Differential Equations I

Week 08 / J. Behrens

BITTE BEACHTEN SIE DIE 3G-REGEL!
PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

–VOLLSTÄNDIG GEIMPFTE
–GENESENE
–GETESTETE
(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

–FULLY VACCINATED
–RECOVERED
–TESTED
(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

① Laplace Transform

$\mathcal{L}[f(t)] = F(z) := \int_0^\infty f(t)e^{-zt}\,dt,\ \ F: D \to \mathbb{C}$

Transformation von f'	$\mathcal{L}[f'(t)] = zF(z) - f(0)$
Transformation von $f^{(n)}$	$\mathcal{L}[f^{(n)}(t)] = z^n F(z) - \sum_{k=1}^{n} z^{n-k} f^{(k-1)}(0)$
Transformation des Integrals	$\mathcal{L}[\int_0^t f(\tau)\,d\tau] = \frac{1}{z}F(z)$
Dämpfung/Verschiebung	$\mathcal{L}[e^{-at}f(t)] = F(z+a)$
Streckung	$\mathcal{L}[f(at)] = \frac{1}{a}F(\frac{z}{a})$
Faltungsregel	$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$
Produkt mit t^n	$\mathcal{L}[(-1)^n t^n f(t)] = F^{(n)}(z)$
Einschaltvorgang bei $t = a$	$\mathcal{L}[h_a(t)f(t-a)] = e^{-az}F(z)$
$y'' + ay' + by = 0$ $y(0) = y_0, y'(0) = y_1$	$z^2F(z) - zy_0 - y_1 + azF(z) - y_0 + bF(z) = 0$ $\mathcal{L}[y(x)] = F(z) = \frac{y_0+y_1+zy_0}{z^2+az+b}$

$f(t)$	$F(z)$	$f(t)$	$F(z)$
1	$\frac{1}{z}$	$t^n,\ n \in \mathbb{N}$	$\frac{n!}{z^{n+1}}$
$t^a,\ a > -1$	$\frac{\Gamma(a+1)}{z^{a+1}}$	e^{at}	$\frac{1}{z-a}$
$\delta(t-t_0)$ bzw. $\delta(t)$	e^{-zt_0} bzw. 1	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{z}}$
$\frac{t^{n-1}e^{at}}{(n-1)!},\ n \in \mathbb{N}$	$\frac{1}{(z-a)^n}$	$\frac{t^{\beta-1}e^{at}}{\Gamma(\beta)},\ \beta > 0$	$\frac{1}{(z-a)^\beta}$
$\sin at$	$\frac{a}{z^2+a^2}$	$\cos at$	$\frac{z}{z^2+a^2}$
$e^{bt}\sin at$	$\frac{a}{(z-b)^2+a^2}$	$e^{bt}\cos at$	$\frac{z-b}{(z-b)^2+a^2}$
$\sinh at$	$\frac{a}{z^2-a^2}$	$\cosh at$	$\frac{z}{z^2-a^2}$
$e^{bt}\sinh at$	$\frac{a}{(z-b)^2-a^2}$	$e^{bt}\cosh at$	$\frac{z-b}{(z-b)^2-a^2}$
$t\sin at$	$\frac{2az}{(z^2+a^2)^2}$	$t\cos at$	$\frac{z^2-a^2}{(z^2+a^2)^2}$
$J_0(at)$	$\frac{1}{\sqrt{z^2+a^2}}$	$\int_0^t f(\tau)\,d\tau$	$\frac{1}{z}F(z)$

Example A):

- Consider: $y'' + 9y = \cos(2x)$, $y(0)=1$, $y(\frac{\pi}{2})=-1$
- Laplace Transform:

$$\mathcal{L}[y''+9y] = \mathcal{L}[y''] + 9\mathcal{L}[y] = \underbrace{\mathcal{L}[\cos(2x)]}_{\frac{z}{z^2+4}}$$

- Computing rules + Table:

$$z^2\mathcal{L}[y] - z\,y(0) - y'(0) + 9\mathcal{L}[y] = \frac{z}{z^2+4}$$

$$\overset{y(0)=1}{\Longrightarrow} \quad (z^2+9)\mathcal{L}[y] - z - y'(0) = \frac{z}{z^2+4}$$

$$\Longrightarrow \quad \mathcal{L}[y] = \frac{z+y'(0)}{(z^2+9)} + \frac{z}{(z^2+9)(z^2+4)}$$

$$\vdots$$

$$= \frac{4}{5}\,\frac{z}{z^2+9} + \frac{y'(0)}{(z^2+9)} + \frac{z}{5(z^2+4)}$$

- Table: $\mathcal{L}[y] = \frac{4}{5}\mathcal{L}[\cos(3x)] + \frac{y'(0)}{3}\mathcal{L}[\sin(3x)] + \frac{1}{5}\mathcal{L}[\cos(2x)]$
- Uniqueness:

$$y(x) = \frac{4}{5}\cos(3x) + \frac{y'(0)}{3}\sin(3x) + \frac{1}{5}\cos(2x)$$

- Determine $y'(0)$ by substituting $y(\frac{\pi}{2})=-1$

$$-1 = -\frac{y'(0)}{3} - \frac{1}{5} \quad \Longrightarrow \quad y'(0) = \frac{12}{5}$$

- Solution: $y(x) = \frac{4}{5}\cos(3x) + \frac{4}{5}\sin(3x) + \frac{1}{5}\cos(2x)$

Example B):

- Consider the system: $$u' = u + 5v \qquad u(0) = 1$$ $$v' = -(u + 3v) \qquad \text{with} \quad v(0) = 0$$

- Laplace Transform: $$-u(0) + z\mathcal{L}[u] = \mathcal{L}[u] + 5\mathcal{L}[v]$$ $$-v(0) + z\mathcal{L}[v] = -\mathcal{L}[u] - 3\mathcal{L}[v]$$

- Use initial Conditions: $$(z-1)\mathcal{L}[u] - 5\mathcal{L}[v] = 1$$ $$\mathcal{L}[u] - (z+3)\mathcal{L}[v] = 0$$

- Solution of lin. system of equations: $$\mathcal{L}[u] = \frac{z+3}{z^2+2z+2}, \qquad \mathcal{L}[v] = \frac{-1}{z^2+2z+2}$$

- quadratic extension: $$\mathcal{L}[u] = \frac{(z+1)}{(z+1)^2+1} + \frac{2}{(z+1)^2+1}$$ $$\mathcal{L}[v] = -\frac{1}{(z+1)^2+1}$$

- Table: $$\mathcal{L}[u] = \mathcal{L}[e^{-x}\cos x + 2e^{-x}\sin x]$$ $$\mathcal{L}[v] = \mathcal{L}[-e^{-x}\sin x]$$

- Uniqueness: $$u(x) = e^{-x}(\cos x + 2\sin x)$$ $$v(x) = -e^{-x}\sin x$$