Differential Equations I

Week 07 / J. Behrens

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BITTE BEACHTEN SIE DIE 3G-REGEL!
PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

–VOLLSTÄNDIG GEIMPFTE
–GENESENE
–GETESTETE
(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

–FULLY VACCINATED
–RECOVERED
–TESTED
(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

1 **Ansatz**:
Let $R_m(x)$ be a polynomial of m^{th} degree, $m \in \mathbb{N}$ and let $\alpha, \beta, \gamma \in \mathbb{R}$.
Consider right hand sides (RHS) of the form

$$R_m(x), \quad R_m(x)e^{\alpha x}, \quad R_m(x)\sin(\beta x), \quad R_m(x)\cos(\gamma x).$$

Then utilize the Approach corresponding to RHS for the particular solution.

Example: Consider $y'' + 5y' + 6y = xe^{-x}$

Ansatz according to RHS: $y_p(x) = ae^{-x} + bxe^{-x}$ (1)

Derivatives: $y_p'(x) = -ae^{-x} + be^{-x} - bxe^{-x}$ (2)

(3)
$y_p''(x) = ae^{-x} - be^{-x} - be^{-x} + bxe^{-x} = ae^{-x} - 2be^{-x} + bxe^{-x}$

Substitute into eq.:

$$\underbrace{ae^{-x}-2be^{-x}+bxe^{-x}}_{③}\underbrace{-5ae^{-x}+5be^{-x}-5bxe^{-x}}_{②}\underbrace{+6ae^{-x}+6bxe^{-x}}_{①}=xe^{-x}$$

$$\Rightarrow (2a+3b)e^{-x} + 2bxe^{-x} = xe^{-x}$$

$$\Rightarrow \underbrace{(2a+3b)}_{\overset{!}{=}0}e^{-x} + \underbrace{(2b-1)}_{\overset{!}{=}0}xe^{-x} = 0$$

Forderung erfüllt: $\overset{!}{=}0$ $\qquad$ $\overset{!}{=}0$

$$\Rightarrow 2a = -\frac{3}{2} \qquad b = \frac{1}{2}$$

$$\Rightarrow a = -\frac{3}{4}$$

general Solution: $y(x) = C_1 e^{-3x} + C_2 e^{-2x} + \frac{1}{2}xe^{-x} - \frac{3}{4}e^{-x}$

② Example:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = g(x)$$

$$g(x) = A e^{\lambda x} \quad , \; A, \lambda \in \mathbb{R}$$

Ansatz: $y_p(x) = B e^{\lambda x}$

Substitute into eq: $B\,P(\lambda)e^{\lambda x} = Ae^{\lambda x}$, $P(\lambda)$ char. Poly.

Part. Solution: $(P(\lambda) \neq 0)$

$$y_p(x) = Be^{\lambda x} = \frac{A}{P(\lambda)}e^{\lambda x}$$

Interpretation: This approach is only possible if $P(\lambda) \neq 0$, i.e. λ is not a root of the Char. Polynomial, therefore, not a solution of the homog. Eq.

$\longrightarrow$ no resonance solution.

If λ is a k-multiple root of the Char. Polynomial, then we may chose

$$y_p(x) = B x^k e^{\lambda x}$$

Then: $$y_p(x) = B x^k e^{\lambda x} = \frac{A}{P^{(k)}(\lambda)} x^k e^{\lambda x}$$

A small explicit example: $$y'' - y = 4e^x$$

Evaluating charact. Poly.: $\lambda^2 - 1 = 0 \implies \lambda_{1,2} = \pm 1$

Fundamental solution of homog. Eq: $y_1(x) = e^x$, $y_2(x) = e^{-x}$

Observation: Since the RHS of the ODE corresp. to a homog. solution
$\longrightarrow$ resonance

$\lambda_1 = 1$ has multiplicity 1

Ansatz: $$y_p(x) = a x e^x$$

Substitute: $y_p'(x) = ae^x + axe^x$, $y_p''(x) = ae^x + ae^x + axe^x = 2ae^x + axe^x$

$$\implies 2ae^x + \cancel{axe^x} - \cancel{axe^x} = 4e^x$$

$$\implies a = 2$$

General solution: $$y(x) = c_1 e^x + c_2 e^{-x} + 2xe^x$$

③ **Observation**: (Structure of Equation)
If

$$y' + xy = x$$

holds, then this also holds after differentiation, so

$$y'' + y + xy' = 1.$$

And $y(x) = ce^{-\frac{x^2}{2}} + 1$ is a solution of this ODE of 2nd order as well.

Consider:

$$y'' + xy' + y = 1 \qquad (*)$$

$$y' + xy = x \qquad (**)$$

Remark: If $(*)$ and $(**)$ describe the same problem in a mathematical modelling approach, we want to obtain the "right" solution to both eq.

Solution $(**)$: $\quad y(x) = c\,e^{-\frac{x^2}{2}} + 1$

Solution: $(*)$

homog. Diff Eq. $\quad y'' + xy' + y = 0$

one solution: $\quad u(x) = e^{-\frac{x^2}{2}}$ is already a solution

Goal: find another (second) fundamental solution $v(x)$

Idea: use ansatz $v(x) = w(x) \cdot u(x)$ (method of order reduction)

Obtain: $\quad w''u + 2w'u' + wu'' + xw'u + xwu' + wu$

$$= w''u + 2w'u' + xw'u + w\underbrace{(u'' + xu' + u)}_{=0} = 0$$

$$\Rightarrow w''u + (2u' + xu)w' = 0$$

Substitute: $\Omega = w'$

$$\Rightarrow \Omega' u + \Omega(2u' + xu) = 0$$

$$\Rightarrow \frac{\Omega'}{\Omega} = -\frac{2u' + xu}{u}$$

Insertion of $u(x) = e^{-\frac{x^2}{2}}$

$$\frac{\Omega'}{\Omega} = x \quad \Rightarrow \Omega = c^* e^{\frac{x^2}{2}}$$

Integrate: $w(x) = c^* \int_0^x e^{\frac{\xi^2}{2}} d\xi \Rightarrow v(x) = e^{-\frac{x^2}{2}}\left[\int_0^x e^{\frac{\xi^2}{2}} d\xi\right]$

Part. Solution: $y_p(x) = 1$

General Solution: $z(x) = C_1 u(x) + C_2 v(x) + 1$

It remains: u, v form a fundamental solution

$$\underline{W}(x) \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} e^{-\frac{x^2}{2}} & e^{-\frac{x^2}{2}} \int_0^x e^{\frac{\xi^2}{2}} d\xi \\ -xe^{-\frac{x^2}{2}} & 1 - xe^{-\frac{x^2}{2}} \int_0^x e^{\frac{\xi^2}{2}} d\xi \end{vmatrix}$$

For $x = 0 \Rightarrow \underline{W}(0) = 1 \neq 0$, so u, v form a fund. system.

Observation: $z(x)$ is only a solution to $(**)$ if $c_2 = 0$!