Exercise 1: (1+4 points)

a) Compute the general solution of the following differential equation by using separation of variables

$$xy' - 3y = 0.$$

b) Solve the following initial value problem for the given Bernoulli differential equation

$$y' - y + 2y^2 = 0 \quad \text{and} \quad y(0) = \frac{1}{3}.$$

Solution:

a) (1 point)

$$xy' - 3y = 0 \quad \Rightarrow \quad y' = \frac{3y}{x} \quad \Rightarrow \quad \frac{y'}{y} = \frac{3}{x} \quad \Rightarrow \quad \int \frac{dy}{y} = 3 \int \frac{dx}{x}$$

$$\Rightarrow \quad \ln|y| = 3\ln|x| + c = \ln|x^3| + c \quad \Rightarrow \quad y = kx^3$$

b) (4 points)

Bernoulli differential equation:

$$y' + a(x)y + b(x)y^\alpha = y' - y + 2y^2 = 0 = 0 \quad \Rightarrow \quad \alpha = 2,\ a(x) = -1,\ b(x) = 2\,.$$

Substitution: $u(x) = y^{(1-\alpha)}(x) = \dfrac{1}{y(x)} \Leftrightarrow y(x) = u^{1/(1-\alpha)}(x) = \dfrac{1}{u(x)}$

Transformed linear inhomogeneous differential equation:

$$u'(x) + (1-\alpha)a(x)u(x) = (\alpha - 1)b(x) \quad \Rightarrow \quad u' + u = 2\,.$$

General homogeneous solution: $u_h(x) = ke^{-x}$ for $k \in \mathbb{R}$.

Ansatz for a particular solution of the inhomogeneous ODE:

$u_p(x) = c \quad \Rightarrow \quad c = 2$

General solution of the transformed equation and back substitution:

$$\Rightarrow \quad u(x) = ke^{-x} + 2 \quad \Rightarrow \quad y(x) = \frac{1}{ke^{-x} + 2}$$

Solution of the initial value problem:

$$\Rightarrow \quad \frac{1}{3} = y(0) = \frac{1}{k+2} \quad \Rightarrow \quad k = 1 \quad \Rightarrow \quad y(x) = \frac{1}{e^{-x} + 2}$$

Exercise 2: (3 points)

Compute the general solution of the following system of differential equations

$$\boldsymbol{y}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \boldsymbol{y}\,.$$

Hint: Note that $(1-\lambda)^2 - 1 = \lambda(\lambda - 2)$.

Solution:

(3 points)

Compute the eigenvalues of the matrix $\boldsymbol{A}$:

$$\begin{aligned} p_{\boldsymbol{A}}(\lambda) &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)((1-\lambda)^2 - 1) = (1-\lambda)\lambda(\lambda-2) \end{aligned}$$

$$\Rightarrow \qquad \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 2$$

Compute the corresponding eigenvectors by $(\boldsymbol{A} - \lambda \boldsymbol{E})\boldsymbol{v} = \boldsymbol{0}$

Eigenvector $\boldsymbol{v}^1$ corresponding to $\lambda_1 = 1$:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad \boldsymbol{v}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvector $\boldsymbol{v}^2$ corresponding to $\lambda_2 = 0$ as well as $\boldsymbol{v}^3$ corresponding to $\lambda_3 = 2$ (the eigenvectors are orthogonal because $\boldsymbol{A}$ is symmetric):

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad \boldsymbol{v}^2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}^3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Alternatively: computation of $\boldsymbol{v}^3$ corresponding to $\lambda_3 = 2$ by:

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad \boldsymbol{v}^3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

For $c_1, c_2, c_3 \in \mathbb{R}$ the general solution is given by

$$\boldsymbol{y}(x) = c_1 e^x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Exercise 3: (4 points)

Solve the following initial value problem

$$y'' + y' - 2y = e^x \,, \quad y(0) = 1 \,,\; y'(0) = \frac{4}{3} \,.$$

Solution:

a) (1 point)
For the general solution of the homogeneous equation $y'' + y' - 2y = 0$ we compute the characteristic polynomial:

$$p(\lambda) = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) \stackrel{!}{=} 0 \;\Rightarrow\; \lambda_1 = 1,\; \lambda_2 = -2$$

and obtain the general homogeneous solution
$y_h(x) = c_1 e^x + c_2 e^{-2x}$ for $c_1, c_2 \in \mathbb{R}$.

b) (1 point)
We use the following ansatz for a particular solution of the inhomogenous ODE: $y_p(x) = axe^x$, as for the inhomogeneity $e^{\mu x} = e^x$ it holds that $\mu = 1 = \lambda_1$.
Plugging the ansatz into the inhomogeneous differential equation yields

$$(axe^x)'' + (axe^x)' - 2axe^x = ((x+2) + (x+1) - 2x)ae^x = 3ae^x \stackrel{!}{=} e^x$$

$$\Rightarrow \quad a = \frac{1}{3} \quad \Rightarrow \quad y_p(x) = \frac{1}{3}xe^x$$

c) (1 point)
The general inhomogeneous solution is given by $y(x) = c_1 e^x + c_2 e^{-2x} + \frac{1}{3}xe^x$

d) (1 point)
$y'(x) = c_1 e^x - 2c_2 e^{-2x} + \frac{1}{3}(x+1)e^x$ and the initial values yield:
$1 = y(0) = c_1 + c_2 \;\Rightarrow\; c_1 = 1 - c_2$,
$\frac{4}{3} = y'(0) = c_1 - 2c_2 + \frac{1}{3} = 1 - c_2 - 2c_2 + \frac{1}{3} \;\Rightarrow\; c_2 = 0 \;\Rightarrow\; c_1 = 1.$
Solution of the initial value problem: $y(x) = e^x + \frac{1}{3}xe^x$

Exercise 4: (3 points)

Consider the linear differential equation

$$y'' + y = 0.$$

a) State a complex-valued fundamental system,

b) compute the real-valued general solution and

c) obtain all solutions of the corresponding boundary value problem with boundary values $y(0) = 2$ and $y(\pi) = -2$.

Solution:

a) (1 point)

The characteristic polynomial: $p(\lambda) = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

yields the complex-valued fundamental system

$$e^{ix} = \cos(x) + i\sin(x)\,, \quad e^{-ix} = \cos(x) - i\sin(x)$$

b) (1 point)

Real and imaginary part lead to the real-valued general solution:

$y(x) = c_1 \cos(x) + c_2 \sin(x)$ for $c_1, c_2 \in \mathbb{R}$

c) (1 point)

The boundary values yield:

$2 = y(0) = c_1 \cos(0) + c_2 \sin(0) \quad \Rightarrow \quad c_1 = 2$ and

$-2 = y(\pi) = c_1 \cos(\pi) + c_2 \sin(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 2$

Solutions of the boundary value problem are given by:

$y(x) = 2\cos(x) + c_2 \sin(x)$ for $c_2 \in \mathbb{R}$

Exercise 5: (5 points)

Consider the following system of linear first-order differential equations:

$$\begin{aligned} \dot{x} &= x + 2y - 4 \\ \dot{y} &= 2x + y - 5\,. \end{aligned}$$

a) State the system in matrix-vector notation ,

b) compute all stationary solutions (equilibria),

c) and determine their stability properties.

d) Compute the general solution of the system of linear differential equations.

Solution:

a) (1 point)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

b) (1 point)

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad \rightarrow \quad \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 5 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -3 \end{array}\right)$$

We obtain the equilibrium $\boldsymbol{x}^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

c) (1 point)

The eigenvalues λ_1, λ_2 are computed by:

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0$$

As $\lambda_1 = -1 < 0 < \lambda_2 = 3$, we conclude that $\boldsymbol{x}^*$ is an instable saddle point .

d) (2 points)

The equilibrium $\boldsymbol{x}^*$ solves the (inhomogeneous) system of differential equations and thus serves as a particular solution. The eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal, due to the symmetry of the matrix.

We compute the eigenvector $\boldsymbol{v}_1$ corresponding to $\lambda_1 = -1$:

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \quad \Rightarrow \quad \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

And obtain the general solution of the system

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}$$