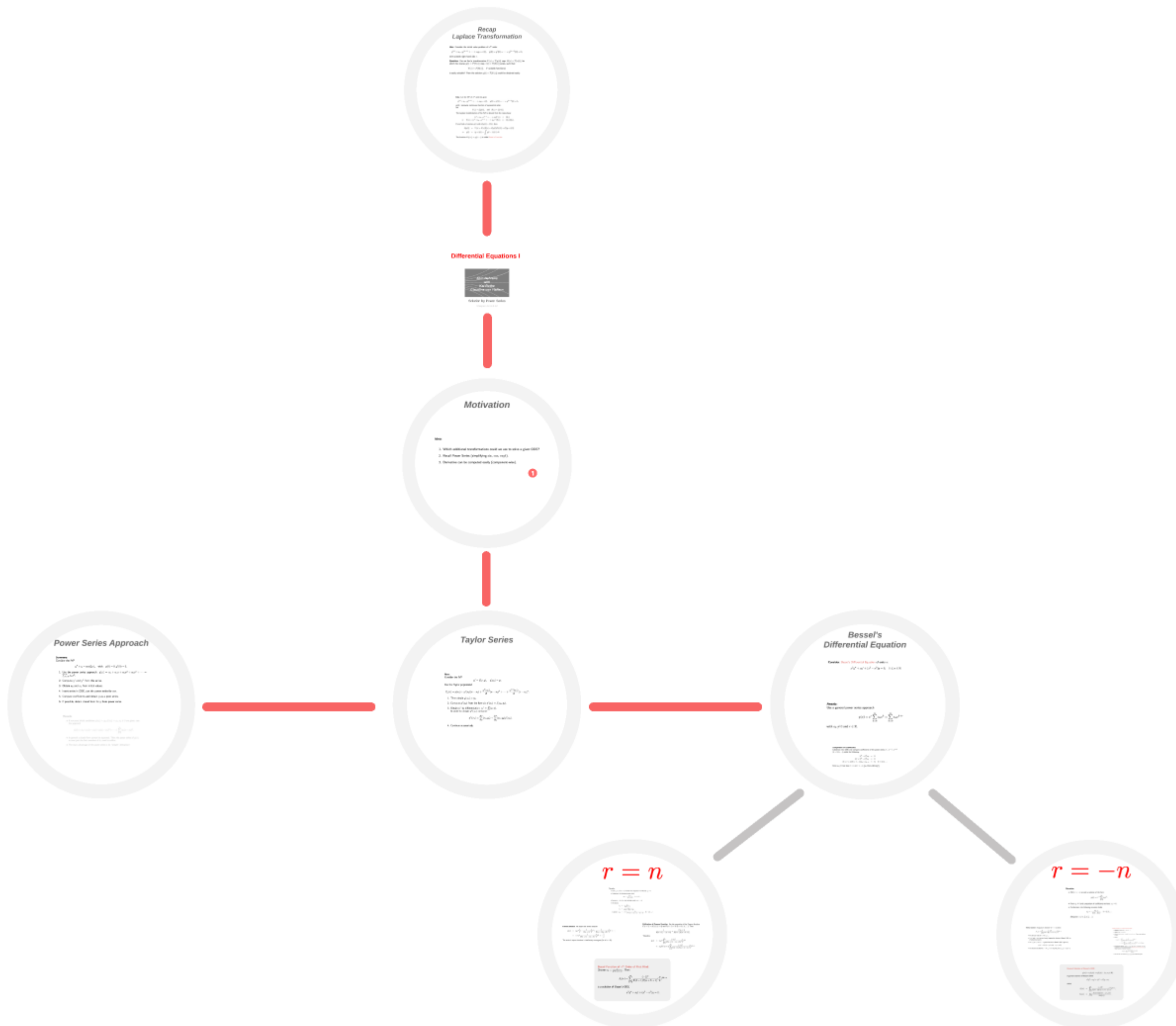
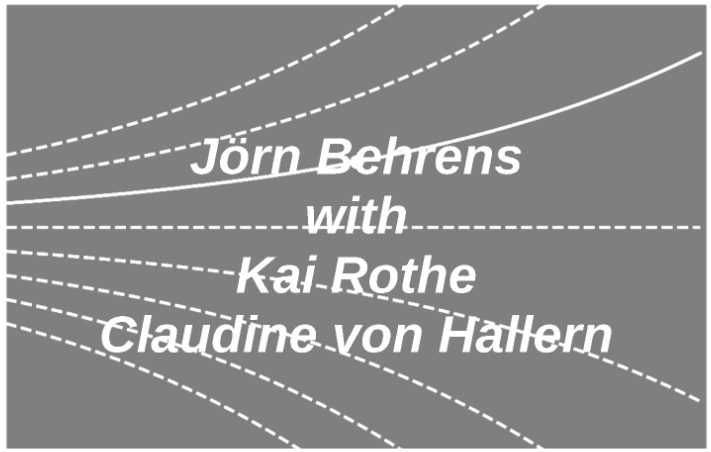

Recap
Laplace Transformation
Differential Equations I
Motivation
Power Series Approach
Taylor Series
Bessel's
Differential Equation
$r = n$
$r = -n$

Differential Equations I



Solution by Power Series

Chapters 6.11-6.12

Recap
Laplace Transformation

Idea: Consider the initial value problem of n^{th} order

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = r(t), \quad y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

with suitable right hand side r.

Question: Can we find a transformation $Y(z) = \mathcal{T}[y(t)]$ resp. $R(z) = \mathcal{T}[r(t)]$, for which the inverse $y(t) = \tilde{\mathcal{T}}[Y(z)]$ resp. $r(t) = \tilde{\mathcal{T}}[R(z)]$ exists, such that

$$Y(z) = F[R(z)], \quad F \text{ suitable functional,}$$

is easily solvable? Then the solution $y(t) = \tilde{\mathcal{T}}[Y(z)]$ could be obtained easily.

Idea: Let the IVP of n^{th} order be given

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = r(t), \quad y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

with r oiecewise continuous function of exponential order.
Set

$$Y(z) = \mathcal{L}[y(t)], \quad \text{and} \quad R(z) = \mathcal{L}[r(t)].$$

The Laplace transformation of the IVP is derived from the rules above:

$$\begin{aligned} (z^n + a_{n-1}z^{n-1} + \cdots + a_0)Y(z) &= R(z) \\ \Rightarrow \quad Y(z) = (z^n + a_{n-1}z^{n-1} + \cdots + a_0)^{-1}R(z) &=: G(z)R(z). \end{aligned}$$

If one finds a function $g(t)$ with $\mathcal{L}[g(t)] = G(t)$, then

$$\begin{aligned} \mathcal{L}[y(t)] &= Y(z) = F(z)R(z) = \mathcal{L}[g(t)]\mathcal{L}[r(t)] = \mathcal{L}[(g * r)(t)] \\ \Rightarrow \quad y(t) &= (g * r)(t) = \int_0^t g(t-\tau)r(\tau)\, d\tau. \end{aligned}$$

The function $K(t,\tau) := g(t-\tau)$ is called Green's Function.

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Question: Can we find a transformation $Y(z) = \mathcal{T}[y(t)]$ resp. $R(z) = \mathcal{T}[r(t)]$, for which the inverse $y(t) = \tilde{\mathcal{T}}[Y(z)]$ resp. $r(t) = \tilde{\mathcal{T}}[R(z)]$ exists, such that

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Motivation

Idea:

1. Which additional transformations could we use to solve a given ODE?
2. Recall Power Series (simplifying $\sin$, $\cos$, $\exp$!).
3. Derivative can be computed easily (component-wise).

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Power Series Approach

Summary:
Consider the IVP

$$y'' + y = \cos(2x), \quad \text{with} \quad y(0) = 0, y'(0) = 1.$$

1. Use the power series approach: $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k$.
2. Compute y' and y'' from this series.
3. Obtain a_0 and a_1 from initial values.
4. Insert series in ODE, use the power series for $\cos$.
5. Compare coefficients and obtain y as a pwer series.
6. If possible, obtain closed form for y from power series.

Remarks:

- If non-zero initial conditions $y(x_0) = y_0, y'(x_0) = y_1, x_0 \neq 0$ are given, use the approach

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

- In general a closed form cannot be expected. Then the power series of $y(x)$, or even just the first members of it, need to suffice.
- The main advantage of the power series is its "simple" derivation!

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Taylor Series

Idea:
Consider the IVP

$$y' = f(x, y), \quad y(x_0) = y_0$$

Use the Taylor polynomial

$$T_n(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n.$$

1. Then obtain $y(x_0) = y_0$.
2. Compute $y'(x_0)$ from the formula $y'(x_0) = f(x_0, y_0)$.
3. Obtain y'' by differentiation: $y'' = \frac{df}{dx}(x, y)$.
   In order to obtain $y''(x_0)$ compute:

   $$y''(x_0) = \frac{\partial f}{\partial x}(x_0 . y_0) + \frac{\partial f}{\partial y}(x_0, y_0) y'(x_0).$$

4. Continue successively.

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Remark: It is not possible to estimate the accuracy, because a general formula for $y^{(n)}(x_0)$ is unknown.

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Bessel's Differential Equation

Consider: Bessel's Differential Equation of order n:

$$x^2y'' + xy' + (x^2 - n^2)y = 0, \quad 0 \leq n \in \mathbb{R}.$$

Ansatz:
Use a general power series approach

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}$$

with $a_0 \neq 0$ and $r \in \mathbb{R}$.

Comparison of Coefficients:
Substitute into ODE and compare coefficients of the power series x^r, x^{r-1}, x^{r+k} $(k = 2, 3, \ldots)$ yields the following:

$$\begin{aligned} (r^2 - n^2)a_0 &= 0 \\ ((r+1)^2 - n^2)a_1 &= 0 \\ (k+r+n)(k+r-n)a_k + a_{k-2} &= 0, \quad k = 2, 3, \ldots \end{aligned}$$

Since $a_0 \neq 0$ we have $r = n$ or $r = -n$ (a_0 then arbitrary!).

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Remark: This ODE can be derived from representing the wave equation in cylindrical coordinates.

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$$r = n$$

Recursion:

- From $a_0 \neq 0$ and $r = n$ it follows with comparison of coefficients: $a_1 = 0$.
- Furthermore, the following recursion holds

$$a_k = -\frac{a_{k-2}}{k(2n+k)}, \quad k = 2, 3, \ldots$$

- Because $a_1 = 0$, all a_k with odd index vanish: $a_{2k-1} = 0$.
- One obtains

$$\begin{aligned} a_2 &= -\frac{a_0}{2^2(n+1)}, \\ a_4 &= \frac{a_0}{2^4 2(n+1)(n+2)}, \cdots \\ \text{in general} \quad a_{2k} &= (-1)^k \frac{a_0}{2^{2k} k!(n+1)(n+2)\cdots(n+k)}, \quad (k = 1, 2, \ldots) \end{aligned}$$

Formal Solution: We obtain the formal Solution

$$\begin{aligned} y(x) &= a_0 x^n \Big[1 - \frac{1}{1!(n+1)} (\frac{x}{2})^2 + \frac{1}{2!(n+1)(n+2)} (\frac{x}{2})^4 + \cdots \\ &+ (-1)^k \frac{1}{k!(n+1)\cdots(n+k)} (\frac{x}{2})^{2k} + \cdots \Big] \end{aligned}$$

The series in square brackets is indefinitely convergent (for all $x \in \mathbb{R}$).

Utilization of Gamma Function: Use the properties of the Gamma function $\Gamma(x+1) = x\Gamma(x)$ $(x > 0)$ and $\Gamma(k+1) = k!$ $(k = 0, 1, 2, \ldots)$. Then

$$\frac{1}{k!(n+1)\cdots(n+k)} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n+k+1)}$$

Therefore

$$\begin{aligned} y(x) &= a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+1)\cdots(n+k)} (\frac{x}{2})^{2k} \\ &= a_0 2^n \Gamma(n+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n+k+1)} (\frac{x}{2})^{2k+n} \end{aligned}$$

Bessel Function of n^{th} Order of First Kind:
Choose $a_0 = \frac{1}{2^n \Gamma(n+1)}$. Then

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n+k+1)} (\frac{x}{2})^{2k+n}$$

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&+ \quad \left. (-1)^k \frac{1}{k!(n+1)\cdots(n+k)} (\frac{x}{2})^{2k} + \cdots \right]
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Proof of indefinite convergence: Set

$$u = (\frac{x}{2})^2, \text{ and}$$
$$b_k = (-1)^k \frac{1}{k!(n+1)\cdots(n+k)}.$$

For the series $\sum_{k=0}^{\infty} b_k u^k$ it holds

$$\frac{|b_k|}{|b_{k+1}|} = (k+1)(n+k+1), \quad \text{so} \quad \lim_{k\to\infty} \frac{|b_k|}{|b_{k+1}|} = \infty.$$

According to proposition (2^{nd} term) the series converges for all $u \in \mathbb{R}$ and thus for all $x \in \mathbb{R}$.

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$$x^2 y'' + xy' + (x^2 - n^2) y = 0.$$

$r = -n$

Recursion:

- With $r = -n$ we seek a solution of the form

$$y(x) = x^{-n} \sum_{k=0}^{\infty} a_k x^k.$$

- From $a_0 \neq 0$ and comparison of coefficients we have: $a_1 = 0$.
- Furthermore, the following recursion holds

$$a_k = -\frac{a_{k-2}}{k(k-2n)}, \quad k = 2, 3, \ldots$$

(Required: $n \neq 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$).

Formal Solution: Analoguous to procedure for $r = n$ we obtain

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(-n+k+1)} (\frac{x}{2})^{2k-n}$$

- The formula holds for $n \neq 0, 1, 2 \ldots$.
- $J_n(x)$ and $J_{-n}(x)$ are two linearly independent solutions of Bessel's ODE, so a fundamental system.
- For $n \geq 0$, $n \neq 0, 1, 2 \ldots$ a general solution to Bessel's ODE is given by:

$$y(x) = c1 J_n(x) + c_2 J_{-n}(x) \quad (c_1, c_2 \in \mathbb{R}).$$

- For solutions bounded for $x \to 0$, $c_2 = 0$ is required, since $J_{-n}(x) = \mathcal{O}(x^{-n})$.

Bessel Function of ν^{th} Order of Second Kind:

- **Question**: Solution for $n = 0, 1, 2, \ldots$?
- **Ansatz**: Set $-n$ in $J_n(x)$.
- Observe: for $0 \leq k \leq n-1$ it is $\Gamma(-n+k+1) = \infty$. Then set the coefficient to zero!
- Obtain:

$$\begin{aligned} J_{-n}(x) &= \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(-n+k+1)} (\frac{x}{2})^{2k-n} \\ &= (-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n+k+1)} (\frac{x}{2})^{2k+n} = (-1)^n J_n(x). \end{aligned}$$

- **Fundamental System**: Since $J_{-n}(x)$ in this case lin. dependent, we need to expand $J_n(x)$ to a fundamental system (Bessel funktion of second kind, or Weber function, or Neumann function):

$$Y_n(x) = \lim_{\nu \to n} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

- One can show: lim exists and $J_n(x), Y_n(x)$ form fundamental system.

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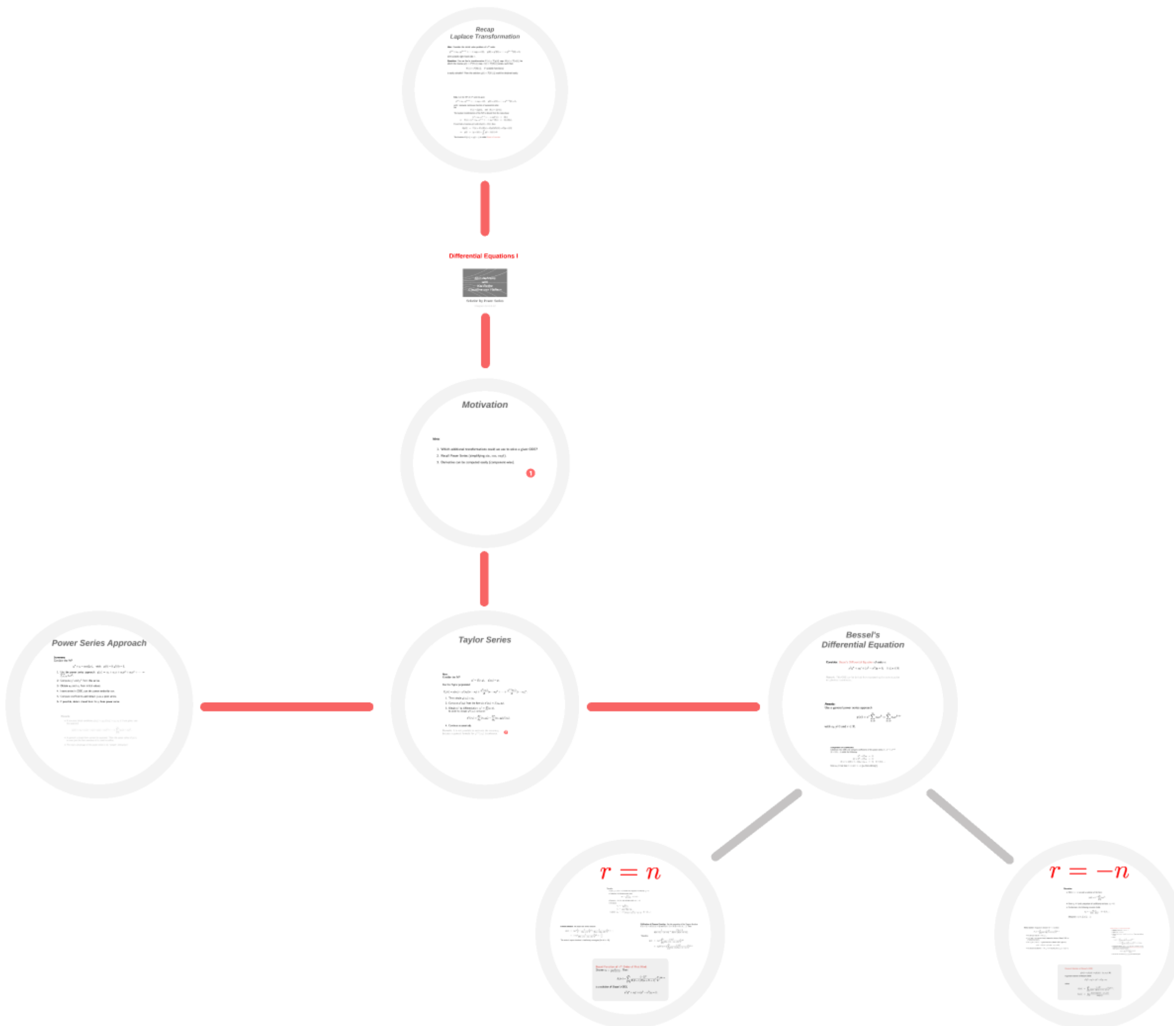

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