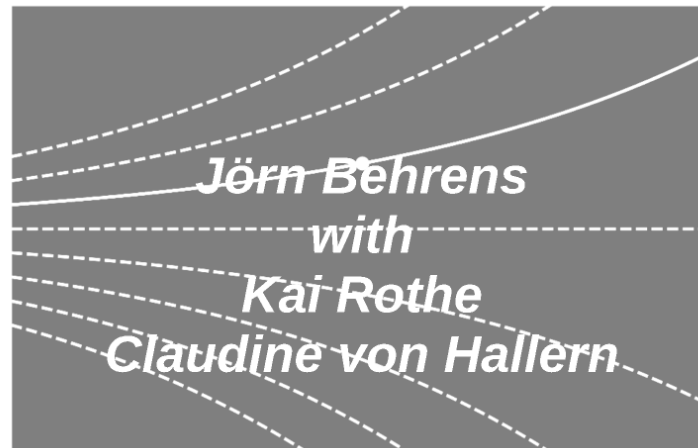


Differential Equations I



Methods for Computing Lin. Systems of ODEs

Chapter 6.8

Where we are right now

Proposition: (Solvability of linear ODE n^{th} order)

Let $a_0(x), \dots, a_{n-1}(x)$ and $g(x)$ continuous functions on $[a, b]$.

1. Then there is a fundamental system y_1, \dots, y_n on $[a, b]$ of

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0$$

and each solution $y_h(x)$ of this homogeneous ODE has the form

$$y_h(x) = c_1 y_1(x) + \dots + c_n y_n(x)$$

with suitable coefficients c_1, \dots, c_n .

2. Each n solutions of the homogeneous ODE form exactly one fundamental system, if $W(y_1, \dots, y_n) \neq 0$ for all $x \in [a, b]$.

3. Let $y_p(x)$ for $x \in [a, b]$ a particular solution of

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$$

if y_1, \dots, y_n is fundamental system of the homogeneous ODE, then by

$$y(x) = y_1(x)c_1 + \dots + y_n(x)c_n + y_p(x), \quad c_i \in \mathbb{R}$$

all solutions of the linear inhomogeneous ODE of n^{th} order are given.

4. If c_1, \dots, c_n and $y_1, \dots, y_n \in \mathbb{R}$, then there is exactly one solution $y(x)$ of the inhomogeneous ODE, which fulfils the initial conditions.

$$y(x) = y_1(x)c_1 + \dots + y_n(x)c_n + y_p(x)$$

The solution exists in the whole interval $[a, b]$.

Proposition: (Solution of system of ODEs with constant coefficients)

Let $A = (a_{ij})$ a constant $n \times n$ -matrix with $a_{ij} \in \mathbb{R}$, λ an eigen value (EV) of A with corresponding eigen vector (EVC) v .

Then

$$y = e^{\lambda x} v$$

is a solution of the homogeneous system of ODEs of n^{th} order $y' = Ay$.

If A has n pairwise different EVs $\lambda_1, \dots, \lambda_n$ with corresponding EVCs v_1, \dots, v_n , the solutions

$$y_i = e^{\lambda_i x} v_i, \quad i = 1, \dots, n$$

form a fundamental system. By linear combination

$$y = \sum_{i=1}^n c_i e^{\lambda_i x} v_i$$

all solutions of the homogeneous system of ODEs are given.

Proposition: (Variation of constants for systems)

Let:

- y_1, \dots, y_n Fundamental system on $[a, b]$,
- Matrix $Y(x) = [y_1 \dots y_n]$,
- Inhomogeneous system $y' = A(x)y + g$ with g component-wise continuous.

Then

$$y_p = Y(x) \cdot c(x)$$

is particular solution of the inhomogeneous system, where $c(x) = \int e^{\lambda(x)} dx$ and $e^{\lambda(x)} = (e^{\lambda_1(x)}, \dots, e^{\lambda_n(x)})^T$ solution of the system of equations

$$Y(x) \cdot c'(x) = g.$$

Summarizing: (Matrix Exponential Solution)

The mapping $y(x) = e^{xA}y(0)$ is solution of the ODE system

$$y' = Ay,$$

where

$$e^{xA} = \sum_{k=0}^{\infty} \frac{x^k}{k!} A^k.$$

Proposition: (Solvability of linear ODE n^{th} order)

Let $a_i(x)$, $i = 0, \dots, n - 1$ and $g(x)$ continuous functions on $]a, b[$.

1. Then there is a fundamental system y_1, \dots, y_n on $]a, b[$ of

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = 0$$

and each solution $y_h(x)$ of this homogeneous ODE has the form

$$y_h(x) = c_1y_1(x) + \dots + c_ny_n(x)$$

with suitable coefficients c_1, \dots, c_n .

2. Each n solutions of the homogeneous ODE form exactly one fundamental system, if $W(x) \neq 0$ for all $x \in]a, b[$.
3. Let $y_p(x)$ for $x \in]a, b[$ a particular solution of

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = g(x)$$

If y_1, \dots, y_n is fundamental system of the homogeneous ODE, then by

$$y(x) = y_p(x) + c_1y_1(x) + \dots + c_ny_n(x), \quad c_i \in \mathbb{R}$$

all solutions of the linear inhomogeneous ODE of n^{th} order are given.

4. If $\xi \in]a, b[$ and $\eta_0, \dots, \eta_{n-1} \in \mathbb{R}$, then there is exactly one solution $y(x)$ of the inhomogeneous ODE, which fulfills the initial conditions

$$y(\xi) = \eta_0, \quad y'(\xi) = \eta_1, \dots, y^{(n-1)}(\xi) = \eta_{n-1}.$$

The solution exists in the whole interval $]a, b[$.

Proposition: (Variation of constants for systems)

Let:

- $\mathbf{y}_1, \dots, \mathbf{y}_n$ Fundamental system on $]a, b[$,
- Matrix $Y(x) = [\mathbf{y}_1 \dots \mathbf{y}_n]$,
- Inhomogeneous system $\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}$ with g component-wise continuous.

Then

$$\mathbf{y}_p = Y(x) \cdot \mathbf{c}(x)$$

is particular solution of the inhomogeneous system, where $\mathbf{c}(x) = \int \mathbf{c}'(x) dx$ and $\mathbf{c}'(x) = (c'_1(x), \dots, c'_n(x))^\top$ solution of the system of equations

$$Y(x) \cdot \mathbf{c}'(x) = \mathbf{g}.$$

Proposition: (Solution of system of ODEs with constant coefficients)

Let $A = (a_{ij})$ a constant $n \times n$ -matrix with $a_{ij} \in \mathbb{R}$, λ an eigen value (EVA) of A with corresponding eigen vector (EVC) \mathbf{v} .

Then

$$\mathbf{y} = e^{\lambda x} \mathbf{v}$$

is a solution of the homogeneous system of ODEs of 1st order $\mathbf{y}' = A\mathbf{y}$.

If A has n pairwise different EVA $\lambda_1, \dots, \lambda_n$ with corresponding EVC $\mathbf{v}_1, \dots, \mathbf{v}_n$, the solutions

$$\mathbf{y}_i = e^{\lambda_i x} \mathbf{v}_i, \quad i = 1, \dots, n$$

form a fundamental system. By linear combination

$$\mathbf{y} = \sum_{i=1}^n c_i e^{\lambda_i x} \mathbf{v}_i$$

all solutions of the homogeneous system of ODEs are given.

Summarizing: (Matrix Exponential Solution)

The mapping $\mathbf{y}(x) = e^{xA}\mathbf{y}(0)$ is solution of the ODE system

$$\mathbf{y}' = A\mathbf{y},$$

where

$$e^{xA} = \sum_{k=0}^{\infty} \frac{x^k}{k!} A^k.$$

Reduction Principle

Principle: (Reduction of Order of Differential Equation)

Let $u(x) \neq 0$ be solution of a linear ODE of n^{th} order

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = 0.$$

The the Product-Ansatz $y(x) = v(x)u(x)$ yields a homogeneous linear ODE of order $n - 1$ for $w := v'$:

$$w^{(n-1)} + b_{n-1}(x)w^{(n-2)} + \dots + b_1(x)w = 0.$$

If w_1, \dots, w_{n-1} is a fundamental system of the ODE of $n-1^{\text{st}}$ order and v_1, \dots, v_{n-1} antiderivatives of w_1, \dots, w_{n-1} , then

$$u, uv_1, \dots, uv_{n-1}$$

form a fundamental system of the ODE of n^{th} order.

Principle: (Reduction of Order of Differential Equation)

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$$u, uv_1, \dots, uv_{n-1}$$

form a fundamental system of the ODE of n^{th} order.

Linear ODEs of Order n with Constant Coefficients

Definition (Linear Differential Equation of n^{th} Order with Constant Coefficients)
Let $a_k \in \mathbb{R}$, $k = 0, \dots, n-1$. Then

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = g(x).$$

is a linear differential equation of n^{th} order with constant coefficients.

Observation:

- Consider the homogeneous ODE of order n .
- Ansatz: $y(x) = e^{\lambda x}$.
- We have: $y^{(k)} = \frac{d^k}{dx^k} e^{\lambda x} = \lambda^k e^{\lambda x}$ and $y = e^{\lambda x} \neq 0$ for $x \in \mathbb{R}$.
- Therefore $y = e^{\lambda x}$ ($g = 0$) is solution, if λ is a root of

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0.$$

Solution Approach:

Investigating the roots of $P(\lambda)$ yields the following cases:

- $P(\lambda)$ has n different real roots $\lambda_1, \dots, \lambda_n$.
- $P(\lambda)$ has a complex root λ_k .
- $P(\lambda)$ has a (real or complex) r -multiple root λ_i ($r \geq 2$).

1

Summary

If λ is a root of the characteristic polynomial of the homogeneous ODE, then it holds:

- If λ has algebraic multiplicity $r \geq 1$, then

$$y_1(x) = e^{\lambda x}, \dots, y_r(x) = x^{r-1} e^{\lambda x}$$
 are fundamental solutions of the ODE.

- If $\lambda = a + ib$ is complex and has algebraic multiplicity $r \geq 1$, then

$$y_1(x) = e^{ax} \cos bx, \dots, y_r(x) = x^{r-1} e^{ax} \cos bx$$
 and

$$y_{r+1}(x) = e^{ax} \sin bx, \dots, y_{2r}(x) = x^{r-1} e^{ax} \sin bx$$
 are complex fundamental solutions. It follows that

$$y_1(x) = e^{ax} \cos bx, \dots, y_r(x) = x^{r-1} e^{ax} \cos bx$$

$$y_{r+1}(x) = e^{ax} \sin bx, \dots, y_{2r}(x) = x^{r-1} e^{ax} \sin bx$$

are real fundamental solutions of the homogeneous ODE.

Notes

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Definition: (Linear Differential Equation of n^{th} Order with Constant Coefficients)

Let $a_k \in \mathbb{R}$, $k = 0, \dots, n - 1$. Then

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = g(x).$$

is a linear differential equation of n^{th} order with constant coefficients.

Remarks:

- For linear ODEs or systems of ODEs with constant coefficients we have a constructive solution theory!
- Define the linear differential operator

$$L[y] := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y,$$

then we can write the equation from the definition in short form $L[y] = g(x)$.

- We call this equation *homogeneous of order n* , resp. *inhomogeneous of order n* (in case of uniqueness).

Observation:

- Consider the homogeneous ODE of order n .
- Ansatz:

$$y(x) = e^{\lambda x}.$$

- We have: $y^{(k)} = \frac{d^k}{dx^k} e^{\lambda x} = \lambda^k e^{\lambda x}$ and $y = e^{\lambda x} \neq 0$ for $x \in \mathbb{R}$.
- Therefore $y = e^{\lambda x}$ ($g = 0$) is solution, iff λ is a root of

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0.$$

Definition: (Characteristic Polynomial)

The polynomial $P(\lambda)$ is called **characteristic polynomial** of the homogeneous ODE $L[y] = 0$.

The equation for finding roots $P(\lambda) = 0$ is called associated **characteristic equation**.

Solution Approach:

Investigating the roots of $P(\lambda)$ yields the following cases:

1. $P(\lambda)$ has n different real roots $\lambda_1, \dots, \lambda_n$.
2. $P(\lambda)$ has a complex root λ_k .
3. $P(\lambda)$ has a (real or complex) r -multiple root λ_1 ($r \geq 2$).



Summary:

If λ is a root of the characteristic polynomial of the homogeneous ODE, then it holds:

1. If λ has algebraic multiplicity $r \geq 1$, then

$$y_1(x) = e^{\lambda x}, \dots, y_r(x) = x^{r-1} e^{\lambda x}$$

are fundamental solutions of the ODE.

2. If $\lambda = a + ib$ is complex and has algebraic multiplicity $r \geq 1$, then

$$z_1(x) = e^{\lambda x}, \dots, z_r(x) = x^{r-1} e^{\lambda x} \quad \text{and} \quad w_1(x) = e^{\bar{\lambda}x}, \dots, w_r(x) = x^{r-1} e^{\bar{\lambda}x}$$

are complex fundamental solutions. It follows that

$$\begin{aligned} y_1(x) &= e^{ax} \cos bx, \dots, y_r(x) = x^{r-1} e^{ax} \cos bx \\ y_{r+1}(x) &= e^{ax} \sin bx, \dots, y_{2r}(x) = x^{r-1} e^{ax} \sin bx \end{aligned}$$

are real fundamental solutions of the homogeneous ODE.

Remarks:

- One finds that there exist always n linearly independent solutions $y_k(x)$ ($k = 1, \dots, n$).
- These solutions form a fundamental system (proof via $W(x) \neq 0$ for solutions of the form $y_k(x) = c_k e^{\lambda_k x}$)



Inhomogeneous ODE of Order n

Preliminary Remarks:

- As an example, consider $y'' + a(x)y' + b(x)y = g(x)$.
- Let $y_1(x)$ and $y_2(x)$ be lin. independent solutions of the homogeneous equation ($g(x) = 0$).
- It holds $\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$.
- The solution of the homogeneous equation is given by $y(x) = C_1 y_1(x) + C_2 y_2(x)$.

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Summary: (Solution of the inhomogeneous ODE of 2nd order)
Consider the inhomogeneous ODE of 2nd order

$$y'' + a(x)y' + b(x)y = g(x).$$

Then the general solution can be written as

$$y(x) = \left[C_1 - \int \frac{y_2(x)g(x)}{W(x)} dx \right] y_1(x) + \left[C_2 + \int \frac{y_1(x)g(x)}{W(x)} dx \right] y_2(x).$$

Remark: Set $C_1 = C_2 = 0$ since only some solution is required.

Generalization (Solution of the inhomogeneous ODE of n^{th} order)

- For the equation of n^{th} order we obtain lin. independent solutions $y_1(x), \dots, y_n(x)$ of the homogeneous equation and vary $C_1(x), \dots, C_n(x)$.

Correspondingly, one assumes

$$C_k'(x)y_1^{(k)} + \dots + C_n'(x)y_n^{(k)} = 0 \quad (k = 0, \dots, n-2).$$

Furthermore, we obtain

$$C_1'(x)y_1^{(n-1)} + \dots + C_n'(x)y_n^{(n-1)} = g(x).$$

This yields a lin. system of equations for $C_1'(x), \dots, C_n'(x)$:

$$\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}$$

Remark: Wronskian Matrix is regular, thus solvable!

Integration yields the solution.

Preliminary Remarks:

- As an example, consider

$$y'' + a(x)y' + b(x)y = g(x).$$

- Let $y_1(x)$ and $y_2(x)$ be lin. independent solutions of the homogeneous equation ($g(x) = 0$).
- It holds

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$$

- The solution of the homogeneous equation is given by

$$y(x) = C_1y_1(x) + C_2y_2(x).$$

3

Summary: (Solution of the inhomogeneous ODE of 2nd order)

Consider the inhomogeneous ODE of 2nd order

$$y'' + a(x)y' + b(x)y = g(x).$$

Then the general solution can be written as

$$y(x) = \left[C_1 - \int \frac{y_2(x)g(x)}{W(x)} dx \right] y_1(x) + \left[C_2 + \int \frac{y_1(x)g(x)}{W(x)} dx \right] y_2(x).$$

Remark: Set $C_3 = C_4 = 0$ since only *some* solution is required.

Generalization: (Solution of the inhomogeneous ODE of n^{th} order)

- For the equation of n^{th} order we obtain lin. independent solutions $y_1(x), \dots, y_n(x)$ of the homogeneous equation and vary $C_1(x), \dots, C_n(x)$.
- Correspondingly, one assumes

$$C_1'(x)y_1^{(k)} + \dots + C_n'(x)y_n^{(k)} = 0 \quad (k = 0, \dots, n-2).$$

- Furthermore, we obtain

$$C_1'(x)y_1^{(n-1)} + \dots + C_n'(x)y_n^{(n-1)} = g(x).$$

- This yields a lin. system of equations for $C_1'(x), \dots, C_n'(x)$:

$$\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}$$

Remarks: Wronski-Matrix is regular, thus solvable!

- Integration yields the solution.

Linear ODEs of Order n with Constant Coefficients

Method: The characteristic equation of the homogeneous ODE is $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$.
 The roots $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix A .
 The general solution is $y = c_1 e^{\lambda_1 x} + \dots + c_n e^{\lambda_n x}$.
 For a particular solution, use the method of undetermined coefficients.

Example: Solve $y'' - 3y' + 2y = 0$.
 Characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$.
 Roots: $\lambda_1 = 1, \lambda_2 = 2$.
 General solution: $y = c_1 e^x + c_2 e^{2x}$.

Reduction Principle

Principle (Reduction of Order of Differential Equation)
 Let $y_1(x)$ be a solution of a linear ODE of order n .
 The **Reduction Principle** states that a homogeneous linear ODE of order n can be reduced to an ODE of order $n-1$ by substituting $y = y_1(x)u(x)$.
 If y_1, \dots, y_{n-1} is a fundamental system of the ODE of order $n-1$ and y_1, \dots, y_{n-1}, y_n is a fundamental system of the ODE of order n .

Differential Equations I

Methods for Computing W -Systems of ODEs

Inhomogeneous ODE of Order n

Method: The general solution is the sum of the homogeneous solution and a particular solution.
 The particular solution can be found using the method of undetermined coefficients or variation of parameters.
 Example: Solve $y'' + y = \sin(x)$.
 Homogeneous solution: $y_h = c_1 \cos(x) + c_2 \sin(x)$.
 Particular solution: $y_p = -\frac{1}{2}x \cos(x)$.
 General solution: $y = c_1 \cos(x) + c_2 \sin(x) - \frac{1}{2}x \cos(x)$.

Where we are right now

Summary of the course content:

- 1. First-order ODEs: Separable, Linear, Bernoulli, Riccati.
- 2. Second-order ODEs: Homogeneous, Inhomogeneous, Variation of Parameters.
- 3. Systems of ODEs: Eigenvalues, Eigenvectors, Matrix Exponential.
- 4. Laplace Transform: Solving ODEs using the Laplace transform.
- 5. Power Series: Solving ODEs using power series.