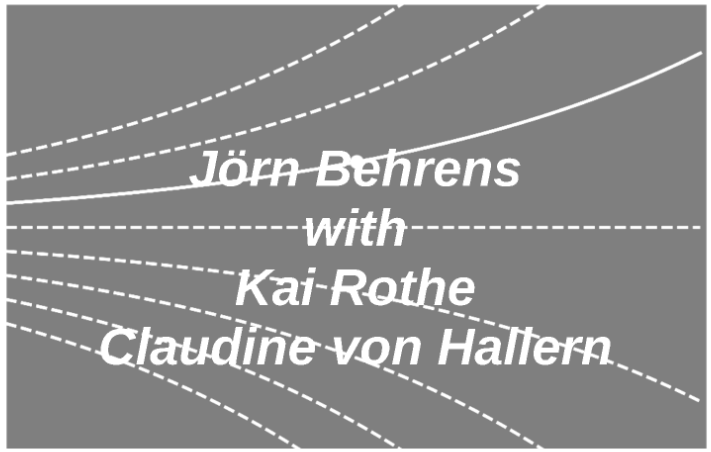
Differential Equations I



Further Methods for Solving linear ODEs

Chapters 6.8-6.9

Recapitulation

Summary:
If λ is a root of the characteristic polynomial of the homogeneous ODE, then it holds:

1. If λ has algebraic multiplicity $r \geq 1$, then

$$y_1(x) = e^{\lambda x}, \ldots, y_r(x) = x^{r-1}e^{\lambda x}$$

   are fundamental solutions of the ODE.

2. If $\lambda = a + ib$ is complex and has algebraic multiplicity $r \geq 1$, then

$$z_1(x) = e^{\lambda x}, \ldots, z_r(x) = x^{r-1}e^{\lambda x} \quad \text{and} \quad w_1(x) = e^{\bar{\lambda} x}, \ldots, w_r(x) = x^{r-1}e^{\bar{\lambda} x}$$

   are complex fundamental solutions. It follows that

$$\begin{aligned} y_1(x) &= e^{ax}\cos bx, \ldots, y_r(x) = x^{r-1}e^{ax}\cos bx \\ y_{r+1}(x) &= e^{ax}\sin bx, \ldots, y_{2r}(x) = x^{r-1}e^{ax}\sin bx \end{aligned}$$

   are real fundamental solutions of the homogeneous ODE.

Generalization: (Solution of the inhomogeneous ODE of n$^{\text{th}}$ order)

- For the equation of n^{th} order we obtain lin. independent solutions $y_1(x), \ldots, y_n(x)$ of the homogeneous equation and vary $C_1(x), \ldots, C_n(x)$.
- Correspondingly, one assumes

$$C_1'(x)y_1^{(k)} + \cdots + C_n'(x)y_n^{(k)} = 0 \quad (k = 0, \ldots, n-2).$$

- Furthermore, we obtain

$$C_1'(x)y_1^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = g(x).$$

- This yields a lin. system of equations for $C_1'(x), \ldots, C_n'(x)$:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}$$

  Remarks: Wronksi-Matrix is regular, thus solvable!

- Integration yields the solution.

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ODEs with simple Inhomogeneities

Remark:
Variation of constants always yields a partikular solution.
However, simplification possible, with special right hand sides!

Beispiel: (Case of Resonance)
Consider an undamped oscillation problem

$$y'' + \omega_0^2 y = K \sin(\omega t).$$

Remark: For the equation of the form $y'' + ry' + \omega_0^2 y = K \sin(\omega t)$, $r > 0$, we obtain a damped system.

- Charakteristic polynomial ($r = 0$): $P(\lambda) = \lambda^2 - \omega_0^2$.
- Roots of $P(\lambda)$: $\lambda_{1,2} = \pm\omega_0 i$.
- General solution of homogeneous problem: $y_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$.
- Ansatz ($\omega \neq \omega_0$): $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$

$$\Rightarrow y_p(t) = \frac{K}{\omega_0^2 - \omega^2} \sin(\omega t).$$

- General solution of inhomogeneous problem:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{K}{\omega_0^2 - \omega^2} \sin(\omega t).$$

Approaches: (Table for diverse right hand sides)
Let $R_m(x)$, $S_m(x)$, $T_m(x)$, and $Q_m(x)$ be polynomials of degree m. For RHS of the form

$$R_m(x), \quad R_m(x)e^{\alpha x}, \quad R_m(x)\sin(\beta x), \quad R_m(x)\cos(\gamma x)$$

($\alpha, \beta, \gamma \in \mathbb{R}$) one may use the following approaches for particular solutions:

$g(x)$	Ansatz for $y_p(x)$	Ansatz for case of resonance
$R_m(x)$	$T_m(x)$	If a summand of the ansatz is solution of homogeneous eq. the approach is multiplied with x so often such that no summand remains as solution to the homog. eq.
$R_m(x)e^{\alpha x}$	$T_m(x)e^{\alpha x}$	
$R_m(x)\sin(\beta x)$ $R_m(x)\cos(\beta x)$	$T_m(x)\sin(\beta x)$ $+Q_m(x)\cos(\beta x)$	
Combination of these functions	Combination of these approaches	Apply above rule only to that part of the approach, which contains the case of resonance.

2

Example: (Case of Resonance)
If $\omega = \omega_0$, then $A\cos(\omega t) + B\sin(\omega t)$ is solution of the homogeneous system.

- Ansatz: $y_p(t) = At\cos(\omega t) + Bt\sin(\omega t)$

$$\Rightarrow y_p(t) = -\frac{K}{2\omega_0} \cos(\omega_0 t).$$

- A Case of Resonance occurs, since the amplitude of y_p grows like t. The frequency ω of the RHS (external force) corresponds to the eigenfrequency ω_0 of the undamped systems.

Definition: (Resonance)
If the RHS or a summand of the RHS of the ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = g(x)$$

is a fundamental solution of the corresponding homogeneous ODE,
we call this Resonance.

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Variation of constants always yields a partikular solution.
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Ansatz:
Let $R_m(x)$ be a polynomial of m^{th} degree, $m \in \mathbb{N}$ and let $\alpha, \beta, \gamma \in \mathbb{R}$.
Consider right hand sides (RHS) of the form

$$R_m(x), \quad R_m(x)e^{\alpha x}, \quad R_m(x)\sin(\beta x), \quad R_m(x)\cos(\gamma x).$$

Then utilize the Approach corresponding to RHS for the particular solution.

1

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2

General Warning

Observation: (Structure of Solution)

- The homogeneous linear ODE of n^{th} order has exactly n linearly independent fundamental solutions.
- The inhomogeneous linear ODE of n^{th} order has general solution of the form
  $$y(x) = y_h(x) + y_p(x)$$
  with $y_h(x)$ linear combination of fundamental solutions and $y_p(x)$ some solution of the inhomogeneous ODE.
- So far we considered:
  $$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = g(x),$$
  where $a_n(x) = 1$ was assumed. If $a_n(x) \neq 1$, but $a_n(x) \neq 0$ for all $x \in D$, w.l.o.g. one may divide by $a_n(x)$ and obtain the above structure.
- If $a_n(x) = 0$, the order of the ODE changes and thus the structure (one fundamental solution is lost).

Observation: (Structure of Equation)
Consider ODE of 1st order
$$y' + xy = x$$

- Solution of homogeneous ODE (separation of variables): $y_h(x) = ce^{-\frac{x^2}{2}}$.
- Particular solution (variation of constants): $y_p(x) = 1$.
- General solution: $y(x) = ce^{-\frac{x^2}{2}} + 1$.

Observation: (Structure of Equation)
If
$$y' + xy = x$$
holds, then this also holds after differentiation, so
$$y'' + y + xy' = 1.$$
And $y(x) = ce^{-\frac{x^2}{2}} + 1$ is a solution of this ODE of 2nd order as well.

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Caution: (Structure of Equation)
When recasting a mathematical model
e.g. by differentiating the ODEs
the solution set must remain the same!

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