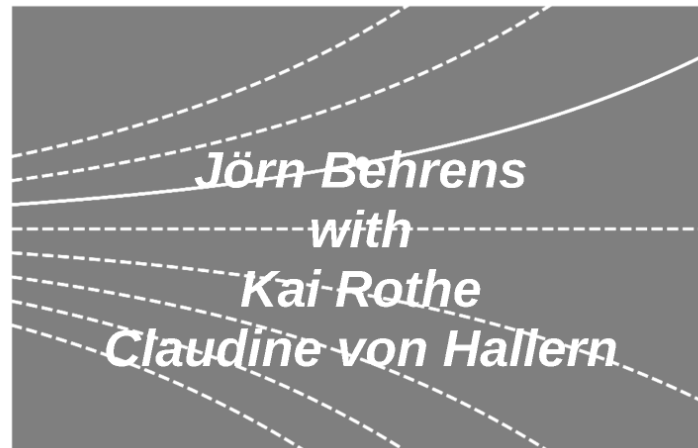


Differential Equations I



Further Methods for Solving linear ODEs

Chapters 6.8-6.9

Recapitulation

Summary:

If λ is a root of the characteristic polynomial of the homogeneous ODE, then it holds:

1. If λ has algebraic multiplicity $r \geq 1$, then

$$y_1(x) = e^{\lambda x}, \dots, y_r(x) = x^{r-1} e^{\lambda x}$$

are fundamental solutions of the ODE.

2. If $\lambda = a + ib$ is complex and has algebraic multiplicity $r \geq 1$, then

$$z_1(x) = e^{\lambda x}, \dots, z_r(x) = x^{r-1} e^{\lambda x} \quad \text{and} \quad w_1(x) = e^{\bar{\lambda}x}, \dots, w_r(x) = x^{r-1} e^{\bar{\lambda}x}$$

are complex fundamental solutions. It follows that

$$\begin{aligned} y_1(x) &= e^{ax} \cos bx, \dots, y_r(x) = x^{r-1} e^{ax} \cos bx \\ y_{r+1}(x) &= e^{ax} \sin bx, \dots, y_{2r}(x) = x^{r-1} e^{ax} \sin bx \end{aligned}$$

are real fundamental solutions of the homogeneous ODE.

Generalization: (Solution of the inhomogeneous ODE of n^{th} order)

- For the equation of n^{th} order we obtain lin. independent solutions $y_1(x), \dots, y_n(x)$ of the homogeneous equation and vary $C_1(x), \dots, C_n(x)$.

- Correspondingly, one assumes

$$C_1'(x)y_1^{(k)} + \dots + C_n'(x)y_n^{(k)} = 0 \quad (k = 0, \dots, n-2).$$

- Furthermore, we obtain

$$C_1'(x)y_1^{(n-1)} + \dots + C_n'(x)y_n^{(n-1)} = g(x).$$

- This yields a lin. system of equations for $C_1'(x), \dots, C_n'(x)$:

$$\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}$$

Remarks: Wronski-Matrix is regular, thus solvable!

- Integration yields the solution.

Summary:

If λ is a root of the characteristic polynomial of the homogeneous ODE, then it holds:

1. If λ has algebraic multiplicity $r \geq 1$, then

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Remarks: Wronski-Matrix is regular, thus solvable!

- Integration yields the solution.

ODEs with simple Inhomogeneities

Remark:
Variation of constants always yields a particular solution.
However, simplification possible, with special right hand sides!

Approaches: (Table for diverse right hand sides)
Let $R_m(x)$, $S_n(x)$, $T_k(x)$, and $Q_m(x)$ be polynomials of degree m . For RHS of the form

$$R_m(x), R_m(x)e^{\alpha x}, R_m(x)\sin(\beta x), R_m(x)\cos(\gamma x)$$

($\alpha, \beta, \gamma \in \mathbb{R}$) one may use the following approaches for particular solutions:

$g(x)$	Ansatz for $y_p(x)$	Ansatz for case of resonance
$R_m(x)$	$T_m(x)$	If a summand of the ansatz is solution of homogeneous eq. the approach is multiplied with x so often such that no summand remains as solution to the homog. eq.
$R_m(x)e^{\alpha x}$	$T_m(x)e^{\alpha x}$	
$R_m(x)\sin(\beta x)$	$T_m(x)\sin(\beta x)$	
$R_m(x)\cos(\beta x)$	$Q_m(x)\cos(\beta x)$	
Combination of these functions	Combination of these approaches	Apply above rule only to that part of the approach, which contains the case of resonance.

2

Definition: (Resonance)
If the RHS or a summand of the RHS of the ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y = g(x)$$

is a fundamental solution of the corresponding homogeneous ODE, we call this **Resonance**.

Definition: (Case of Resonance)
Consider an undamped oscillation problem

$$y'' + \omega_0^2 y = K \sin(\omega t)$$

Remark: For the equation of the form $y'' + \gamma y' + \omega_0^2 y = K \sin(\omega t)$, $\gamma > 0$, we obtain a damped system.

- Characteristic polynomial ($r = 0$): $P(r) = r^2 + \omega_0^2$
- Roots of $P(r)$: $\lambda_{1,2} = \pm i\omega_0$
- General solution of homogeneous problem: $y_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$
- Ansatz ($\omega \neq \omega_0$): $y_p(t) = A \cos(\omega t) + B \sin(\omega t)$
 $\Rightarrow y_p(t) = \frac{K}{\omega_0^2 - \omega^2} \sin(\omega t)$

- General solution of inhomogeneous problem
 $y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{K}{\omega_0^2 - \omega^2} \sin(\omega t)$

Example: (Case of Resonance)
If $\omega = \omega_0$, then $A \cos(\omega t) + B \sin(\omega t)$ is solution of the homogeneous system.

- Ansatz: $y_p(t) = At \cos(\omega t) + Bt \sin(\omega t)$
 $\Rightarrow y_p(t) = -\frac{K}{2\omega_0} \cos(\omega_0 t)$
- A **Case of Resonance** occurs, since the amplitude of y_p grows like t . The frequency ω of the RHS (external force) corresponds to the eigenfrequency ω_0 of the undamped systems.



Remark:

Variation of constants always yields a particular solution.

However, simplification possible, with special right hand sides!

Ansatz:

Let $R_m(x)$ be a polynomial of m^{th} degree, $m \in \mathbb{N}$ and let $\alpha, \beta, \gamma \in \mathbb{R}$.

Consider right hand sides (RHS) of the form

$$R_m(x), \quad R_m(x)e^{\alpha x}, \quad R_m(x) \sin(\beta x), \quad R_m(x) \cos(\gamma x).$$

Then utilize the **Approach corresponding to RHS** for the particular solution.

1

Beispiel: (Case of Resonance)

Consider an undamped oscillation problem

$$y'' + \omega_0^2 y = K \sin(\omega t).$$

Remark: For the equation of the form $y'' + ry' + \omega_0^2 y = K \sin(\omega t)$, $r > 0$, we obtain a damped system.

- Characteristic polynomial ($r = 0$): $P(\lambda) = \lambda^2 - \omega_0^2$.
- Roots of $P(\lambda)$: $\lambda_{1,2} = \pm \omega_0 i$.
- General solution of homogeneous problem: $y_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$.
- Ansatz ($\omega \neq \omega_0$): $y_p(t) = A \cos(\omega t) + B \sin(\omega t)$

$$\Rightarrow y_p(t) = \frac{K}{\omega_0^2 - \omega^2} \sin(\omega t).$$

- General solution of inhomogeneous problem:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{K}{\omega_0^2 - \omega^2} \sin(\omega t).$$

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$(\alpha, \beta, \gamma \in \mathbb{R})$ one may use the following approaches for particular solutions:

$g(x)$	Ansatz for $y_p(x)$	Ansatz for case of resonance
$R_m(x)$	$T_m(x)$	If a summand of the ansatz is solution of homogeneous eq. the approach is multiplied with x so often such that no summand remains as solution to the homog. eq.
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2

General Warning

Observation: (Structure of Solution)

- The homogeneous linear ODE of n^{th} order has exactly n linearly independent fundamental solutions.
- The inhomogeneous linear ODE of n^{th} order has general solution of the form

$$y(x) = y_h(x) + y_p(x)$$

with $y_h(x)$ linear combination of fundamental solutions and $y_p(x)$ some solution of the inhomogeneous ODE.

- So far we considered

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = g(x),$$

where $a_n(x) = 1$ was assumed. If $a_n(x) \neq 1$, but $a_n(x) \neq 0$ for all $x \in D$, we can divide by $a_n(x)$ and obtain the above structure.

- If $a_n(x) = 0$, the order of the ODE changes and thus the structure (see fundamental solution is lost).

Observation: (Structure of Equation)

Consider ODE of 1st order

$$y' + xy = x$$

- Solution of homogeneous ODE (separation of variables): $y_h(x) = ce^{-x^2/2}$.
- Particular solution (variation of constants): $y_p(x) = 1$.
- General solution: $y(x) = ce^{-x^2/2} + 1$.

Observation: (Structure of Equation)

If

$$y' + xy = x$$

holds, then this also holds after differentiation, so

$$y'' + y + xy' = 1.$$

And $y(x) = ce^{-x^2/2} + 1$ is a solution of this ODE of 2nd order as well.

3

Caution: (Structure of Equation)
 When recasting a mathematical model
 e.g. by differentiating the ODEs
 the solution set must remain the same!

Observation: (Structure of Solution)

- The homogeneous linear ODE of n^{th} order has exactly n linearly independent fundamental solutions.
- The inhomogeneous linear ODE of n^{th} order has general solution of the form

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with $y_h(x)$ linear combination of fundamental solutions and $y_p(x)$ some solution of the inhomogeneous ODE.

- So far we considered:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = g(x),$$

where $a_n(x) = 1$ was assumed. If $a_n(x) \neq 1$, but $a_n(x) \neq 0$ for all $x \in D$, w.l.o.g. one may divide by $a_n(x)$ and obtain the above structure.

- If $a_n(x) = 0$, the order of the ODE changes and thus the structure (one fundamental solution is lost).

Observation: (Structure of Equation)

Consider ODE of 1st order

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- Solution of homogeneous ODE (separation of variables): $y_h(x) = ce^{-\frac{x^2}{2}}$.
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If

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holds, then this also holds after differentiation, so

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And $y(x) = ce^{-\frac{x^2}{2}} + 1$ is a solution of this ODE of 2nd order as well.

3

Caution: (Structure of Equation)

When recasting a mathematical model

e.g. by differentiating the ODEs

the solution set must remain the same!

Recapitulation

Definition: Consider the homogeneous ODE of order n

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = 0$$
 where $p_i(x)$ are continuous functions on an interval I . Then:

1. The fundamental solutions y_1, \dots, y_n are linearly independent solutions of the ODE.
2. The general solution of the homogeneous ODE is given by

$$y_h(x) = c_1 y_1(x) + \dots + c_n y_n(x)$$
 where c_1, \dots, c_n are arbitrary constants.

Definition: Consider the inhomogeneous ODE of order n

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = q(x)$$
 where $q(x)$ is a continuous function on an interval I . Then:

1. The general solution of the inhomogeneous ODE is given by

$$y(x) = y_h(x) + y_p(x)$$
 where $y_h(x)$ is the general solution of the homogeneous ODE and $y_p(x)$ is a particular solution of the inhomogeneous ODE.

Definition: Consider the inhomogeneous ODE of order n

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = q(x)$$
 where $q(x) = P_m(x)e^{ax}$ and $P_m(x)$ is a polynomial of degree m . Then:

1. The form of the particular solution $y_p(x)$ is

$$y_p(x) = x^s Q_m(x) e^{ax}$$
 where s is the multiplicity of a as a root of the characteristic equation and $Q_m(x)$ is a polynomial of degree m .

Definition: Consider the inhomogeneous ODE of order n

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = q(x)$$
 where $q(x) = P_m(x)\cos(ax) + R_m(x)\sin(ax)$ and P_m, R_m are polynomials of degree m . Then:

1. The form of the particular solution $y_p(x)$ is

$$y_p(x) = x^s [U_m(x)\cos(ax) + V_m(x)\sin(ax)]$$
 where s is the multiplicity of $\pm ia$ as roots of the characteristic equation and U_m, V_m are polynomials of degree m .

Differential Equations I



Patrick McInnis, for teaching with a goal

General Warning

Warning: When solving a differential equation, it is essential to check the domain of the solution. The domain of the solution must be the largest interval containing the initial condition where the solution is defined.

Example: Consider the differential equation

$$y' = y^2 - 1$$
 with initial condition $y(0) = 1$. The general solution is

$$y(x) = \frac{1 + e^{2x}}{1 - e^{2x}}$$
 which is defined for $x < 0$. The solution is not defined for $x > 0$ because the denominator becomes zero. This is why the solution set must remain the same!

Observation: (Structure of Equation) When recasting a mathematical model, the solution set must remain the same!

Caution: (Structure of Equation) When recasting a mathematical model, the solution set must remain the same!

ODEs with simple Inhomogeneities

Example: Consider the differential equation

$$y'' + y = \sin(x)$$
 with initial conditions $y(0) = 0$ and $y'(0) = 1$. The general solution is

$$y(x) = \frac{1}{2}x^2 \cos(x) + \frac{1}{2}x \sin(x)$$
 which is defined for all x .

Example: Consider the differential equation

$$y'' + y = \cos(x)$$
 with initial conditions $y(0) = 1$ and $y'(0) = 0$. The general solution is

$$y(x) = \frac{1}{2}x^2 \sin(x) + \frac{1}{2}x \cos(x)$$
 which is defined for all x .

Example: Consider the differential equation

$$y'' + y = e^{ix}$$
 with initial conditions $y(0) = 0$ and $y'(0) = 1$. The general solution is

$$y(x) = \frac{1}{2}x^2 e^{ix} + \frac{1}{2}x e^{-ix}$$
 which is defined for all x .

Definition: (Structure of Equation) When recasting a mathematical model, the solution set must remain the same!