Differential Equations I

Week 09 / J. Behrens

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BITTE BEACHTEN SIE DIE 3G-REGEL!
PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

–VOLLSTÄNDIG GEIMPFTE
–GENESENE
–GETESTETE
(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

–FULLY VACCINATED
–RECOVERED
–TESTED
(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

① Example:

- Consider: $y'' + y = \cos(2x)$ with $y(0)=0,\ y'(0)=1$
- Ansatz: $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots = \sum_{k=0}^{\infty} a_k x^k$
- Initial Conditions: $y(0) = 0 \Rightarrow a_0 = 0$

  $y'(0) = 1 \Rightarrow a_1 = 1$
- Derivatives: $y'(x) = 1 + 2a_2 \underline{x} + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$

  $y''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \dots$
- Right hand side: $\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + - \dots$

- Substitute into ODE:

$$y'': \quad 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots$$

$$y: \quad x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\cos: \quad = 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 - \frac{64}{6!}x^6 + - \cdots$$

- Comparison of Coefficients:

$$2a_2 = 1, \quad 6a_3 + 1 = 0, \quad 12a_4 + a_2 = -\frac{4}{2!}$$

$$20a_5 + a_3 = 0$$

$$\Rightarrow \quad a_2 = \frac{1}{2}, \quad a_3 = -\frac{1}{6}, \quad a_4 = -\frac{5}{24}, \quad a_5 = \frac{1}{120}$$

$$a_0 = 0, \quad a_1 = 1$$

- Intermediate for y: $\quad y(x) = x + \frac{1}{2}x^2 - \frac{x^3}{6} - \frac{5}{24}x^4 + \frac{x^5}{120} + \cdots$

- In general (from comparison of coefficients):

$$x^{2j}: \quad a_{2j} + (2j+1)(2j+2)a_{2j+2} = (-1)^j \frac{2^{2j}}{(2j)!}$$

$$x^{2j+1}: \quad a_{2j+1} + (2j+2)(2j+3)a_{2j+3} = 0$$

- For the odd indices:

$$a_{2j+3} = -\frac{a_{2j+1}}{(2j+2)(2j+3)} \qquad \text{with } a_1 = 1$$

$$\Rightarrow a_{2j+3} = \frac{(-1)^{j+1}}{(2j+3)!} \qquad (j = 0, 1, 2, \cdots)$$

- Power series for $\sin x$:

$$a_1 x + a_3 x^3 + a_5 x^5 + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j+1)!} x^{2j+1} = \sin x$$

- So: $y(x) = \sin x + \left(\frac{1}{2}x^2 - \frac{5}{24}x^4 + \cdots\right)$
- Without detailed derivation but analogously:

$$\frac{1}{2}x^2 - \frac{5}{24}x^4 + \cdots = \frac{1}{3}(\cos x - \cos(2x))$$

- Overall solution:

$$y(x) = \sin x + \frac{1}{3}(\cos x - \cos(2x))$$

② Example: (Taylor Series)

- Consider: $y' = x + y^2 \quad , \quad y(0) = 1$
- Ansatz: Taylor-Polynomial of degree 4 (approximation to y)

$\rightarrow y'(0),\ y''(0),\ y'''(0),\ y^{IV}(0)$ compute successively.

- Derivatives:

$$y'' = 1 + 2yy'$$

$$y''' = 2y'y' + 2yy''$$

$$y^{IV} = 2y''y' + 2y'y'' + 2y'y'' + 2yy''' = 6y''y' + 2y'''y$$

- Initial Cond.:

$$y'(0) = 1$$

$$y''(0) = 3$$

$$y'''(0) = 8$$

$$y^{IV}(0) = 18 + 16 = 34$$

- Taylor-Polynomial: $y(x) \approx T_4(x) = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{34}{24}x^4$

in a neighborhood of $x = 0$