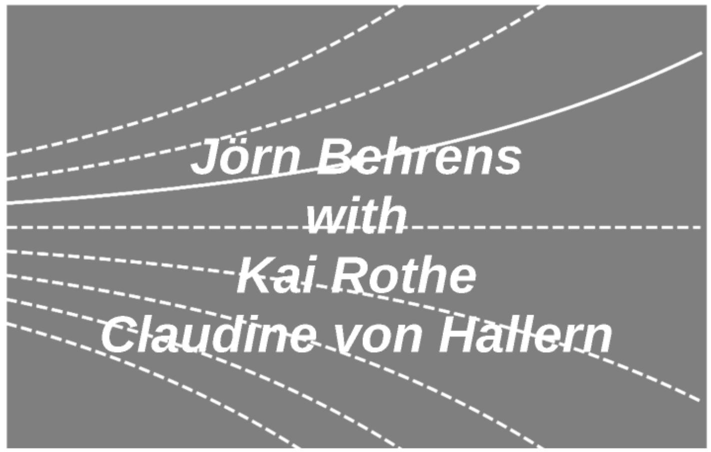
Differential Equations I


Jörn Behrens
with
Kai Rothe
Claudine von Hallern

Linear Systems of ODEs of 1st Order

Chapter 6.7

Linear Systems of Differential Equations

Motivation: Examples of Systems of ODEs

- 2-Mass-Oscillation

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1), \quad (1)$$
$$m_2 x_2'' = k_2(x_1 - x_2) - k_3 x_2, \quad (2)$$

where x_1, x_2 are coordinates of the point masses, m_1, m_2 masses and k_1, k_2, k_3 spring rates.

- Predator-prey system (Lottka-Volterra Equations)

$$x_1' = k_1 x_1 - k_2 x_1 x_2, \quad (3)$$
$$x_2' = k_3 x_1 x_2 - k_4 x_2, \quad (4)$$

where x_1, x_2 number of individuals of each species (predator, prey, resp.) and k_i $(i = 1, \ldots, 4)$ growth and mortality rates.

Definition: (Linear System of ODEs of 1st Order)
A linear system of ODEs of 1st order is an equation

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x) + \mathbf{g}, \quad A(x) = [a_{ij}(x)]_{i,j=1,\ldots,n}$$

where the $a_{ij}(x)$ are functions, and $\mathbf{y}$ and $\mathbf{g}$ column vectors of n components, depending on x.

If $\mathbf{g} \equiv 0$, then the system is called homogeneous, otherwise inhomogeneous.

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Remarks:

- Differential equations of order k can be reduced to systems of k equations of order 1!
  Idea: $x_1 = y$, $x_2 = y'$, $x_3 = y''$, etc.
- If $n = 1$, then we have a linear ODE.

Solvability

Proposition: (Solvability of linear systems of 1st order ODEs)
Let the elements $a_{ij}(x)$ of matrix $A(x)$ and the components of $\mathbf{g}$ be continuous in intervall $]a,b[$. Further, let $x_0 \in]a,b[$ and $\mathbf{y}_0 = (y_{01}, \ldots, y_{0n})^\top$ be given arbitrarily. Then the initial value problem

$$\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}, \quad \mathbf{y}(x_0) = \mathbf{y}_0,$$

has a unique solution on $]a,b[$.

Proposition: (Solution of homogeneous linear systems of ODEs of 1st 0rder)
If the elements $a_{ij}(x)$ of matrix $A(x)$ are continuous in $]a,b[$, then the homogenous system

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has exactly n linear independent solutions on $]a,b[$.

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Remarks:

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Wronski-Matrix
Wronski-Test

Question: Had we found n solutions $\mathbf{y}_1, \ldots, \mathbf{y}_n$ (a_{ij} continuous), could we then decide, if they form a fundamental system?

Proposition: (Wronski Test)
Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be solutions of the system $\mathbf{y}' = A(x)\mathbf{y}$ on $]a, b[$.
If $a_{ij}(x)$ continuous in $]a, b[$, then

1. $W(x) \equiv 0$ or $W(x) \neq 0$ for all $x \in]a, b[$.
2. The solutions $\mathbf{y}_1, \ldots, \mathbf{y}_n$ form a fundamental system on $]a, b[$ if and only if (iff) $W(x) \neq 0$.

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Holonomic Solutions

Proposition: (Holonomic Solution)
Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be a fundamental system on $]a, b[$ of

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Then any solution $\mathbf{y}$ on $]a, b[$ can be written in the form

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{y}_i, \quad \text{const.} \equiv c_i \in \mathbb{R} \text{ or } \mathbb{C}.$$

This above $\mathbf{y}$ is called holonomic solution of the homogeneous system of differential equations.

Remark: The linear combinations are solutions of $\mathbf{y}' = A(x)\mathbf{y}$, since $\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{y}_i$ yields:

$$\mathbf{y}' = \sum_{i=1}^{n} c_i \mathbf{y}_i' = \sum_{i=1}^{n} c_i A(x)\mathbf{y}_i = A(x)\sum_{i=1}^{n} c_i \mathbf{y}_i = A(x)\mathbf{y}.$$

Proposition: (Solution from Generalized Eigenvektor)
Let λ be eigen value of the $n \times n$-matrix A with algebraic multiplicity σ and $\mathbf{v}_1, \ldots, \mathbf{v}_\sigma$ linearly independent solutions of the linear system

$$(A - \lambda E)^\sigma \mathbf{v} = \mathbf{0}.$$

Then

$$\mathbf{y}_k = e^{\lambda k} \sum_{j=0}^{\sigma-1} \frac{x^j}{j!}(A - \lambda E)^j \mathbf{v}_k \quad (k = 1, \ldots, \sigma)$$

are linearly independent solutions of the 1^{st} order system of ODEs $\mathbf{y}' = A\mathbf{y}$.

Proposition: (Solution of system of ODEs with constant coefficients)
Let $A = (a_{ij})$ a constant $n \times n$-matrix with $a_{ij} \in \mathbb{R}$, λ an eigen value (EVa) of A with corresponding eigen vector (EVc) $\mathbf{v}$.
Then

$$\mathbf{y} = e^{\lambda x}\mathbf{v}$$

is a solution of the homogeneous system of ODEs of 1^{st} order $\mathbf{y}' = A\mathbf{y}$.

If A has n pairwise different EVa $\lambda_1, \ldots, \lambda_n$ with corresponding EVc $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the solutions

$$\mathbf{y}_i = e^{\lambda_i x}\mathbf{v}_i, \quad i = 1, \ldots, n$$

form a fundamental system. By linear combination

$$\mathbf{y} = \sum_{i=1}^{n} c_i e^{\lambda_i x}\mathbf{v}_i$$

all solutions of the homogeneous system of ODEs are given.

Remarks: (Application of Linear Algebra)

- Matrices not always have pairwise different EVa, multiplicity > 1 is possible. Therefore, construction of a fundamental system is only possible, if algebraic and geometric multiplicity correspond.
- If the algebraic multiplicity $\sigma_k < n$ corresponding to EVa λ_k equals the geometric multiplicity, then there exists σ_k linearly independent EVc $\mathbf{v}_{k_1}, \ldots, \mathbf{v}_{k_{\sigma_k}}$ and thus σ_k linearly independent solutions

$$\mathbf{y}_{k_1} = e^{\lambda_k x}\mathbf{v}_{k_1}, \ldots, \mathbf{y}_{k_{\sigma_k}} = e^{\lambda_k x}\mathbf{v}_{k_{\sigma_k}}.$$

- In this case for m different EVa $\lambda_1, \ldots, \lambda_m$ with multiplicities $\sigma_1, \ldots, \sigma_m$ there are n linearly independent solutions (fundamental system)

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since $\sum_{k=1}^{m} \sigma_k = n$.

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