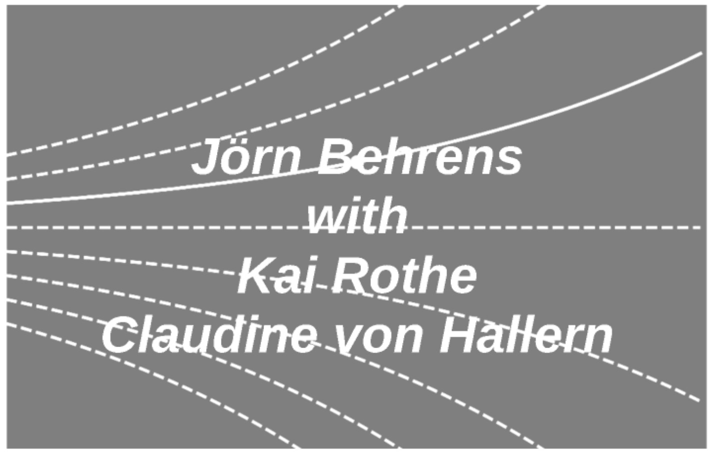
Differential Equations I



Jörn Behrens
with
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Linear Systems of ODEs – Matrix Exponential Solution
Linear ODE of order n

Chapter 6.7-6.8

Recap: Linear System of ODEs of Order 1

Definition: (Linear System of ODEs of 1st Order)
A linear system of ODEs of 1st order is an equation

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x) + \mathbf{g}, \quad A(x) = [a_{ij}(x)]_{i,j=1,\dots,n}$$

where the $a_{ij}(x)$ are functions, and $\mathbf{y}$ and $\mathbf{g}$ column vectors of n components, depending on x.

If $\mathbf{g} \equiv 0$, then the system is called homogeneous, otherwise inhomogeneous.

Remarks:

- Differential equations of order k can be reduced to systems of k equations of order 1!
  Idea: $x_1 = y$, $x_2 = y'$, $x_3 = y''$, etc.
- If $n = 1$, then we have a linear ODE.

Proposition: (Solvability of linear systems of 1st order ODEs)
Let the elements $a_{ij}(x)$ of matrix $A(x)$ and the components of $\mathbf{g}$ be continuous in intervall $]a,b[$. Further, let $x_0 \in]a,b[$ and $\mathbf{y}_0 = (y_{01},\dots,y_{0n})^\top$ be given arbitrarily. Then the initial value problem

$$\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}, \quad \mathbf{y}(x_0) = \mathbf{y}_0,$$

has a unique solution on $]a,b[$.

Proposition: (Solution of homogeneous linear systems of ODEs of 1st 0rder)
If the elements $a_{ij}(x)$ of matrix $A(x)$ are continuous in $]a,b[$, then the homogenous system

$$\mathbf{y}' = A(x)\mathbf{y}$$

has exactly n linear independent solutions on $]a,b[$.

Proposition: (Wronski Test)
Let $\mathbf{y}_1,\dots,\mathbf{y}_n$ be solutions of the system $\mathbf{y}' = A(x)\mathbf{y}$ on $]a,b[$.
If $a_{ij}(x)$ continuous in $]a,b[$, then

1. $W(x) \equiv 0$ or $W(x) \neq 0$ for all $x \in]a,b[$.
2. The solutions $\mathbf{y}_1,\dots,\mathbf{y}_n$ form a fundamental system on $]a,b[$ if and only if (iff) $W(x) \neq 0$.

Proposition: (Solution from Generalized Eigenvektor)
Let λ be eigen value of the $n \times n$-matrix A with algebraic multiplicity σ and $\mathbf{v}_1,\dots,\mathbf{v}_\sigma$ linearly independent solutions of the linear system

$$(A - \lambda E)^\sigma \mathbf{v} = \mathbf{0}.$$

Then

$$\mathbf{y}_k = e^{\lambda k} \sum_{j=0}^{\sigma-1} \frac{x^j}{j!}(A - \lambda E)^j \mathbf{v}_k \quad (k = 1,\dots,\sigma)$$

are linearly independent solutions of the 1st order system of ODEs $\mathbf{y}' = A\mathbf{y}$.

Proposition: (Solution of system of ODEs with constant coefficients)
Let $A = (a_{ij})$ a constant $n \times n$-matrix with $a_{ij} \in \mathbb{R}$, λ an eigen value (EVa) of A with corresponding eigen vector (EVc) $\mathbf{v}$.
Then

$$\mathbf{y} = e^{\lambda x}\mathbf{v}$$

is a solution of the homogeneous system of ODEs of 1st order $\mathbf{y}' = A\mathbf{y}$.

If A has n pairwise different EVa $\lambda_1,\dots,\lambda_n$ with corresponding EVc $\mathbf{v}_1,\dots,\mathbf{v}_n$, the solutions

$$\mathbf{y}_i = e^{\lambda_i x}\mathbf{v}_i, \quad i = 1,\dots,n$$

form a fundamental system. By linear combination

$$\mathbf{y} = \sum_{i=1}^{n} c_i e^{\lambda_i x}\mathbf{v}_i$$

all solutions of the homogeneous system of ODEs are given.

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Matrix Exponential Solution

Preliminary Remark: (Exponential Solution)

- For linear ODE $y' = ay$ the solution is $y(x) = e^{ax}y(0)$.
- **Aim:** Transfer this solution to a system

$$\mathbf{y}' = A\mathbf{y}.$$

- Utilize the Matrix Exponential Function

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k.$$

- e^B is a $(n \times n)$-matrix, if B is one.
- This sequence converges!

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Summarizing: (Matrix Exponential Solution)
The mapping $\mathbf{y}(x) = e^{xA}\mathbf{y}(0)$ is solution of the ODE system

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where

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Inhomogeneous Systems of Order 1

Preliminary Remark: Solve in two steps:

1. Solve homogeneous system
2. Determine particular solution

Proposition: (Solution structure of inhomogeneous system)
Let:

- Inhomogeneous linear system: $\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}$
- Homogeneous lineare system: $\mathbf{y}' = A(x)\mathbf{y}$
- Fundamental system of homogeneous system: $\mathbf{y}_1, \ldots, \mathbf{y}_n$
- Solution of homogeneous system: $\mathbf{y}_h = c_1\mathbf{y}_1 + \cdots + c_n\mathbf{y}_n$
- Some solution of inhomogeneous system: $\mathbf{y}_p$

Then each solution of the inhomogeneous linear system is of the form

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + \cdots + c_n\mathbf{y}_n = \mathbf{y}_p + \mathbf{y}_h$$

with constants $c_1, \ldots, c_n \in \mathbb{R}|\mathbb{C}$.

Proposition: (Variation of constants for systems)
Let:

- $\mathbf{y}_1, \ldots, \mathbf{y}_n$ Fundamental system on $]a, b[$,
- Matrix $Y(x) = [\mathbf{y}_1 \ldots \mathbf{y}_n]$,
- Inhomogeneous system $\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}$ with g component-wise continuous.

Then

$$\mathbf{y}_p = Y(x) \cdot \mathbf{c}(x)$$

is particular solution of the inhomogeneous system, where $\mathbf{c}(x) = \int \mathbf{c}'(x)\, dx$ and $\mathbf{c}'(x) = (c_1'(x), \ldots, c_n'(x))^\top$ solution of the system of equations

$$Y(x) \cdot \mathbf{c}'(x) = \mathbf{g}.$$

3

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3

Linear ODEs of Order n

Definition: (Linear ODE of n^{th} order)
A linear differential equation of n^{th} order is given by

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x),$$

with $a_0(x), \ldots, a_{n-1}(x), g(x)$ defined on $]a, b[$.

Remark: (Linear ODE of n^{th} order – system of 1^{st} order)
Introduce functions

$$y_1 := y,\ y_2 := y', \ldots, y_n = y^{(n-1)}$$

and obtain system of 1^{st} order with n equations:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= -a_0(x)y_1 - a_1(x)y_2 - \cdots - a_{n-1}(x)y_n + g(x). \end{aligned}$$

Proposition: (Solvability of linear ODE n^{th} order)
Let $a_i(x)$, $i = 0, \ldots, n-1$ and $g(x)$ continuous functions on $]a, b[$.

1. Then there is a fundamental system $y_1, \ldots, y_n$ on $]a, b[$ of
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$
and each solution $y_h(x)$ of this homogeneous ODE has the form
$$y_h(x) = c_1y_1(x) + \cdots + c_ny_n(x)$$
with suitable coefficients $c_1, \ldots, c_n$.
2. Each n solutions of the homogeneous ODE form exactly one fundamental system, if $W(x) \neq 0$ for all $x \in]a, b[$.
3. Let $y_p(x)$ for $x \in]a, b[$ a particular solution of
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x)$$
If $y_1, \ldots, y_n$ is fundamental system of the homogeneous ODE, then by
$$y(x) = y_p(x) + c_1y_1(x) + \cdots + c_ny_n(x), \quad c_i \in \mathbb{R}$$
all solutions of the linear inhomogeneous ODE of n^{th} order are given.
4. If $\xi \in]a, b[$ and $\eta_0, \ldots, \eta_{n-1} \in \mathbb{R}$, then there is exactly one solution $y(x)$ of the inhomogeneous ODE, which fulfills the initial conditions
$$y(\xi) = \eta_0,\ y'(\xi) = \eta_1, \ldots, y^{(n-1)}(\xi) = \eta_{n-1}.$$
The solution exists in the whole interval $]a, b[$.

Remark: (Solvability)
Consider homogeneous case: $g(x) = 0$. Then $y(x)$ is solution of the linear ODE of n^{th} order, if

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix}$$

is solution of the homogeneous system $\mathbf{y}' = A(x)\mathbf{y}$. If initial conditions

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are given for the equation of n^{th} order, then

$$\mathbf{y}(\xi) = (\eta_0, \eta_1, \ldots, \eta_{n-1})^T$$

are the initial conditions for the system.

Definition: (Wronski-Determinant of n solutions of linear ODE of n^{th} order)
Let $y_1, \ldots, y_n$ on $]a, b[$ be arbitrary solutions of the homogeneous ODE of n^{th} order. Then

$$W(x) := \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

is called Wronski-Determinant of these n solutions.
Remark: It can be shown:

$$W(x) \neq 0\ \forall x \in]a, b[\quad \Leftrightarrow \quad \exists x_0 \in]a, b[:\ W(x_0) \neq 0.$$

Definition: (Fundamental system of linear ODE) Let

- $y_1, \ldots, y_n$ solutions of the homogeneous n^{th} order OCE on $]a, b[$,
- $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ coefficients, and
- let hold: for all $x \in]a, b[$ if
$$\alpha_1y_1(x) + \alpha_2y_2(x) + \cdots + \alpha_ny_n(x) = 0$$
then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

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\end{aligned}$$

Remark: With

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0(x) & -a_1(x) & -a_2(x) & \dots & -a_{n-1}(x) \end{pmatrix}, \quad \mathbf{g}(x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(x) \end{pmatrix} \quad \text{and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

the linear ODE of n^{th} order corresponds to the system

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Remark: Differentiation (for $k = 1, 2, \dots, n-1$) of

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