Differential Equations I

Week 05 / J. Behrens

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BITTE BEACHTEN SIE DIE 3G-REGEL!

PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

– VOLLSTÄNDIG GEIMPFTE
– GENESENE
– GETESTETE

(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

– FULLY VACCINATED
– RECOVERED
– TESTED

(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

1 **Preliminary Remark**: (Exponential Solution)

- For linear ODE $y' = ay$ the solution is $y(x) = e^{ax}y(0)$.
- **Aim:** Transfer this solution to a system

$$\mathbf{y}' = A\mathbf{y}.$$

- Utilize the Matrix Exponential Function

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k.$$

- e^B is a $(n \times n)$-matrix, if B is one.
- This sequence converges!

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- $e^{B} = \sum_{k=0}^{\infty} \frac{1}{k!} B^k$ $\qquad B \in \mathbb{R}/\mathbb{C}^{n\times n}$

- $B = xA \to e^{xA} = \sum_{k=0}^{\infty} \frac{x^k}{k!} A^k$ $\qquad x \in \mathbb{R}/\mathbb{C}$

- component-wise differentiation + shift of indices:

$$(e^{xA})' = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} \underbrace{A^k}_{A\cdot A^{(k-1)}} = A \sum_{k=0}^{\infty} \frac{x^k}{k!} A^k = A\, e^{xA}$$

$A \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} A^{k-1}$

- So, we obtain $\vec{y}(x) = e^{xA}\vec{y}(0)$ is solution of $\vec{y}\,' = A\vec{y}$

- Let λ EVa of A corresp. to EVc $\vec{v}$, i.e. $A\vec{v} = \lambda\vec{v}$

$$\Rightarrow e^{xA}\vec{v} = \sum_{k=0}^{\infty} \frac{x^k}{k!} A^k \vec{v} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \lambda^k \vec{v} = \sum_{k=0}^{\infty} \frac{(x\lambda)^k}{k!} \vec{v} = e^{\lambda x}\vec{v}$$

- Statement: The matrix exp. solution is equivalent to an eigenvalue Pobl.
  we don't need to comput matrix exponentials!

- For arbitrary λ: $\qquad$ we assume E is the unit matrix

$$e^{xA}\vec{v} = e^{\lambda x E + x(A-\lambda E)}\vec{v} = e^{\lambda x E}\, e^{x(A-\lambda E)}\vec{v}$$

$$\overset{\text{(remark)}}{=} e^{\lambda x} \sum_{k=0}^{\infty} \frac{x^k}{k!} (A-\lambda E)^k \vec{v}$$

remark:

$$e^{\lambda x E} = \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} E^k = e^{\lambda x} E$$

- If $\vec{v}$ is a principle vector of $(A-\lambda E)^k = 0$

$$\Rightarrow e^{xA}\vec{v} = e^{\lambda x} \sum_{j=0}^{k-1} \frac{x^j}{j!} (A-\lambda E)^j \vec{v} \quad \text{is finite!}$$

thus $y = e^{xA}\vec{v}$ is a principle vector solution.

② <u>Example:</u> Assume: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix}$ with

$\vec{y} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad$ $\vec{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- look at A and its potences:

$$A^0 = E, \quad A^1 = A, \quad A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^5 = A, \quad A^6 = A^2 \ \dots \quad A^{k+4} = A^k \qquad (k \geqslant 0)$$

- Matrix Exponential Solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \left[\boxed{\frac{t^0}{0!}} E + \frac{t}{1!} A + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \dots \right] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - + \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + - \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + \dots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

③ **Proposition**: (Variation of constants for systems)
Let:

- $\mathbf{y}_1, \ldots, \mathbf{y}_n$ Fundamental system on $]a, b[$,
- Matrix $Y(x) = [\mathbf{y}_1 \ldots \mathbf{y}_n]$,
- Inhomogeneous system $\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}$ with g component-wise continuous.

Then

$$\mathbf{y}_p = Y(x) \cdot \mathbf{c}(x)$$

is particular solution of the inhomogeneous system, where $\mathbf{c}(x) = \int \mathbf{c}'(x)\, dx$ and $\mathbf{c}'(x) = (c_1'(x), \ldots, c_n'(x))^\top$ solution of the system of equations

$$Y(x) \cdot \mathbf{c}'(x) = \mathbf{g}.$$

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- We want to consider component-wise integration for $\int \vec{c}\,'(x)\, dx$:

$$\int \vec{c}\,'(x)\, dx = \begin{pmatrix} \int c_1'(x)\, dx \\ \vdots \\ \int c_n'(x)\, dx \end{pmatrix}$$

- differentiation of y_p:

$$\begin{aligned} y_p'(x) &= \left(c_1(x)\vec{y}_1 + \cdots + c_n(x)\vec{y}_n \right)' \\ &= c_1'(x)\,\vec{y}_1 + \cdots + c_n'(x)\,\vec{y}_n + c_1(x)\vec{y}_1\,' + \cdots + c_n(x)\vec{y}_n\,' \\ &= Y(x) \cdot \vec{c}\,'(x) + c_1(x) A\vec{y}_1 + \cdots \; c_n(x) A \vec{y}_n \\ &= Y(x) \cdot \vec{c}\,'(x) + A\big(\underbrace{c_1\vec{y}_1 + \cdots + c_n\vec{y}_n}_{y_p}\big) \\ &= \underbrace{Y(x) \cdot \vec{c}\,'(x)}_{=g} + A\, y_p \quad = A\, y_p(x) + g(x) \end{aligned}$$