Differential Equations I

Winter 2021/22 / J. Behrens

TUHH Technische Universität Hamburg

BITTE BEACHTEN SIE DIE 3G-REGEL!

PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

–VOLLSTÄNDIG GEIMPFTE
–GENESENE
–GETESTETE
(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

–FULLY VACCINATED
–RECOVERED
–TESTED
(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

1 Recall:

- **Type**: Consider ODE of the form

$$F(x, y', y'') = 0$$

Idea:

- **Substitution**: using $v := y'$ we obtain ODE of 1st order:

$$F(x, v, v') = 0$$

- **Integration**: If $v = \Psi(x, C)$ is a general solution to the 1st order ODE, then

$$y(x) = \int \Psi(\zeta, C)\, d\zeta + C_1, \quad C, C_1 \in \mathbb{R}$$

is a general solution to the 2nd order ODE.

1

Example: $y'' = 5\,y' \ln x \qquad x > 0$

or $\quad y'' - 5y' \ln x =: F(x, y', y'') = 0$

- Substitution: $v = y' \Rightarrow v' = 5 \ln x \; v$

Side Comp:

$v' = \frac{5 \ln x}{\frac{1}{v}}$

$\int \frac{dv}{v} = \int 5 \ln x \, dx$

$\int \ln x \, dx = \int 1 \cdot \ln x \, dx$

Integr. by parts

$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx$

$= x \cdot \ln x - x + C$

- Separation of Variables:

$$\int \frac{dv}{v} = 5x \ln x - 5x + C$$

$$\Rightarrow \ln|v| = 5x \ln x - 5x + C$$

$$\Rightarrow v = C_1 e^{5x \ln x - 5x}$$

$$= C_1 x^{5x} e^{-5x} = C_1 \left(\frac{x}{e}\right)^{5x}$$

- Integration of v:

$$y(x) = C_1 \int x^{5x} e^{-5x} \, dx + C_2$$

So this yields the Solution of 2^{nd} order ODE.

2

Remark:
Let the 2^{nd} order ODE be given (note that x does not appear explicitly):

$$F(y, y', y'') = 0$$

Solution Idea:

- **Substitution**: with $v(y) := y'$ and the chain rule we obtain:

$$y'' = \frac{d}{dx} v(y) = \frac{dv}{dy}\frac{dy}{dx} = v'(y)y' = v'(y)v(y)$$

  This yields a 1^{st} order ODE for v: $F(y, v, v'v) = 0$.

- **Integration**: If $v = \Psi(x, C)$ is general solution of 1^{st} order ODE, then with $v(y) = y'$ we obtain

$$y' = \Psi(y, C)$$

  an ODE with separable variables for y, with general implicit solution

$$\int_{y_0}^{y} \frac{d\zeta}{\Psi(\zeta, C)} = x + C_1, \quad C, C_1 \in \mathbb{R}.$$

2

Example: $y'' = -\frac{y'^2}{5y}$

$y > 0$

- Substitution: $v(y) = y'$

  or $y'' = v' \cdot v$

  $\Rightarrow v v' = -\frac{v^2}{5y}$ ODE of 1^{st} order

  $\Rightarrow \frac{v'}{v} = -\frac{1}{5y}$

Side Comp:

$v(y) = y'(x)$

$\frac{dv}{dx} = v'(y) \cdot y'(x)$

$= v' \cdot v$ Subst.

- Seperation of Variables:

  $\int \frac{dv}{v} = -\frac{1}{5} \ln|y| + C$

  $\Rightarrow v(y) = C_1 \, y^{-\frac{1}{5}}$

- Since $y' = v(y)$

$$\Rightarrow y' = c_1 y^{-\frac{1}{5}}$$

- Again apply sep. of variables:

$$\int y^{\frac{1}{5}} dy = C_1 x + C_2$$

$$\Rightarrow \frac{5}{6} y^{\frac{6}{5}} = C_1 x + \tilde{C}_2$$

- With this we obtain the solution:

$$y(x) = (C_3 x + C_4)^{\frac{5}{6}}$$

3 **Consider**:
ODE of the form $y' = \phi(\frac{y}{x})$, with $x \neq 0$ and ϕ continuous.

Solution Idea:

- **Substitution**: $u = \frac{y}{x}$ yields:

$$y = xu \quad \Rightarrow \quad y' = u + xu' = \phi(u)$$

  Therefore

$$xh' = \phi(u) - u \quad \Rightarrow \quad u' = \frac{\phi(u) - u}{x}.$$

- **Separation of Variables**: We obtain as solution

$$\frac{du}{\phi(u) - u} = \frac{dx}{x} \quad \Rightarrow \quad \int \frac{du}{\phi(u) - u} = \ln|x| + C.$$

3

Example: $y' = \frac{xy}{x^2 - y^2} = \frac{\frac{y}{x}}{1 - (\frac{y}{x})^2} = \phi(\frac{y}{x})$

- Substitution: $\phi(u) = \frac{u}{1-u^2}$ where $u = \frac{y}{x}$

- We obtain: $\int \frac{du}{\frac{u}{1-u^2} - u} = \ln|x| + C$

$$\Rightarrow \int \frac{1-u^2}{u - u(1-u^2)}\, du = \ln|x| + C$$

$$\Rightarrow \int \frac{1-u^2}{u^3}\, du = \ln|x| + C$$

$$\Rightarrow -\frac{1}{2u^2} - \ln|u| = \ln|x| + C$$

- Back Substitution

$$-\frac{x^2}{2y^2} = \ln|y| + C \Rightarrow |y| = e^{-\frac{x^2}{2y^2} - C}$$

$$\Rightarrow y = C_1 e^{-\frac{x^2}{2y^2}}$$

Computation for case $k = 2$:

4

- Euler's ODE (homogenous): $a_0 y + a_1 x y' + a_2 x^2 y'' = 0$.
- Substitution yields: $a_0 + a_1 r + a_2 r(r-1) = 0$, quadratic polynomial.
- Differentiation proves: $y = x^r$ is solution of homogenous Euler's ODE, if r root of polynomial.
- If $r_1 \neq r_2$ are real roots of polynomial, then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ are solutions of ODE.
- If $r_1, r_2 \in \mathbb{C}$ are complex roots, then if $r_1 = a + ib$ is root, so is $r_2 = \bar{r}_1 = a - ib$.
- Complex solution for $y = x^r$:

$$x^{a+ib} = e^{\ln x^{a+ib}} = e^{(a+ib)\ln x} = e^{a \ln x} e^{ib \ln x} = x^a[\cos(b \ln x) + i \sin(b \ln x)]$$

- For complex solutions of the problem one finds

$$y_1(x) = x^a \cos(b \ln x) \quad \text{and} \quad y_2(x) = x^a \sin(b \ln x)$$

two solutions of the homogenous Euler's ODE.

- General solution: due to linearity the general solution is

$$y(x) = c_1 x^a \cos(b \ln x) + c_2 x^a \sin(b \ln x).$$

$y = x^r \quad y = x^2 \quad r = 2$

$y' = 2x$

$y'' = 2$

$a_0 x^r + a_1 r x^{r-1} + a_2 r(r-1) x^{r-2}$

$a_0 x^2 + a_1 2x + a_2 2 \cdot 1 \cdot x^0$

Example: $x^2 y'' + 4xy' + 2y = 0$ Subst. $x^r = y(x)$

$\Rightarrow 2 + 4r + r(r-1) = r^2 + 3r + 2 \stackrel{!}{=} 0$

$\Rightarrow r_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = -\frac{3}{2} \pm \frac{1}{2}$

$\Rightarrow r_1 = -1$ and $r_2 = -2$

$\Rightarrow y(x) = c_1 \frac{1}{x} + c_2 \frac{1}{x^2}$.