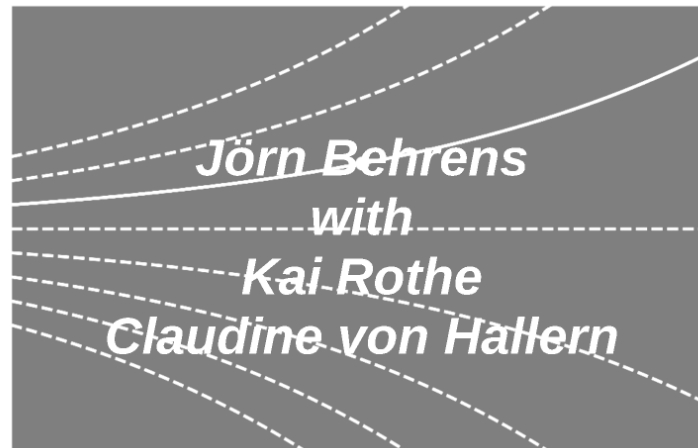


# Differential Equations I



Linear Systems of ODEs of 1st Order

Chapter 6.7

# Linear Systems of Differential Equations

**Motivation:** Examples of Systems of ODEs

- 2-Mass-Oscillation

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1), \quad (1)$$

$$m_2 x_2'' = k_2(x_1 - x_2) - k_3 x_2, \quad (2)$$

where  $x_1, x_2$  are coordinates of the point masses,  $m_1, m_2$  masses and  $k_1, k_2, k_3$  spring rates.

- Predator-prey system (Lottka-Volterra Equations)

$$x_1' = k_1 x_1 - k_2 x_1 x_2, \quad (3)$$

$$x_2' = k_3 x_1 x_2 - k_4 x_2, \quad (4)$$

where  $x_1, x_2$  number of individuals of each species (predator, prey, resp.) and  $k_i$  ( $i = 1, \dots, 4$ ) growth and mortality rates.

**Definition:** (Linear System of ODEs of 1<sup>st</sup> Order)

A linear system of ODEs of 1<sup>st</sup> order is an equation

$$y'(x) = A(x)y(x) + g, \quad A(x) = [a_{ij}(x)]_{i,j=1,\dots,n}$$

where the  $a_{ij}(x)$  are functions, and  $y$  and  $g$  column vectors of  $n$  components, depending on  $x$ .

If  $g \equiv 0$ , then the system is called **homogeneous**, otherwise **inhomogeneous**.

## Motivation: Examples of Systems of ODEs

- 2-Mass-Oscillation

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1), \quad (1)$$

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**Definition:** (Linear System of ODEs of 1<sup>st</sup> Order)

A **linear system of ODEs of 1<sup>st</sup> order** is an equation

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x) + \mathbf{g}, \quad A(x) = [a_{ij}(x)]_{i,j=1,\dots,n}$$

where the  $a_{ij}(x)$  are functions, and  $\mathbf{y}$  and  $\mathbf{g}$  column vectors of  $n$  components, depending on  $x$ .

If  $\mathbf{g} \equiv 0$ , then the system is called **homogeneous**, otherwise **inhomogeneous**.

**Remarks:**

- Differential equations of order  $k$  can be reduced to systems of  $k$  equations of order 1!  
Idea:  $x_1 = y$ ,  $x_2 = y'$ ,  $x_3 = y''$ , etc.
- If  $n = 1$ , then we have a linear ODE.

# Solvability

**Proposition:** (Solvability of linear systems of 1<sup>st</sup> order ODEs)

Let the elements  $a_{ij}(x)$  of matrix  $A(x)$  and the components of  $\mathbf{g}$  be continuous in interval  $]a, b[$ . Further, let  $x_0 \in ]a, b[$  and  $\mathbf{y}_0 = (y_{01}, \dots, y_{0n})^T$  be given arbitrarily. Then the initial value problem

$$\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}, \quad \mathbf{y}(x_0) = \mathbf{y}_0,$$

has a unique solution on  $]a, b[$ .

**Proposition:** (Solution of homogeneous linear systems of ODEs of 1<sup>st</sup> order)

If the elements  $a_{ij}(x)$  of matrix  $A(x)$  are continuous in  $]a, b[$ , then the homogeneous system

$$\mathbf{y}' = A(x)\mathbf{y}$$

has exactly  $n$  linear independent solutions on  $]a, b[$ .

—

**Proposition:** (Solvability of linear systems of 1<sup>st</sup> order ODEs)

Let the elements  $a_{ij}(x)$  of matrix  $A(x)$  and the components of  $\mathbf{g}$  be continuous in interval  $]a, b[$ . Further, let  $x_0 \in ]a, b[$  and  $\mathbf{y}_0 = (y_{01}, \dots, y_{0n})^\top$  be given arbitrarily. Then the initial value problem

$$\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}, \quad \mathbf{y}(x_0) = \mathbf{y}_0,$$

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**Proposition:** (Solution of homogeneous linear systems of ODEs of 1<sup>st</sup> Order)  
If the elements  $a_{ij}(x)$  of matrix  $A(x)$  are continuous in  $]a, b[$ , then the homogenous system

$$\mathbf{y}' = A(x)\mathbf{y}$$

has exactly  $n$  linear independent solutions on  $]a, b[$ .

Remarks:

- A system of  $n$  linearly independent solutions of the system is called **fundamental system** or **basis** of solutions.
- The elements of the basis are called **fundamental solutions** (or **holonomic functions**).

# Wronski-Matrix

## Wronski-Test

**Question:** Had we found  $n$  solutions  $y_1, \dots, y_n$  ( $a_{ij}$  continuous), could we then decide, if they form a fundamental system?

**Proposition:** (Wronski Test)

Let  $y_1, \dots, y_n$  be solutions of the system  $y' = A(x)y$  on  $]a, b[$ .

If  $a_{ij}(x)$  continuous in  $]a, b[$ , then

1.  $W(x) \equiv 0$  or  $W(x) \neq 0$  for all  $x \in ]a, b[$ .
2. The solutions  $y_1, \dots, y_n$  form a fundamental system on  $]a, b[$  if and only if (iff)  $W(x) \neq 0$ .



**Question:** Had we found  $n$  solutions  $\mathbf{y}_1, \dots, \mathbf{y}_n$  ( $a_{ij}$  continuous), could we then decide, if they form a fundamental system?

**Definition:** (Wronski Matrix and Wronski Determinant)

The **Wronski matrix**  $Y(x)$  is formed by the columns of the fundamental system:

$$Y(x) := [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n].$$

We define the **Wronski determinant** of the system of solutions  $\mathbf{y}_1, \dots, \mathbf{y}_n$  of the system  $\mathbf{y}' = A(x)\mathbf{y}$  as

$$W(x) := \det Y(x).$$

**Proposition:** (Wronski Test)

Let  $\mathbf{y}_1, \dots, \mathbf{y}_n$  be solutions of the system  $\mathbf{y}' = A(x)\mathbf{y}$  on  $]a, b[$ .

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2. The solutions  $\mathbf{y}_1, \dots, \mathbf{y}_n$  form a fundamental system on  $]a, b[$  if and only if (iff)  $W(x) \neq 0$ .

# Holonomic Solutions

**Proposition: (Holonomic Solution)**

Let  $y_1, \dots, y_n$  be a fundamental system on  $]a, b[$  of

$$y' = A(x)y.$$

Then any solution  $y$  on  $]a, b[$  can be written in the form

$$y = \sum_{i=1}^n c_i y_i, \quad \text{const. } \equiv c_i \in \mathbb{R} \text{ or } \mathbb{C}.$$

This above  $y$  is called **holonomic solution** of the homogeneous system of differential equations.

**Remark:** The linear combinations are solutions of  $y' = A(x)y$ , since  $y = \sum_{i=1}^n c_i y_i$  yields

$$y' = \sum_{i=1}^n c_i y_i' = \sum_{i=1}^n c_i A(x)y_i = A(x) \sum_{i=1}^n c_i y_i = A(x)y.$$

**Proposition: (Solution from Generalized Eigenvector)**

Let  $\lambda$  be eigen value of the  $n \times n$ -matrix  $A$  with algebraic multiplicity  $\sigma$  and  $v_1, \dots, v_\sigma$  linearly independent solutions of the linear system

$$(A - \lambda E)^{\sigma} v = 0.$$

Then

$$y_k = e^{\lambda x} \sum_{j=0}^{\sigma-k} \frac{x^j}{j!} (A - \lambda E)^j v_k \quad (k = 1, \dots, \sigma)$$

are linearly independent solutions of the 1<sup>st</sup> order system of ODEs  $y' = Ay$ .

**Proposition: (Solution of system of ODEs with constant coefficients)**

Let  $A = (a_{ij})$  a constant  $n \times n$ -matrix with  $a_{ij} \in \mathbb{R}$ ,  $\lambda$  an eigen value (EV) of  $A$  with corresponding eigen vector (EV)  $v$ .

Then

$$y = e^{\lambda x} v$$

is a solution of the homogeneous system of ODEs of 1<sup>st</sup> order  $y' = Ay$ .

If  $A$  has  $n$  pairwise different EVs  $\lambda_1, \dots, \lambda_n$  with corresponding EVs  $v_1, \dots, v_n$ , the solutions

$$y_i = e^{\lambda_i x} v_i, \quad i = 1, \dots, n$$

form a fundamental system. By linear combination

$$y = \sum_{i=1}^n c_i e^{\lambda_i x} v_i$$

all solutions of the homogeneous system of ODEs are given.

**Remarks: (Solutions of linear Algebra)**

- If the algebraic multiplicity of the eigenvalue  $\lambda$  is  $\sigma$ , then there are  $\sigma$  linearly independent generalized eigenvectors  $v_1, \dots, v_\sigma$ .
- If the algebraic multiplicity  $\sigma = 1$  corresponding to EV  $\lambda$ , then the given system has the fundamental system  $y_1 = e^{\lambda x} v_1$  and that is, linearly independent solution.

$$y_k = e^{\lambda x} \sum_{j=0}^{\sigma-k} \frac{x^j}{j!} (A - \lambda E)^j v_k$$

- In the case for  $n$  different EVs  $\lambda_1, \dots, \lambda_n$  with multiplicity  $\sigma_1, \dots, \sigma_n$ , there are  $n$  linearly independent solutions  $y_1, \dots, y_n$ .

$$y_i = e^{\lambda_i x} \sum_{j=0}^{\sigma_i-1} \frac{x^j}{j!} (A - \lambda_i E)^j v_i \quad (i = 1, \dots, n)$$

and  $\sum_{i=1}^n \sigma_i = n$ .

**Proposition:** (Holonomic Solution)

Let  $\mathbf{y}_1, \dots, \mathbf{y}_n$  be a fundamental system on  $]a, b[$  of

$$\mathbf{y}' = A(x)\mathbf{y}.$$

Then any solution  $\mathbf{y}$  on  $]a, b[$  can be written in the form

$$\mathbf{y} = \sum_{i=1}^n c_i \mathbf{y}_i, \quad \text{const.} \equiv c_i \in \mathbb{R} \text{ or } \mathbb{C}.$$

This above  $\mathbf{y}$  is called **holonomic solution** of the homogeneous system of differential equations.

**Remark:** The linear combinations are solutions of  $\mathbf{y}' = A(x)\mathbf{y}$ , since  $\mathbf{y} = \sum_{i=1}^n c_i \mathbf{y}_i$  yields:

$$\mathbf{y}' = \sum_{i=1}^n c_i \mathbf{y}'_i = \sum_{i=1}^n c_i A(x) \mathbf{y}_i = A(x) \sum_{i=1}^n c_i \mathbf{y}_i = A(x) \mathbf{y}.$$

1

**Proposition:** (Solution of system of ODEs with constant coefficients)

Let  $A = (a_{ij})$  a constant  $n \times n$ -matrix with  $a_{ij} \in \mathbb{R}$ ,  $\lambda$  an eigen value (EVa) of  $A$  with corresponding eigen vector (EVC)  $\mathbf{v}$ .

Then

$$\mathbf{y} = e^{\lambda x} \mathbf{v}$$

is a solution of the homogeneous system of ODEs of 1<sup>st</sup> order  $\mathbf{y}' = A\mathbf{y}$ .

If  $A$  has  $n$  pairwise different EVa  $\lambda_1, \dots, \lambda_n$  with corresponding EVC  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , the solutions

$$\mathbf{y}_i = e^{\lambda_i x} \mathbf{v}_i, \quad i = 1, \dots, n$$

form a fundamental system. By linear combination

$$\mathbf{y} = \sum_{i=1}^n c_i e^{\lambda_i x} \mathbf{v}_i$$

all solutions of the homogeneous system of ODEs are given.

**Remarks:** (Application of Linear Algebra)

- Matrices not always have pairwise different EVa, multiplicity  $> 1$  is possible. Therefore, construction of a fundamental system is only possible, if algebraic and geometric multiplicity correspond.
- If the algebraic multiplicity  $\sigma_k < n$  corresponding to EVa  $\lambda_k$  equals the geometric multiplicity, then there exists  $\sigma_k$  linearly independent EVc  $\mathbf{v}_{k_1}, \dots, \mathbf{v}_{k_{\sigma_k}}$ , and thus  $\sigma_k$  linearly independent solutions

$$\mathbf{y}_{k_1} = e^{\lambda_k x} \mathbf{v}_{k_1}, \dots, \mathbf{y}_{k_{\sigma_k}} = e^{\lambda_k x} \mathbf{v}_{k_{\sigma_k}}.$$

- In this case for  $m$  different EVa  $\lambda_1, \dots, \lambda_m$  with multiplicities  $\sigma_1, \dots, \sigma_m$  there are  $n$  linearly independent solutions (fundamental system)

$$\mathbf{y}_{k_1} = e^{\lambda_k x} \mathbf{v}_{k_1}, \dots, \mathbf{y}_{k_{\sigma_k}} = e^{\lambda_k x} \mathbf{v}_{k_{\sigma_k}}, \quad (k = 1, \dots, m),$$

since  $\sum_{k=1}^m \sigma_k = n$ .

2

**Proposition:** (Solution from Generalized Eigenvektor)

Let  $\lambda$  be eigen value of the  $n \times n$ -matrix  $A$  with algebraic multiplicity  $\sigma$  and  $\mathbf{v}_1, \dots, \mathbf{v}_\sigma$  linearly independent solutions of the linear system

$$(A - \lambda E)^\sigma \mathbf{v} = \mathbf{0}.$$

Then

$$\mathbf{y}_k = e^{\lambda k} \sum_{j=0}^{\sigma-1} \frac{x^j}{j!} (A - \lambda E)^j \mathbf{v}_k \quad (k = 1, \dots, \sigma)$$

are linearly independent solutions of the 1<sup>st</sup> order system of ODEs  $\mathbf{y}' = A\mathbf{y}$ .



## Linear Systems of Differential Equations

**Motivation:** Example of Systems of ODEs

• 2-Mass Oscillation

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \quad (1)$$

$$m_2 \ddot{x}_2 = k_2(x_1 - x_2) - k_3 x_2 \quad (2)$$

where  $x_1, x_2$  are coordinates of the point masses,  $m_1, m_2$  masses and  $k_1, k_2, k_3$  spring rates.

• Predator-prey system (Lotka-Volterra Equations)

$$\dot{x}_1 = \lambda_1 x_1 - \alpha x_1 x_2 \quad (3)$$

$$\dot{x}_2 = \beta x_1 x_2 - \lambda_2 x_2 \quad (4)$$

where  $x_1, x_2$  number of individuals of each species (prey, resp.) and  $\lambda_1, \lambda_2 > 0$  growth and mortality rates.

**Definition:** (Linear System of ODEs of 1<sup>st</sup> Order)

A linear system of ODEs of 1<sup>st</sup> order is an equation

$$x'(x) = A(x)x(x) + g(x)$$

where the  $A(x)$  and  $g(x)$  are functions and  $x$  and  $g$  column vectors of a components, depending on  $x$ .

If  $g(x) = 0$ , then the system is called **homogeneous**, otherwise **inhomogeneous**.

**Remarks:**

• A homogeneous equation of order  $n$  can be reduced to a system of  $n$  equations of order 1.

• If  $n = 1$ , then we have a linear ODE.

## Solvability

**Proposition:** (Solvability of linear systems of 1<sup>st</sup> order ODEs)

Let the elements  $a_{ij}(x)$  of matrix  $A(x)$  and the components of  $g$  be continuous in interval  $[a, b]$ . Further, let  $x_0 \in [a, b]$  and  $y_0 = (y_1, \dots, y_n)^T$  be given arbitrarily. Then the initial value problem

$$y' = A(x)y + g(x), \quad y(x_0) = y_0$$

has a unique solution on  $[a, b]$ .

**Proposition:** (Solution of homogeneous linear systems of ODEs of 1<sup>st</sup> Order)

If the elements  $a_{ij}(x)$  of matrix  $A(x)$  are continuous in  $[a, b]$ , then the homogeneous system

$$y' = A(x)y$$

has exactly  $n$  linear independent solutions on  $[a, b]$ .

**Remarks:**

• A system of  $n$  linearly independent solutions of the system is called **fundamental system** or **basis of solutions**.

• The elements of the basis are called **fundamental solutions** or **fundamental functions**.

## Differential Equations I



Linear System of ODEs of 1st Order

## Holonomic Solutions

**Proposition:** (Holonomic Solution)

Let  $y_1, \dots, y_n$  be a fundamental system on  $[a, b]$  of

$$y' = A(x)y$$

Then any solution  $y$  on  $[a, b]$  can be written in the form

$$y = \sum_{i=1}^n c_i y_i, \quad \text{with } c_i \in \mathbb{R} \text{ or } \mathbb{C}.$$

This solution is called **holonomic solution** of the homogeneous system of differential equations.

**Proposition:** (Wronski Test)

Let  $y_1, \dots, y_n$  be solutions of the system  $y' = A(x)y$  on  $[a, b]$ . If  $W(x)$  is continuous in  $[a, b]$ , then

$$W(x) \equiv 0 \text{ or } W(x) \neq 0 \text{ for all } x \in [a, b].$$

and the solutions  $y_1, \dots, y_n$  form a fundamental system if and only if  $W(x) \neq 0$ .

Proof:

Let  $y_1, \dots, y_n$  be solutions of the system  $y' = A(x)y$  on  $[a, b]$ . Then  $W(x)$  satisfies the equation

$$W'(x) = -\text{tr}(A(x))W(x).$$

Since  $W(x)$  is continuous in  $[a, b]$ , it follows that  $W(x) \equiv 0$  or  $W(x) \neq 0$  for all  $x \in [a, b]$ .

If  $W(x) \neq 0$ , then the solutions  $y_1, \dots, y_n$  form a fundamental system.

□

## Wronski-Matrix Wronski-Test

**Question:** How to find  $n$  solutions  $y_1, \dots, y_n$  ( $n_0$  unknown) such as the basis of the homogeneous system?

**Definition:** (Wronski Matrix and Wronski Determinant)

The **Wronski Matrix**  $W(x)$  is formed by the columns of the fundamental system:

$$W(x) = (y_1 \quad \dots \quad y_n).$$

We define the **Wronski determinant** of the system of solutions  $y_1, \dots, y_n$  of the system  $y' = A(x)y$  as

$$W(x) = \det W(x).$$

**Proposition:** (Wronski Test)

Let  $y_1, \dots, y_n$  be solutions of the system  $y' = A(x)y$  on  $[a, b]$ . If  $W(x)$  is continuous in  $[a, b]$ , then

1.  $W(x) \equiv 0$  or  $W(x) \neq 0$  for all  $x \in [a, b]$ .

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