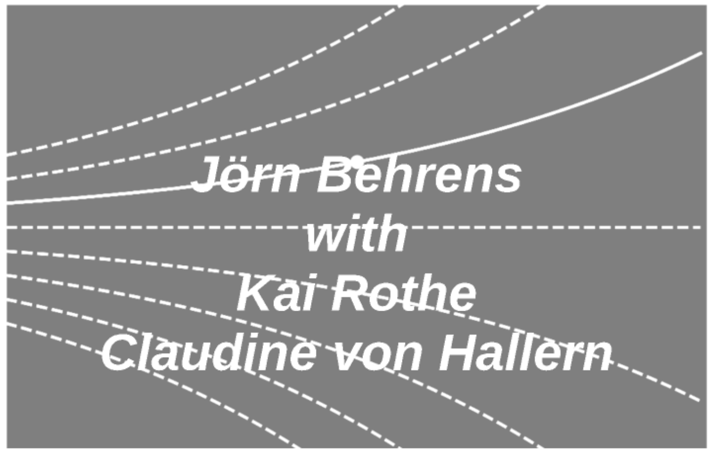
Differential Equations I



Methods for Computing Lin. Systems of ODEs

Chapter 6.8

Where we are right now

Proposition: (Solvability of linear ODE n^{th} order)
Let $a_i(x)$, $i = 0, \dots, n-1$ and $g(x)$ continuous functions on $]a, b[$.

1. Then there is a fundamental system $y_1, \dots, y_n$ on $]a, b[$ of
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$
and each solution $y_h(x)$ of this homogeneous ODE has the form
$$y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$
with suitable coefficients $c_1, \dots, c_n$.
2. Each n solutions of the homogeneous ODE form exactly one fundamental system, if $W(x) \neq 0$ for all $x \in]a, b[$.
3. Let $y_p(x)$ for $x \in]a, b[$ a particular solution of
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x)$$
If $y_1, \dots, y_n$ is fundamental system of the homogeneous ODE, then by
$$y(x) = y_p(x) + c_1 y_1(x) + \cdots + c_n y_n(x), \quad c_i \in \mathbb{R}$$
all solutions of the linear inhomogeneous ODE of n^{th} order are given.
4. If $\xi \in]a, b[$ and $\eta_0, \dots, \eta_{n-1} \in \mathbb{R}$, then there is exactly one solution $y(x)$ of the inhomogeneous ODE, which fulfills the initial conditions
$$y(\xi) = \eta_0,\ y'(\xi) = \eta_1, \dots, y^{(n-1)}(\xi) = \eta_{n-1}.$$
The solution exists in the whole interval $]a, b[$.

Proposition: (Solution of system of ODEs with constant coefficients)
Let $A = (a_{ij})$ a constant $n \times n$-matrix with $a_{ij} \in \mathbb{R}$, λ an eigen value (EVa) of A with corresponding eigen vector (EVc) $\mathbf{v}$.
Then
$$\mathbf{y} = e^{\lambda x}\mathbf{v}$$
is a solution of the homogeneous system of ODEs of 1st order $\mathbf{y}' = A\mathbf{y}$.

If A has n pairwise different EVa $\lambda_1, \dots, \lambda_n$ with corresponding EVc $\mathbf{v}_1, \dots, \mathbf{v}_n$, the solutions
$$\mathbf{y}_i = e^{\lambda_i x}\mathbf{v}_i, \quad i = 1, \dots, n$$
form a fundamental system. By linear combination
$$\mathbf{y} = \sum_{i=1}^{n} c_i e^{\lambda_i x}\mathbf{v}_i$$
all solutions of the homogeneous system of ODEs are given.

Proposition: (Variation of constants for systems)
Let:

- $\mathbf{y}_1, \dots, \mathbf{y}_n$ Fundamental system on $]a, b[$,
- Matrix $Y(x) = [\mathbf{y}_1 \dots \mathbf{y}_n]$,
- Inhomogeneous system $\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}$ with g component-wise continuous.

Then
$$\mathbf{y}_p = Y(x) \cdot \mathbf{c}(x)$$
is particular solution of the inhomogeneous system, where $\mathbf{c}(x) = \int \mathbf{c}'(x)\, dx$ and $\mathbf{c}'(x) = (c_1'(x), \dots, c_n'(x))^\top$ solution of the system of equations
$$Y(x) \cdot \mathbf{c}'(x) = \mathbf{g}.$$

Summarizing: (Matrix Exponential Solution)
The mapping $\mathbf{y}(x) = e^{xA}\mathbf{y}(0)$ is solution of the ODE system
$$\mathbf{y}' = A\mathbf{y},$$
where
$$e^{xA} = \sum_{k=0}^{\infty} \frac{x^k}{k!} A^k.$$

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where

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Reduction Principle

Principle: (Reduction of Order of Differential Equation)
Let $u(x) \neq 0$ be solution of a linear ODE of n^{th} order

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0.$$

The the Product-Ansatz $y(x) = v(x)u(x)$ yields a homogeneous linear ODE of order $n-1$ for $w := v'$:

$$w^{(n-1)} + b_{n-1}(x)w^{(n-2)} + \cdots + b_1(x)w = 0.$$

If $w_1, \ldots, w_{n-1}$ is a fundamental system of the ODE of $n-1^{\text{st}}$ order and $v_1, \ldots, v_{n-1}$ antiderivatives of $w_1, \ldots, w_{n-1}$, then

$$u, uv_1, \ldots, uv_{n-1}$$

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Linear ODEs of Order n with Constant Coefficients

Definition: (Linear Differential Equation of n^{th} Order with Constant Coefficients)
Let $a_k \in \mathbb{R}$, $k = 0, \ldots, n-1$. Then

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = g(x).$$

is a linear differential equation of n^{th} order with constant coefficients.

Observation:

- Consider the homogeneous ODE of order n.
- Ansatz:
$$y(x) = e^{\lambda x}.$$
- We have: $y^{(k)} = \frac{d^k}{dx^k} e^{\lambda x} = \lambda^k e^{\lambda x}$ and $y = e^{\lambda x} \neq 0$ for $x \in \mathbb{R}$.
- Therefore $y = e^{\lambda x}$ $(g = 0)$ is solution, iff λ is a root of
$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0.$$

Solution Approach:
Ivestigating the roos of $P(\lambda)$ yields the following cases:

1. $P(\lambda)$ has n different real roots $\lambda_1, \ldots, \lambda_n$.
2. $P(\lambda)$ has a complex root λ_k.
3. $P(\lambda)$ has a (real or complex) r-multiple root λ_1 $(r \geq 2)$.

1

Summary:
If λ is a root of the characteristic polynomial of the homogeneous ODE, then it holds:

1. If λ has algebraic multiplicity $r \geq 1$, then
$$y_1(x) = e^{\lambda x}, \ldots, y_r(x) = x^{r-1}e^{\lambda x}$$
are fundamental solutions of the ODE.
2. If $\lambda = a + ib$ is complex and has algebraic multiplicity $r \geq 1$, then
$$z_1(x) = e^{\lambda x}, \ldots, z_r(x) = x^{r-1}e^{\lambda x} \quad \text{and} \quad w_1(x) = e^{\bar{\lambda} x}, \ldots, w_r(x) = x^{r-1}e^{\bar{\lambda} x}$$
are complex fundamental solutions. It follows that
$$\begin{aligned} y_1(x) &= e^{ax}\cos bx, \ldots, y_r(x) = x^{r-1}e^{ax}\cos bx \\ y_{r+1}(x) &= e^{ax}\sin bx, \ldots, y_{2r}(x) = x^{r-1}e^{ax}\sin bx \end{aligned}$$
are real fundamental solutions of the homogeneous ODE.

Remarks:

2

Definition: (Linear Differential Equation of n^{th} Order with Constant Coefficients)
Let $a_k \in \mathbb{R}$, $k = 0, \dots, n-1$. Then

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = g(x).$$

is a linear differential equation of n^{th} order with constant coefficients.

Remarks:

- For linear ODEs or systems of ODEs with constant coefficients we have a constructive solution theory!
- Define the linear differential operator

$$L[y] := y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y,$$

  then we can write the equation from the definition in short form $L[y] = g(x)$.
- We call this equation *homogeneous of order* n, resp. *inhomogeneous of order* n (in case of uniqueness).

Observation:

- Consider the homogeneous ODE of order n.
- Ansatz:

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- Therefore $y = e^{\lambda x}$ $(g = 0)$ is solution, iff λ is a root of

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0.$$

Definition: (Characteristic Polynomial)
The polynomial $P(\lambda)$ is called characteristic polynomial of the homogeneous ODE $L[y] = 0$.
The equation for finding roos $P(\lambda) = 0$ is called associated characteristic equation.

Solution Approach:
Ivestigating the roos of $P(\lambda)$ yields the following cases:

1. $P(\lambda)$ has n different real roots $\lambda_1, \ldots, \lambda_n$.
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are real fundamental solutions of the homogeneous ODE.

Remarks:

- One finds that there exist always n linearly independent solutions $y_k(x)$ $(k = 1, \ldots, n)$.
- These solutions form a fundamental system (proof via $W(x) \neq 0$ for solutions of the form $y_k(x) = c_k e^{\lambda_k x}$)

Inhomogeneous ODE of Order n

Preliminary Remarks:

- As an example, consider

$$y'' + a(x)y' + b(x)y = g(x).$$

- Let $y_1(x)$ and $y_2(x)$ be lin. independent solutions of the homogeneous equation ($g(x) = 0$).
- It holds

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$$

- The solution of the homogeneous equation is given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

3

Summary: (Solution of the inhomogeneous ODE of 2nd order)
Consider the inhomogeneous ODE of 2nd order

$$y'' + a(x)y' + b(x)y = g(x).$$

Then the general solution can be written as

$$y(x) = \left[C_1 - \int \frac{y_2(x)g(x)}{W(x)}dx\right] y_1(x) + \left[C_2 + \int \frac{y_1(x)g(x)}{W(x)}dx\right] y_2(x).$$

Remark: Set $C_3 = C_4 = 0$ since only *some* solution is required.

Generalization: (Solution of the inhomogeneous ODE of nth order)

- For the equation of n^{th} order we obtain lin. independent solutions $y_1(x), \ldots, y_n(x)$ of the homogeneous equation and vary $C_1(x), \ldots, C_n(x)$.
- Correspondingly, one assumes

$$C_1'(x)y_1^{(k)} + \cdots + C_n'(x)y_n^{(k)} = 0 \quad (k = 0, \ldots, n-2).$$

- Furthermore, we obtain

$$C_1'(x)y_1^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = g(x).$$

- This yields a lin. system of equations for $C_1'(x), \ldots, C_n'(x)$:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}$$

Remarks: Wronksi-Matrix is regular, thus solvable!

- Integration yields the solution.

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